Few-atom problem & quantum Townes solitons

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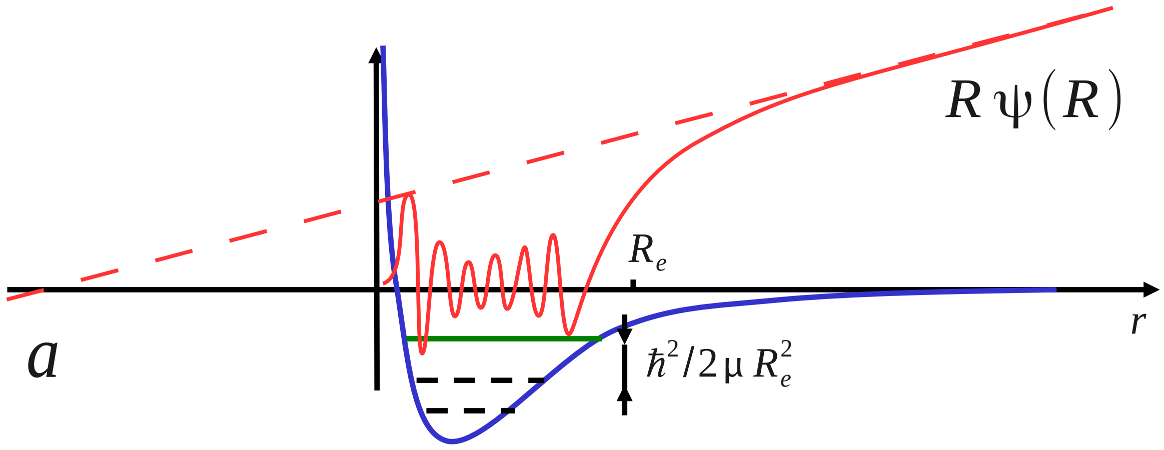
Introduction to the few-body problem: Zero-range method and Efimov effect

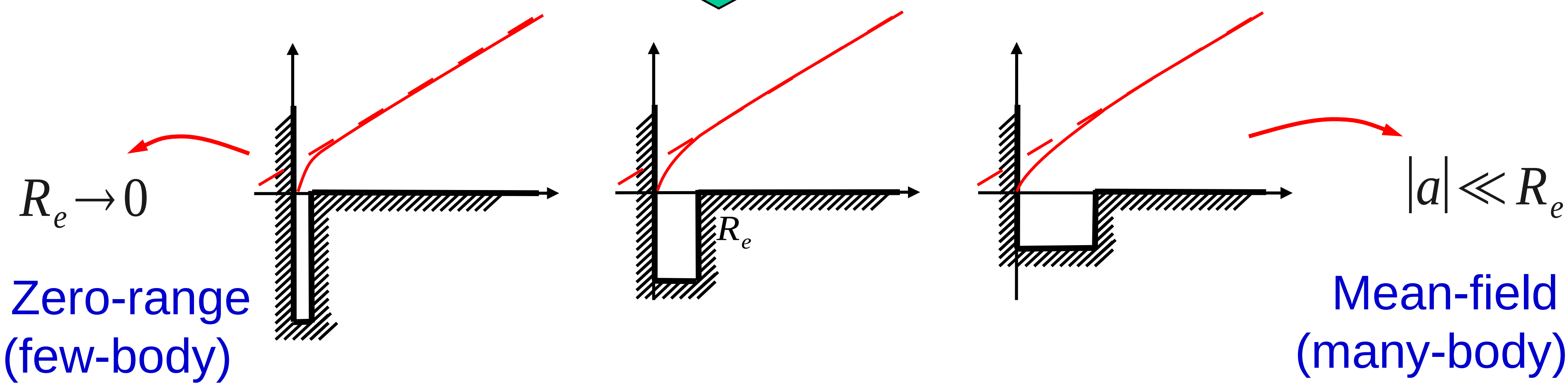
From van der Waals to effective potential



$R_e \sim 40 \div 100\, a_0$

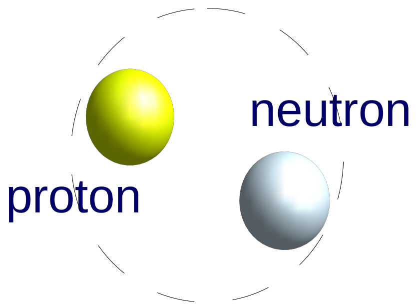
$1/k \sim 10^4\, a_0$

Replace by equivalent potentials as long as a is the same and $kR_e \ll 1$

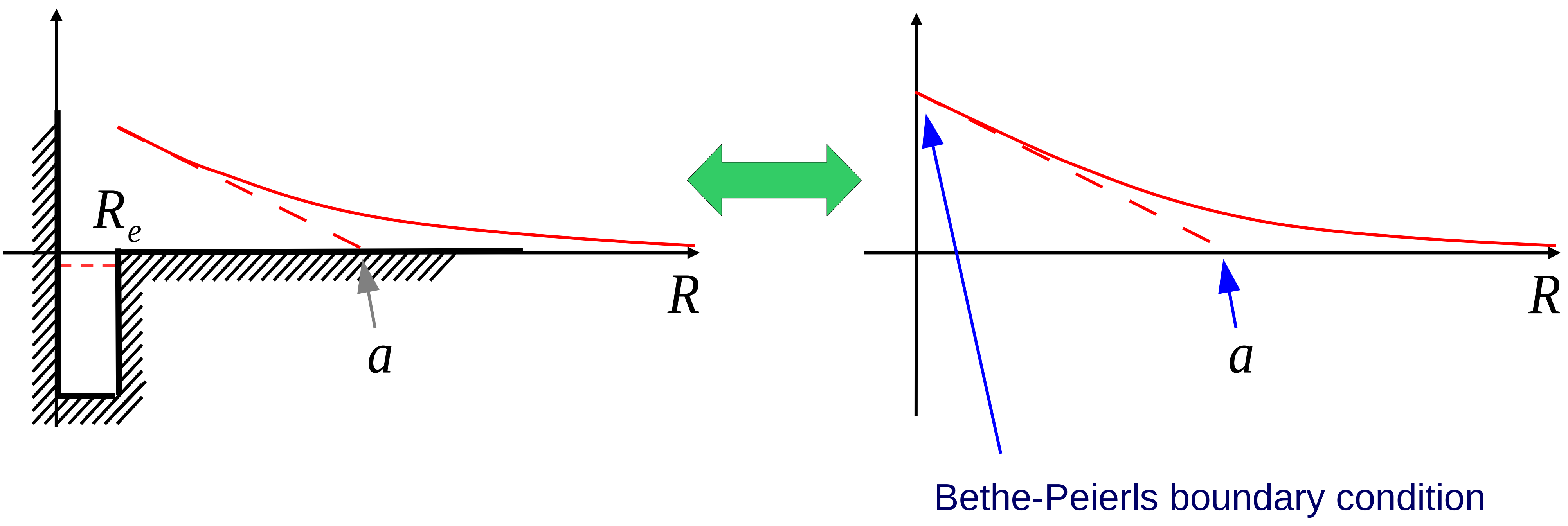


Bethe – Peierls boundary conditions

Bethe and Peierls (1934), "Quantum theory of the diplon"



Diplon (deuteron) is a weakly bound dimer with $R_e \approx 10^{-13}\,\mathrm{cm}\,,\ a \approx 4.5 \cdot 10^{-13}\,\mathrm{cm}$



$$\frac{(R\psi(R))'}{R\psi(R)} = -\frac{1}{a} \quad \Longleftrightarrow \quad \psi(R) \underset{R\to 0}{\rightarrow} C\left(\frac{1}{R} - \frac{1}{a}\right)$$

Ultracold gases + Feshbach resonances

Nuclear matter:

$$R_e \approx 10^{-13}\,\mathrm{cm}\,,\;\; a \approx 4.5\cdot 10^{-13}\,\mathrm{cm}$$

⇩

Ultracold atoms:

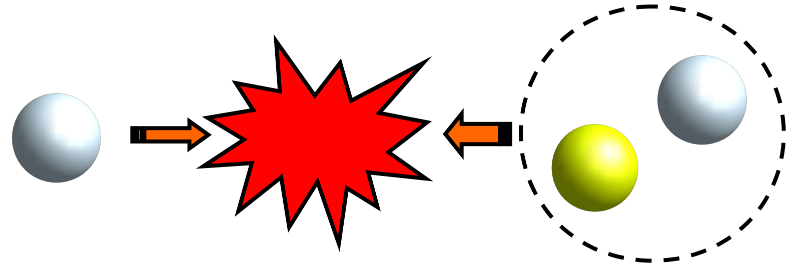
$$R_e \sim 0.5\cdot 10^{-6}\,\mathrm{cm}\,;\;\; \lambda_{dB}, a > 10^{-5}\,\mathrm{cm}$$

⇩

Golden age of the zero-range approach!

It becomes quantitative

STM approach, neutron-deuteron scattering



SOVIET PHYSICS JETP VOLUME 4, NUMBER 5 JUNE, 1957

Three Body Problem for Short Range Forces. I. Scattering of Low Energy Neutrons by Deuterons

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Substituting Eq. (11) in Eq. (10), we get an integral equation for $\chi(\mathbf{k})$:

$$(\alpha - \gamma_k)\chi(\mathbf{k}) + 8\pi \int \frac{d\mathbf{k}'}{(2\pi)^3} \frac{\chi(\mathbf{k}')}{k^2 + k'^2 + \mathbf{k}\mathbf{k}' - (ME/\hbar^2) - i\tau} = 0. \tag{12}$$

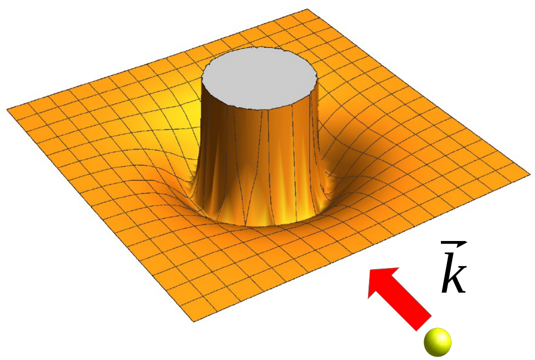
STM equation

The solution of this equation determines the wave function of the system, in accord with Eq. (11). For states with a definite quantity of momentum, Eq. (12) reduces to an equation for a function which depends on one independent variable, which can be solved numerically.

The idea for this consideration of the three body problem was supplied by L. D. Landau. !

Zero-range method

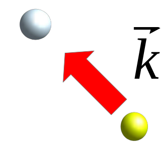
$$\left[-\nabla_{\vec{r}}^2 + 2\mu\, U(|\vec{r}-\vec{R}|) - k^2\right]\psi(\vec{r}) = 0$$



VS.

$$\left[-\nabla_{\vec{r}}^2 - k^2\right]\psi(\vec{r}) = 0$$

$$\psi(\vec{r}) \to \frac{1}{|\vec{r}-\vec{R}|} - \frac{1}{a}$$



expand in spherical waves

set of 1D radial equations

etc.

$$\psi(\vec{r}) = \exp[i\vec{k}(\vec{r}-\vec{R})] + C\,\frac{\exp(ik|\vec{r}-\vec{R}|)}{4\pi|\vec{r}-\vec{R}|}$$

Plane wave

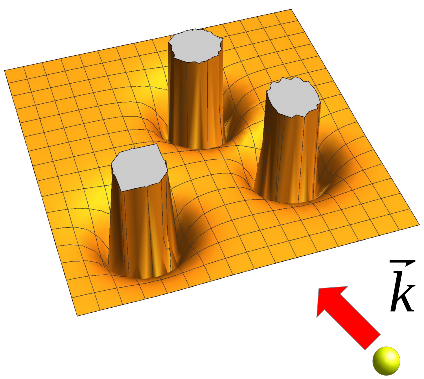
Green's function

The only number that we need to know

$$C = \frac{4\pi}{-1/a - ik}$$

Zero-range method

$$\left[-\nabla_{\vec{r}}^2 + 2\mu \sum_j U(|\vec{r}-\vec{R}_j|) - k^2\right]\psi(\vec{r}) = 0$$



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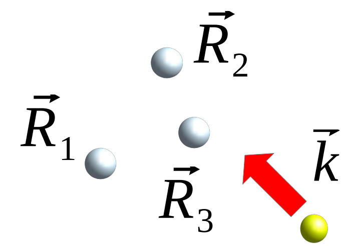


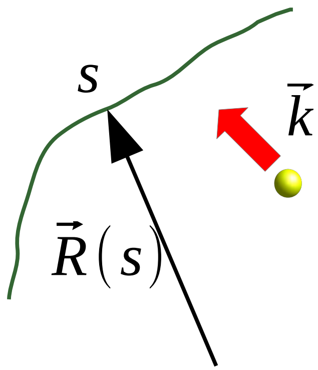
?

$$\psi(\vec{r}) = \exp[i\vec{k}\vec{r}] + \sum_j C_j \frac{\exp(ik|\vec{r}-\vec{R}_j|)}{4\pi|\vec{r}-\vec{R}_j|}$$

$$\hat{M}_{k,a,\{R_{ij}\}}\begin{pmatrix}\vdots\\ C_j\\ \vdots\end{pmatrix} = \begin{bmatrix}\vdots\\ e^{i\vec{k}\vec{R}_j}\\ \vdots\end{bmatrix}$$

Demkov & Ostrovskii « Zero-range potentials and their applications in atomic physics » ~1975

Scattering by a curve or hypersurface → STM



Helmholtz equation

$$\left[-\nabla_{\vec{r}}^{2}-k^{2}\right]\psi(\vec{r})=0$$

+ a boundary condition on the curve

$$\psi(\vec{r})=\exp[i\vec{k}\vec{r}]+\int ds\,C(s)\frac{\exp(ik|\vec{r}-\vec{R}(s)|)}{4\pi|\vec{r}-\vec{R}(s)|}$$

$$\int ds\,K(s,s')C(s')=F(s)$$

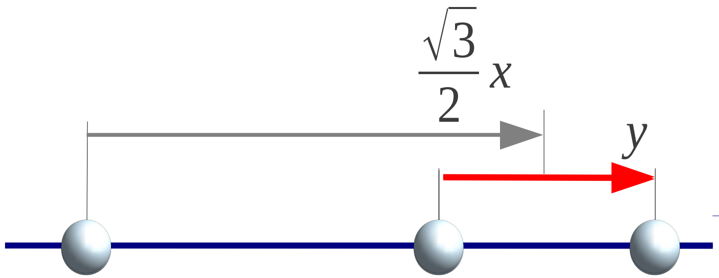
← Fredholm equation (1 dimensional)

3D 3-body scattering = 6D 1-body scattering by a 3D hyperspace

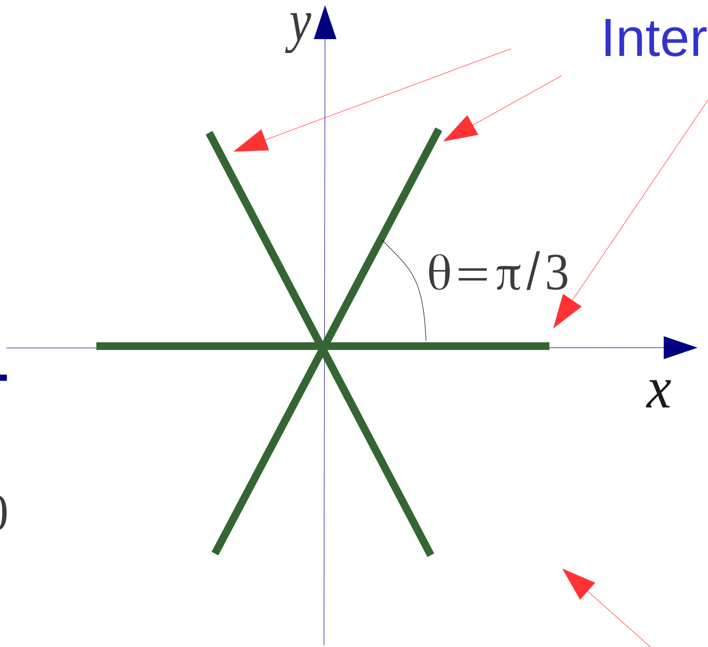
6D Schroedinger equation → 3D Fredholm equation (STM)

Examples

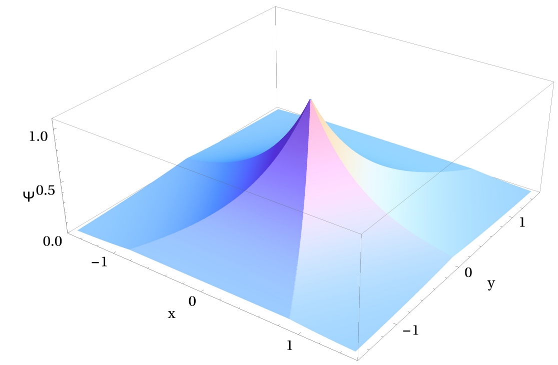
3 one-dimensional bosons :



$$(-\partial^2/\partial x^2 - \partial^2/\partial y^2 - mE)\psi(\boldsymbol{\rho}) = 0$$

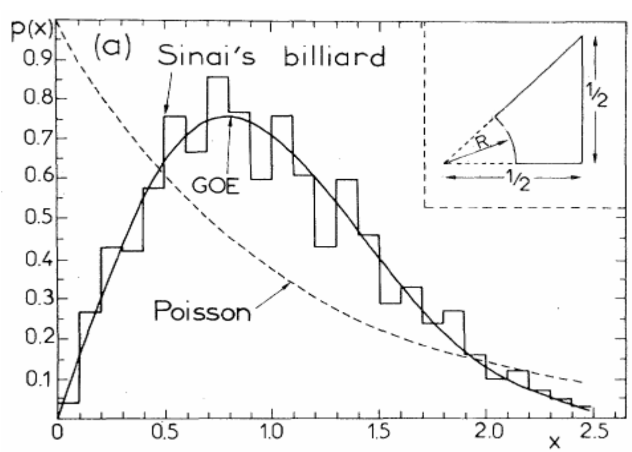


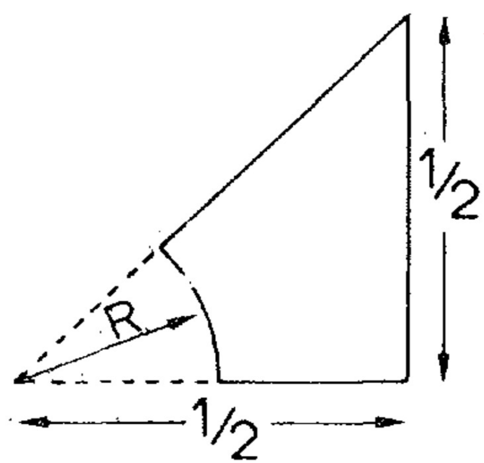
Interactions



$$\int ds\, K(s,s')C(s') = F(s)$$

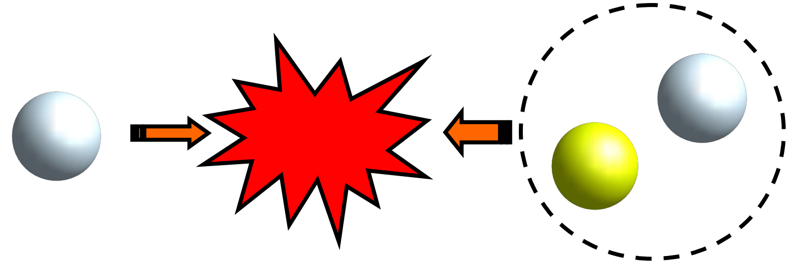
Sinai billiard [Bohigas,Giannoni,Schmit'1984]:





Is zero-range model sufficient for three atoms?
Is *a* sufficient?

STM approach, neutron-deuteron scattering



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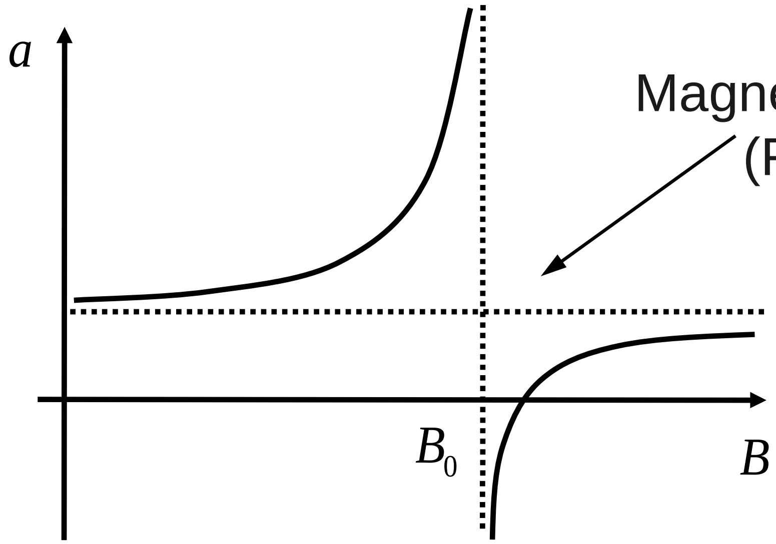
Correct

Wrong [Danilov'1961]

Ultracold gas with large scattering strength



Magnetically tunable resonance
(Feshbach resonance)

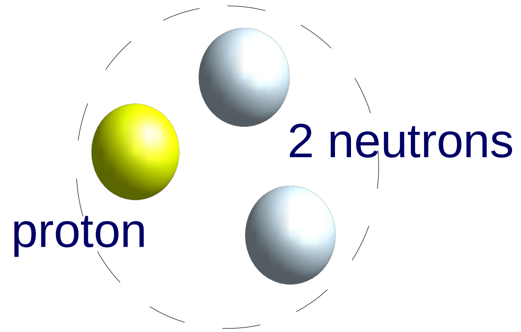
BEC & large *a*	Deg. Fermi gas & large *a*
⇩	⇩
Collapse and inelastic losses. Very short lifetime ☹	No collapse & long lifetime ☺
^{23}Na (MIT), ^{85}Rb (JILA), ^{87}Rb (MPQ), ^{7}Li (Rice), ^{135}Cs (Innsbruck)	^{40}K (JILA), ^{6}Li (Duke, MIT, ENS, Innsbruck, Rice)

What is the difference?

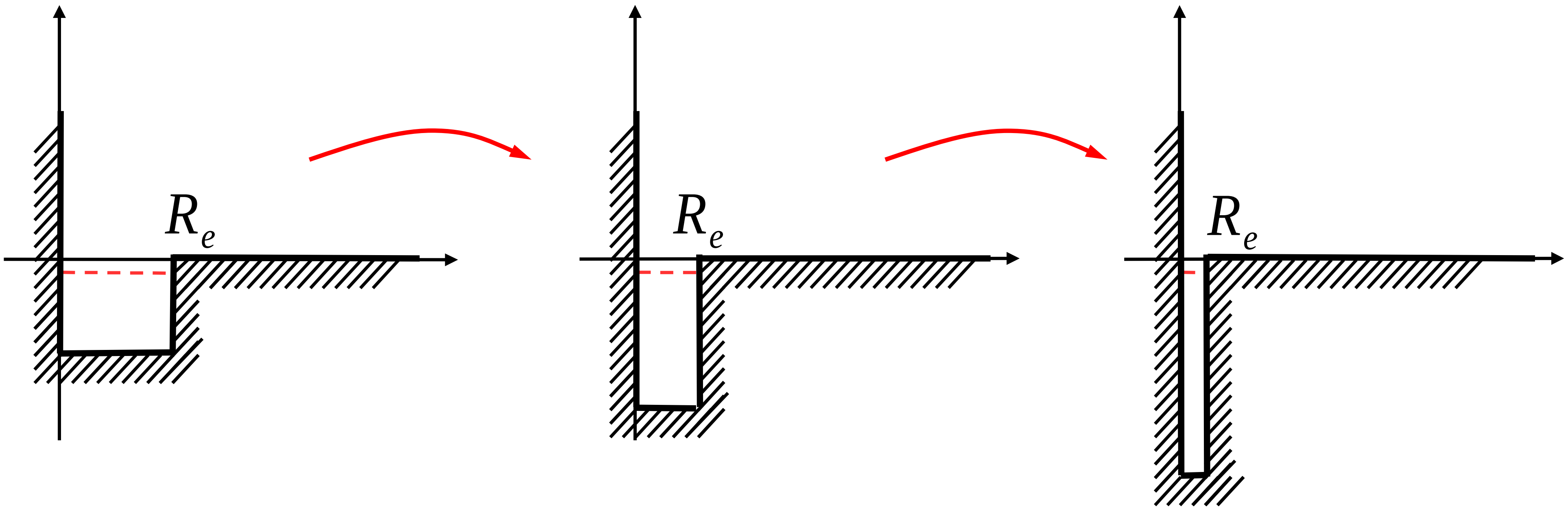
Efimovian vs nonefimovian regimes

Thomas effect

Thomas (1935), "The interaction between a neutron and a proton and the structure of ^3H"



Decrease the range of the proton-neutron potential keeping their binding energy constant



The trimer binding energy tends to infinity! ⇨ Thomas effect or Thomas collapse

Example: three ^4He atoms form much deeper bound molecule than two ^4He atoms

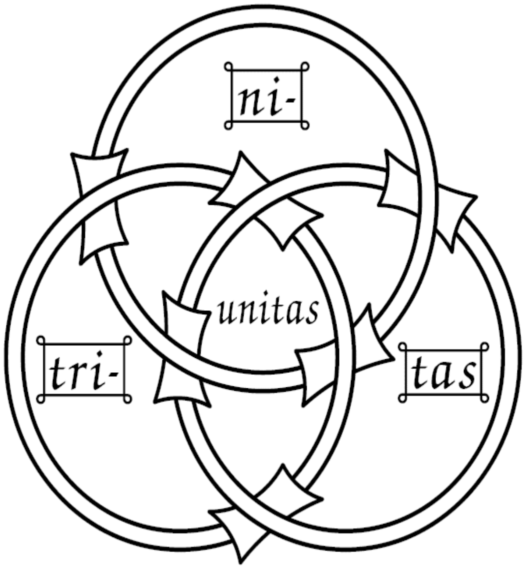
Borromean binding

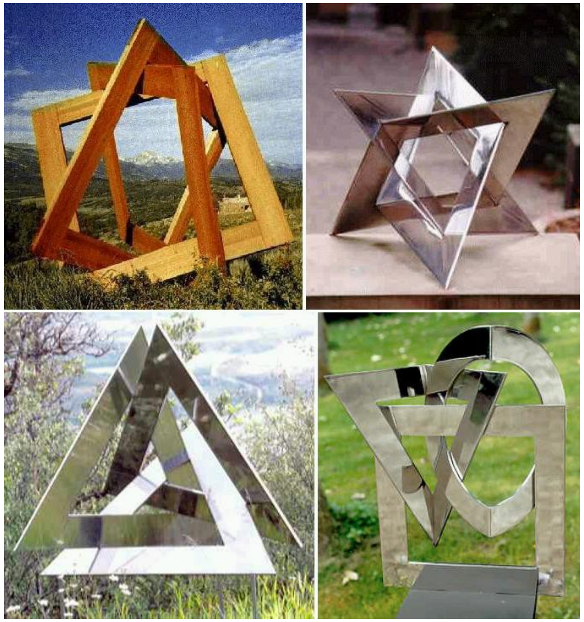
Borromean rings – symbol of strength in unity.
Remove one ring and the other two fall apart

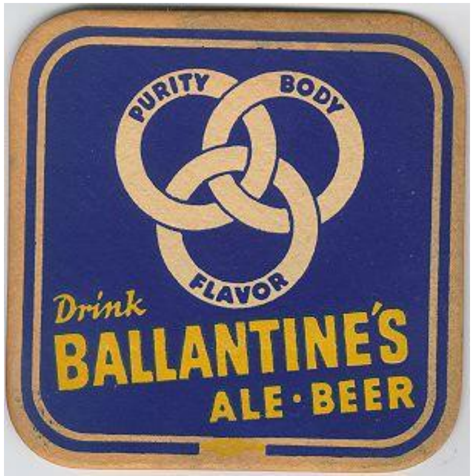
The symbol is used in a number of other applications

Borromean sculptures (John Robinson)

Christian Trinity



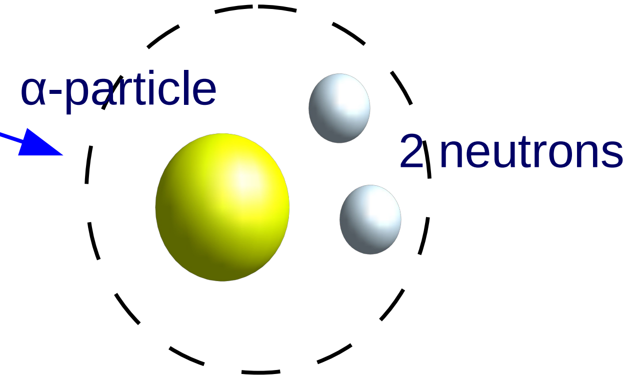


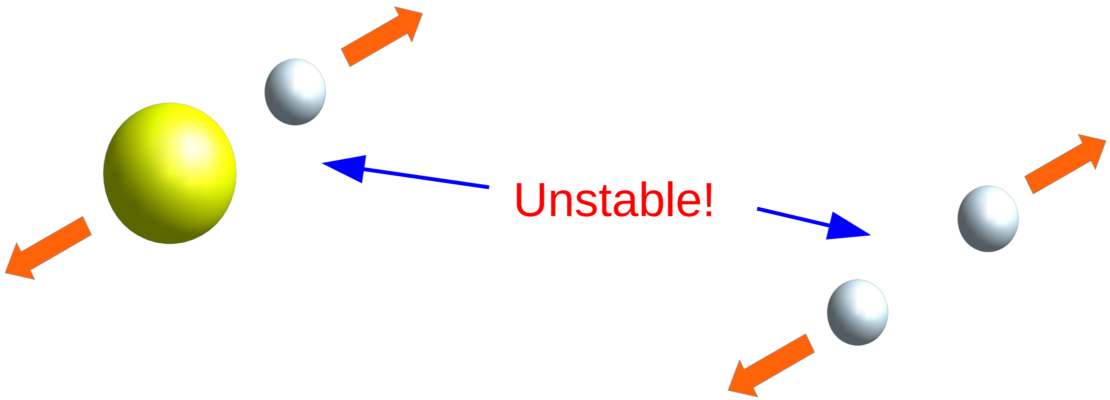


Borromean binding

... in nuclear physics (see halo nuclei, neutron-rich nuclei)

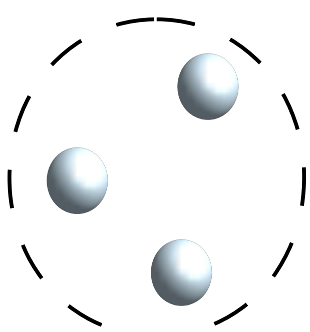
Example: stable ^{6}He nucleus

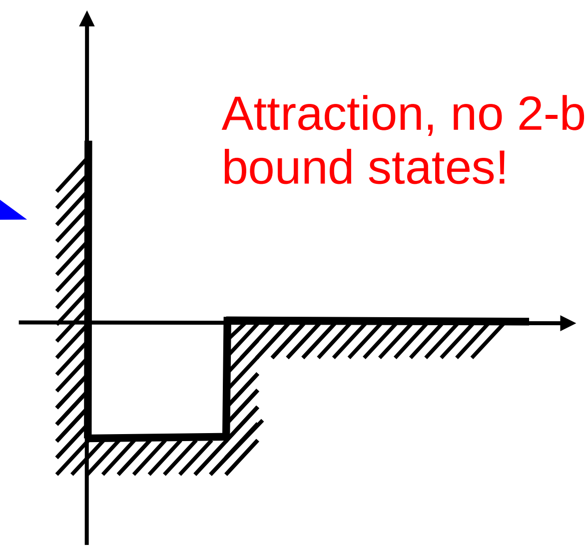




Unstable!

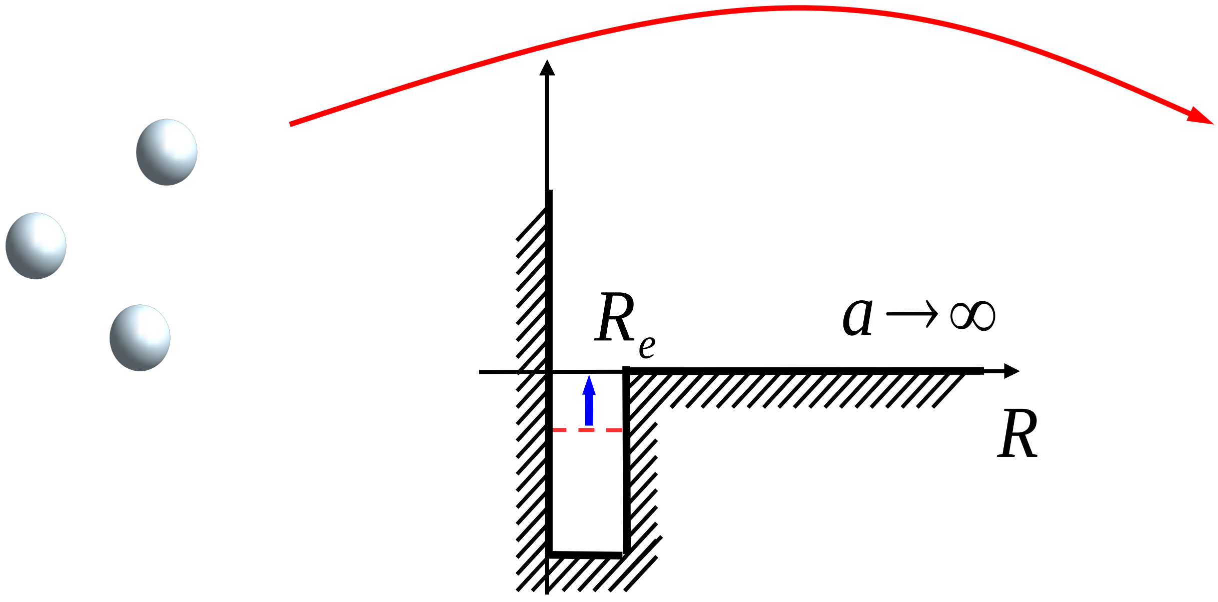
Not a big surprise. Three bosons attracting each other via this potential could form a trimer state





Attraction, no 2-body bound states!

Efimov effect



Efimov (1970) found that the number of trimer states

$$N_{tr} \sim \frac{s_0}{\pi} \log(|a|/R_e) \to \infty$$

with the accumulation point at

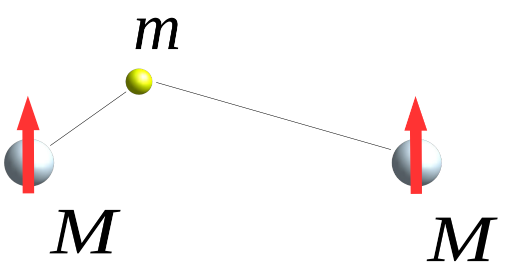
$$E = 0$$

Efimov state – weakly bound trimer state

Discrete scaling symmetry: three-body observables depend on R_e, but if $R'_e = R_e \exp(\pm\pi/s_0)$, they do not change

For three identical bosons $s_0 \approx 1.00624$ This number depends on the symmetry (Fermi, Bose) and on the masses of particles

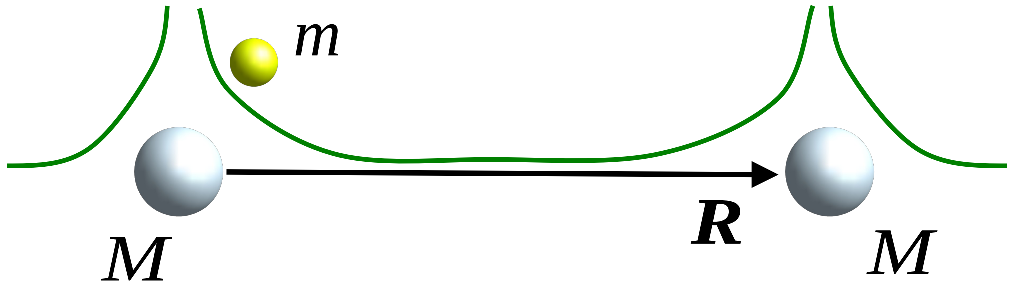
(2+1)-body problem



Heavy bosons → Efimovian system

Heavy fermions → non-Efimovian-to-Efimovian transition for *M/m*~13 [Efimov'1973]

Born-Oppenheimer approximation



Effective interaction between heavy atoms is provided by exchange of the fast light particle. Born-Oppenheimer approximation [Fonseca,Redish,Shanley'1979].

Light atom wavefunction: $$\psi(\boldsymbol{r}) = \frac{\exp(-\kappa|\boldsymbol{r}-\boldsymbol{R}/2|)}{|\boldsymbol{r}-\boldsymbol{R}/2|} + \frac{\exp(-\kappa|\boldsymbol{r}+\boldsymbol{R}/2|)}{|\boldsymbol{r}+\boldsymbol{R}/2|}$$

Bethe-Peierls boundary condition gives: $$\frac{1}{a} - \kappa + \frac{\exp(-\kappa R)}{R} = 0$$

$$R \ll a \Rightarrow \kappa \approx \frac{0.567}{R}$$

$$\Downarrow$$

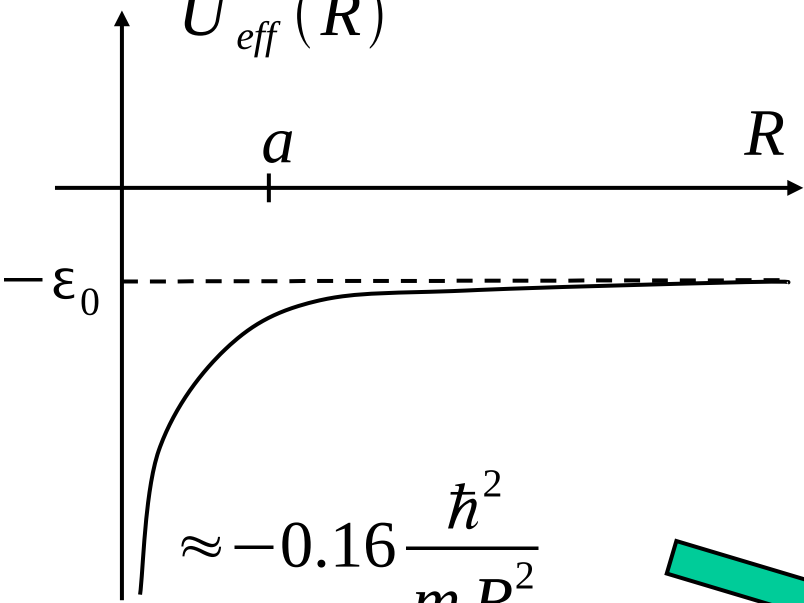
$$U_{eff}(R) = -\frac{\hbar^2 \kappa^2}{2m} \approx -0.16 \frac{\hbar^2}{m R^2}$$

$$R \gg a \Rightarrow \kappa \approx \frac{1}{a}$$

$$\Downarrow$$

$$U_{eff}(R) \approx -\frac{\hbar^2}{2 m a^2} = -|\epsilon_0|$$

$$\left[-\frac{\hbar^2}{M}\frac{\partial^2}{\partial R^2}+\widetilde{U}_{eff}(R)\right]\chi(R)=E\chi(R)$$



$$\approx -0.16\frac{\hbar^2}{mR^2}$$

Solving the Schrödinger equation for the heavy atoms we take into account their statistics:

$$\widetilde{U}_{eff}(R)\approx U_{eff}+\frac{\hbar^2 l(l+1)}{MR^2}$$

Heavy fermions ⇒ $l=1,3,5\ldots$

Heavy bosons ⇒ $l=0,2,4\ldots$

$$\widetilde{U}_{eff}(R)\approx\frac{\hbar^2}{MR^2}\underbrace{\left(l(l+1)-0.16\frac{M}{m}\right)}_{\beta}$$

$R\ll a,\ E=0$ ⇒ $(-\partial^2/\partial R^2+\beta/R^2)\chi(R)=0$

$\chi(R)=R^{\nu}$ $\qquad \nu_{\pm}=1/2\pm\sqrt{\beta+1/4}$

Efimovian vs non-Efimovian regimes

$$R \ll a, E=0 \Rightarrow (-\partial^2/\partial R^2 + \beta/R^2)\chi(R)=0$$

$$\chi(R) \propto R^{\nu} \qquad \nu_{\pm} = 1/2 \pm \sqrt{\beta + 1/4}$$

$$\beta = l(l+1) - 0.16\frac{M}{m}$$

Identical fermions $\Rightarrow l=1 \Rightarrow \tilde{U}_{eff}(R) \approx \frac{\hbar^2}{MR^2}\left(2 - 0.16\frac{M}{m}\right)$

$M/m < 13.6 \Rightarrow \beta > -1/4$

$$\chi(R) \propto R^{\nu_+}$$

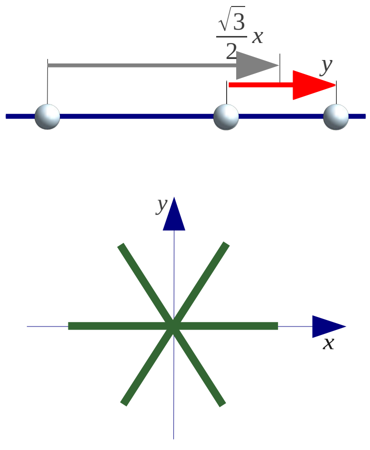
Unique wavefunction without zeros $\Rightarrow$ no Efimov states. Centrifugal barrier is stronger than the induced attraction. Non-Efimovian case

$M/m > 13.6 \Rightarrow \beta < -1/4$

$$\chi(R) \propto \sqrt{R}\sin(s_0 \log R/R_0), s_0 = \sqrt{-1/4 - \beta}$$

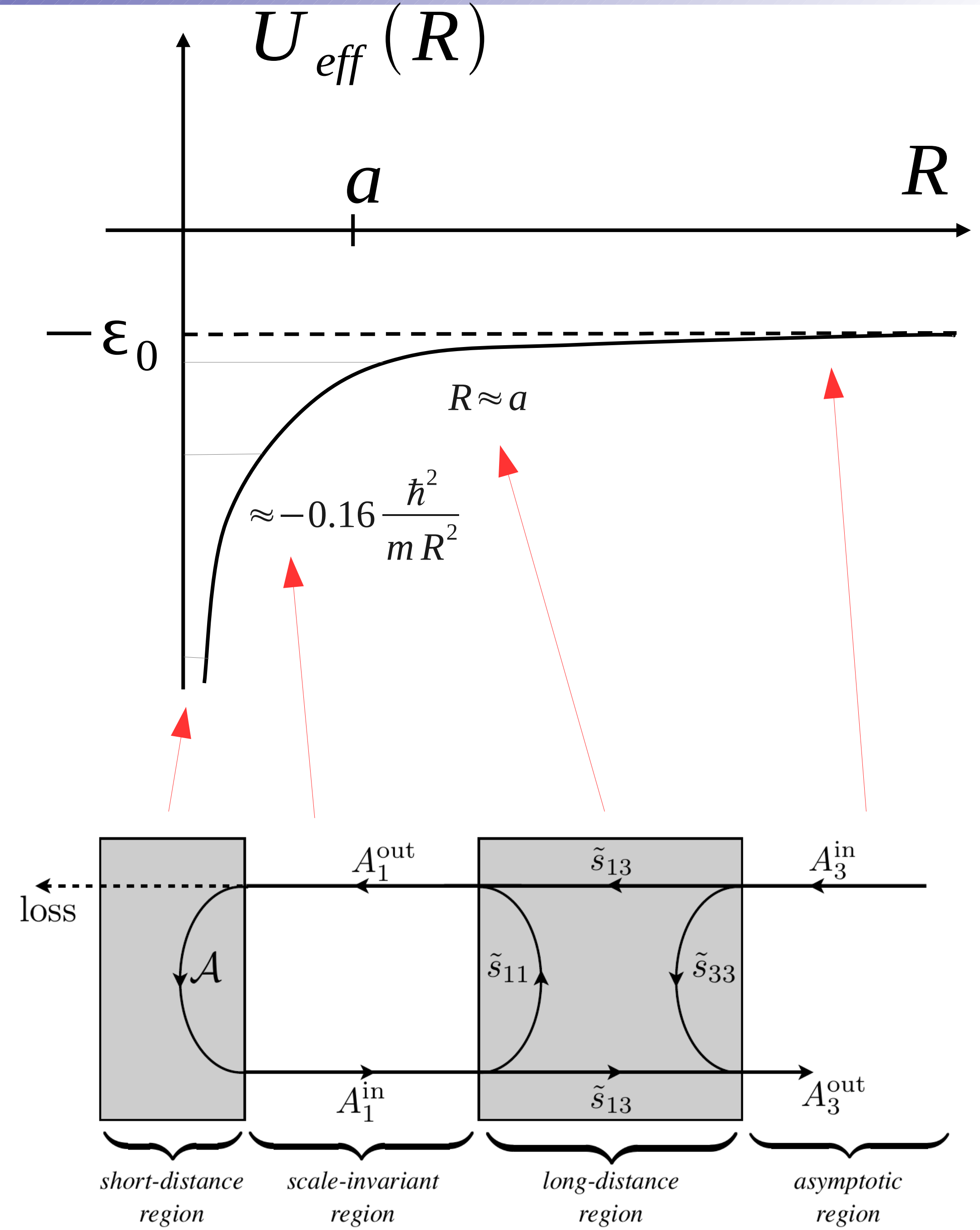
"Fall of a particle to the center in R^{-2} potential". Infinite number of zeros of the wavefunction. Infinite number of trimer states. Efimovian case

Efimov radial law and Fabri-Perot interferometer



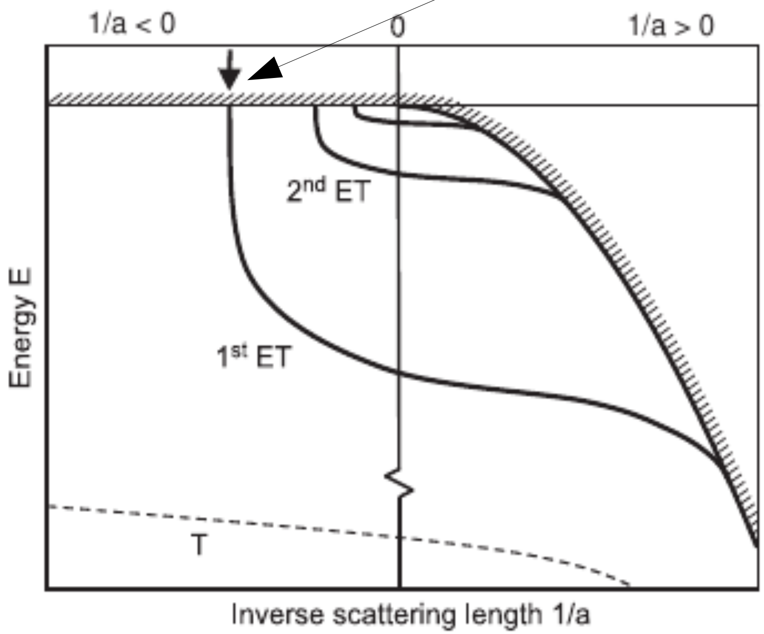
R - hyperradius

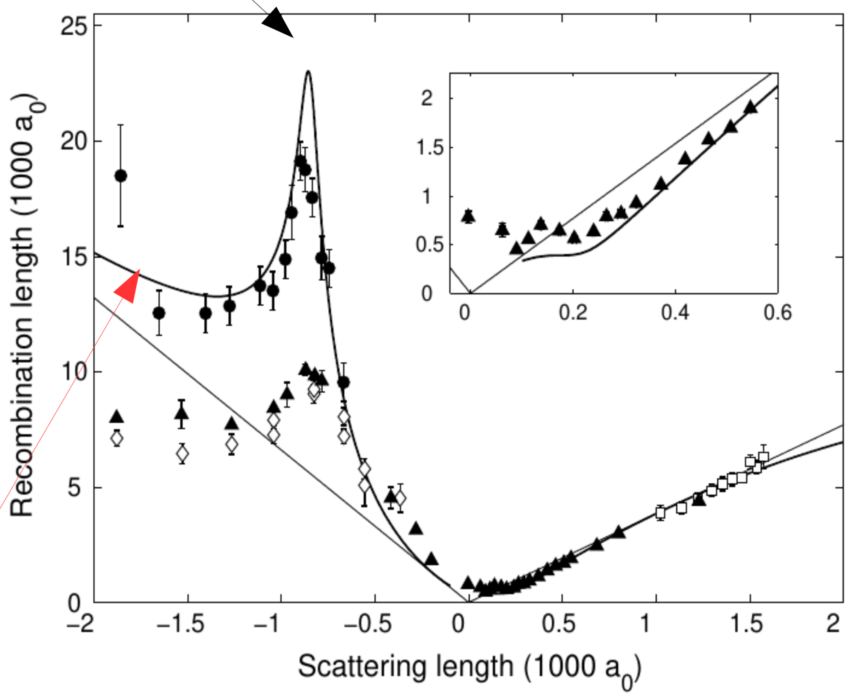
Efimov radial law
[Efimov'79,Braaten &Hammer'06]



First observation of Efimov trimer in Cs [Kraemer et al.'2006]

Efimov loss resonance





T=0 universal theory [Braaten&Hammer'2006]

Fitting parameters - real and imaginary parts of R_0

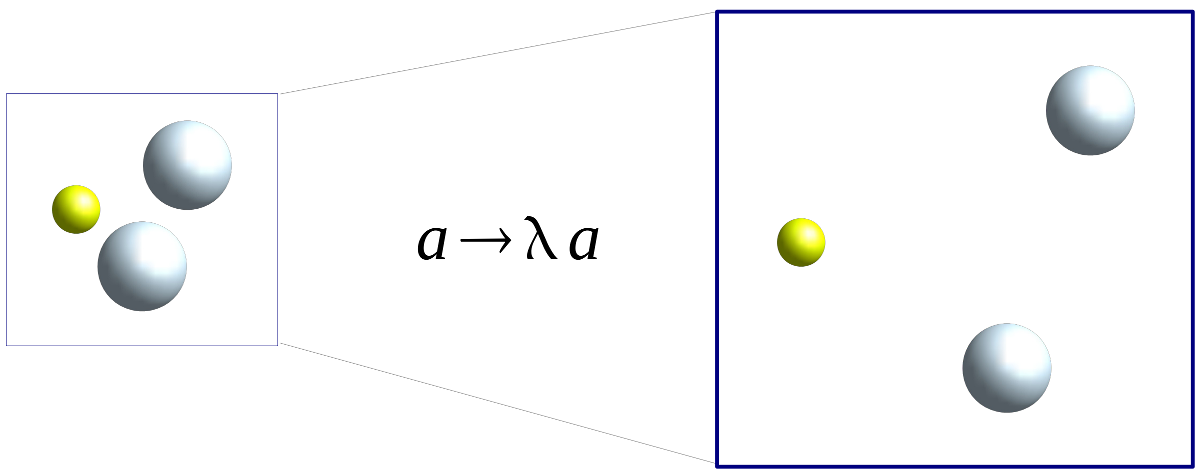
Efimov effect. Discrete scaling symmetry

$$\beta = \left| l(l+1) - 0.16\frac{M}{m} \right| < -1/4 \;\Rightarrow\; \nu_{\pm} = 1/2 \pm i s_0 \;\Rightarrow\; \chi(R) \propto \sqrt{R}\sin(s_0 \log R/R_0)$$

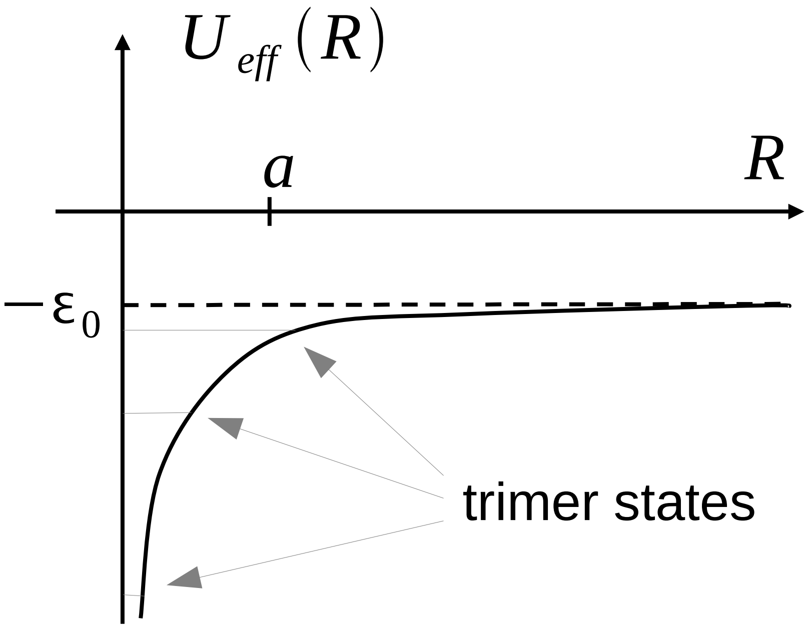
"Fall of a particle to the center in R^{-2} potential". Infinite number of zeros of the wavefunction. Infinite number of trimer states. Efimov effect

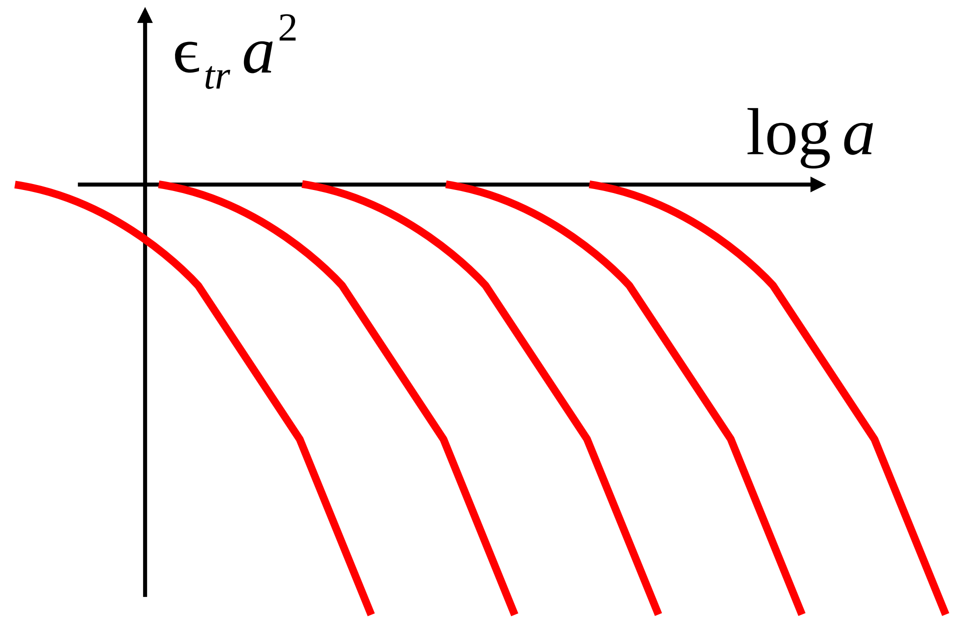
Discrete scaling symmetry.
Multiplicative factor $\lambda = \exp(\pi/s_0)$

$$\chi(\lambda R) \propto \chi(R)$$

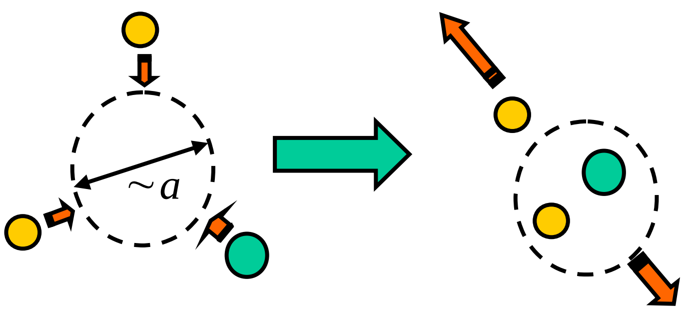


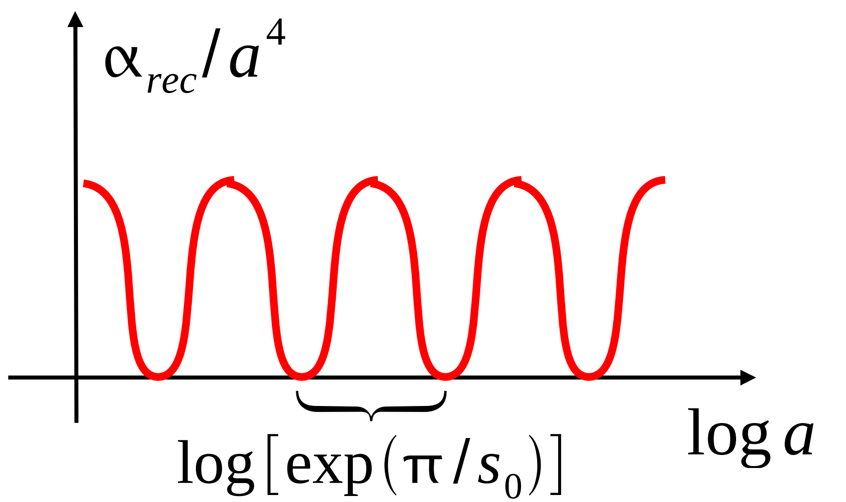
Manifestation of discrete scaling symmetry



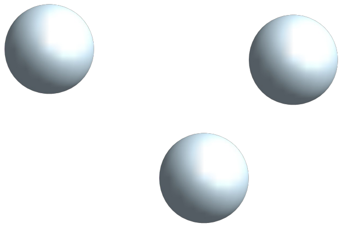


3-body recombination





Observation of discrete scaling symmetry

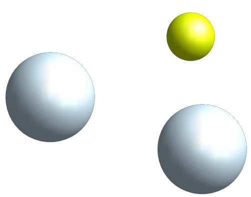


Cs-Cs-Cs, K-K-K, Li-Li-Li → $\lambda \approx 23$

Innsbruck, Florence, Heidelberg,
PennState, Rice, Ramat Gan

Rb-Rb-K → $\lambda \approx 120$

Florence, JILA

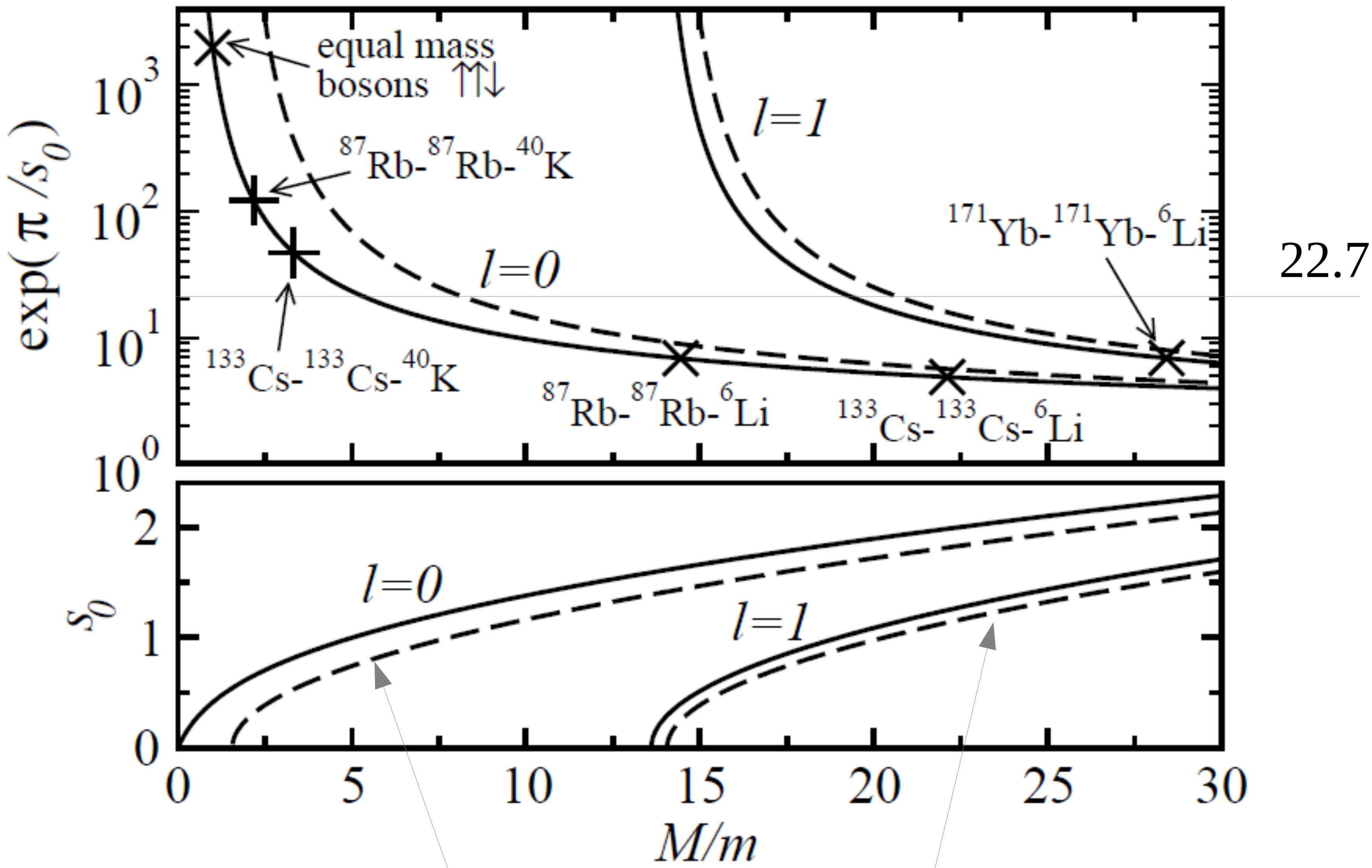


Rb-Rb-Li and Yb-Yb-Li → $\lambda \approx 7$

Cs-Cs-Li → $\lambda \approx 5$

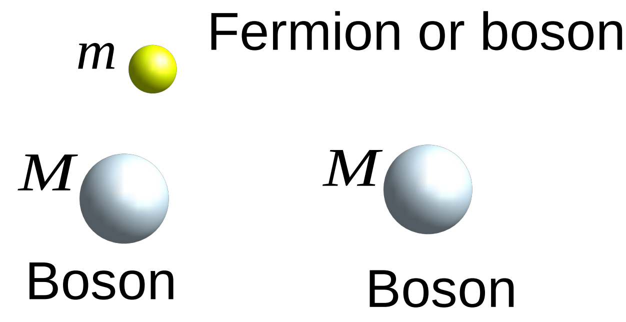
Direct measurements of 3-body observables (energies of Efimov states, 3-body recombination, atom-dimer scattering amplitude)

Values of s_0 and the scaling parameter $\lambda = \exp(\pi / s_0)$



$$s_0(BO) = \sqrt{-\beta - 1/4} = \sqrt{-l(l+1) + 0.16\, M/m - 1/4}$$

Recombination loss rates in a cold mixture:



Rate constant for three-body recombination to a weakly bound molecular level

$$\alpha_s = C(M/m)\frac{\sin^2[s_0\log(a/a_{0*})]+\sinh^2\eta_*}{\sinh^2(\pi s_0+\eta_*)+\cos^2[s_0\log(a/a_{0*})]}\frac{\hbar a^4}{m}$$

… and to deeply bound states

$$\alpha_d(a>0) = C(M/m)\frac{\coth(\pi s_0)\sinh(\eta_*)\cosh(\eta_*)}{\sinh^2(\pi s_0+\eta_*)+\cos^2[s_0\log(a/a_{0*})]}\frac{\hbar a^4}{m}$$

$$\alpha_d(a<0) = C(M/m)\frac{\coth(\pi s_0)\sinh(\eta_*)\cosh(\eta_*)}{\sinh^2\eta_*+\cos^2[s_0\log(|a|/a_{0*})]}\frac{\hbar a^4}{m}$$

where

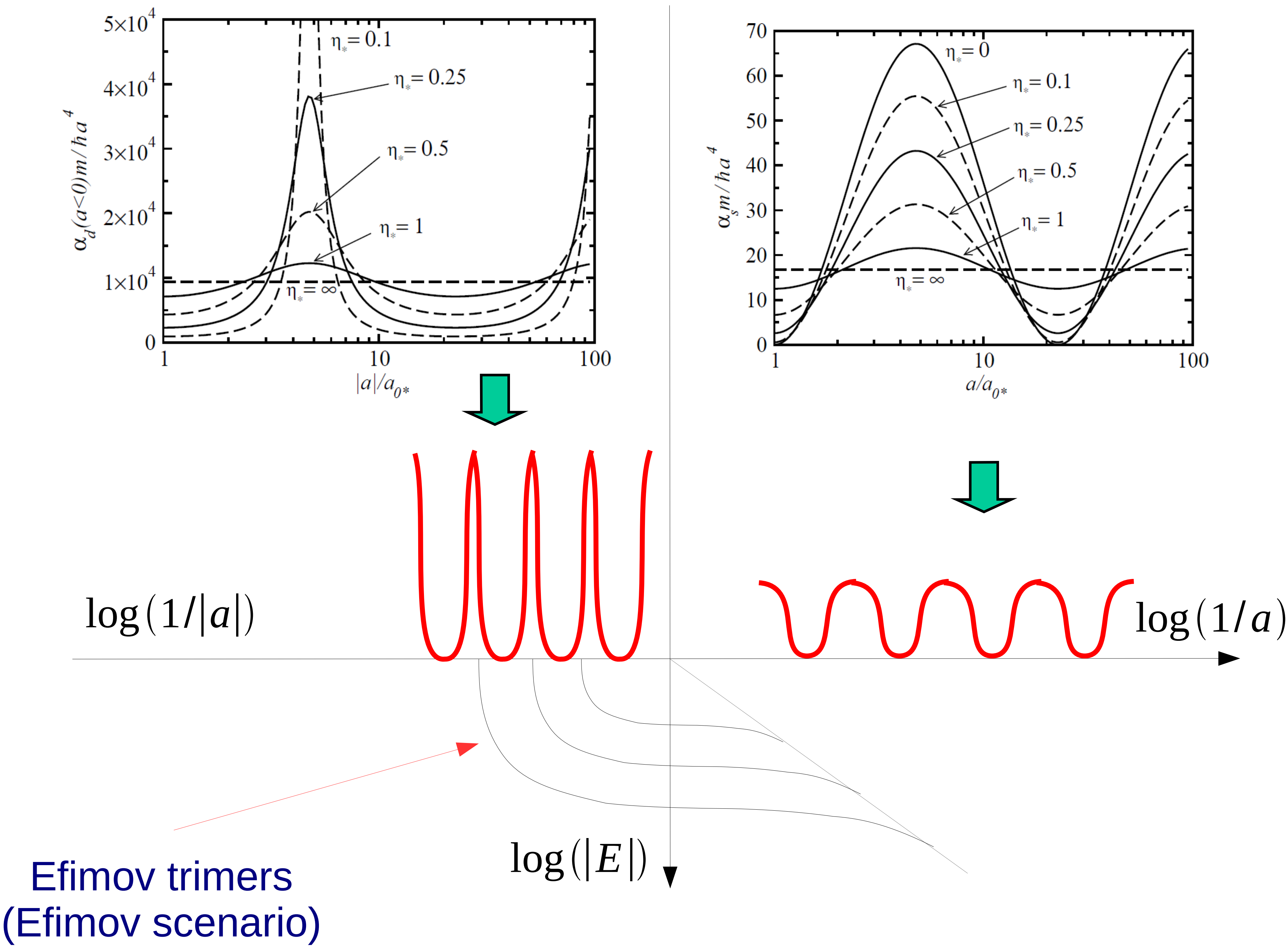
$$C(M/m) = 64\left[\left(\frac{M+m}{M}\right)^2\arcsin\left(\frac{M}{M+m}\right)-\frac{\sqrt{m(2M+m)}}{M}\right]$$

η_* - (in)elasticity parameter

a_{0*} - the value of a where $\alpha_s(a)$ reaches its minimum

Recombination maxima symmetric with respect to the resonance center!

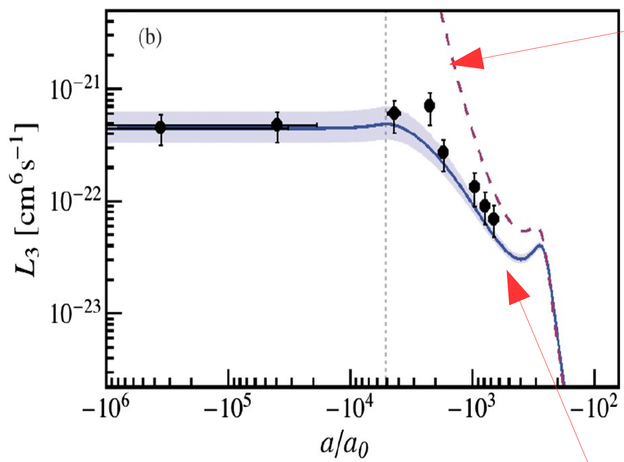
Helfrich, Hammer, Petrov,
PRA 81, 042715 (2010)



5×10^4
4×10^4
3×10^4
2×10^4
1×10^4
0
α_d(a<0)m/ħa^4
η_* = 0.1
η_* = 0.25
η_* = 0.5
η_* = 1
η_* = ∞
1
10
100
|a|/a_0*
70
60
50
40
30
20
10
0
α_s m/ħa^4
η_* = 0
η_* = 0.1
η_* = 0.25
η_* = 0.5
η_* = 1
η_* = ∞
1
10
100
a/a_0*
log(1/|a|)
log(1/a)
log(|E|)
Efimov trimers
(Efimov scenario)

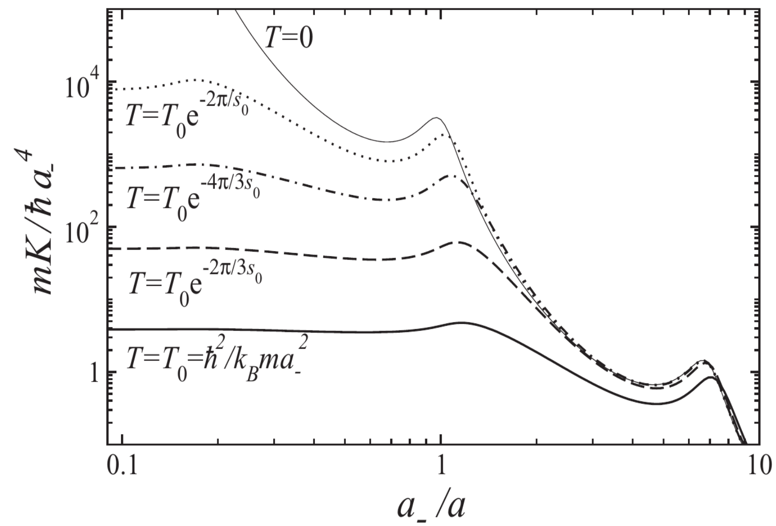
Finite-temperature effects

ENS ^{7}Li experiment [Rem et al.'2013]



T=0

Rb-Rb-Li theory [Petrov&Werner'2015]



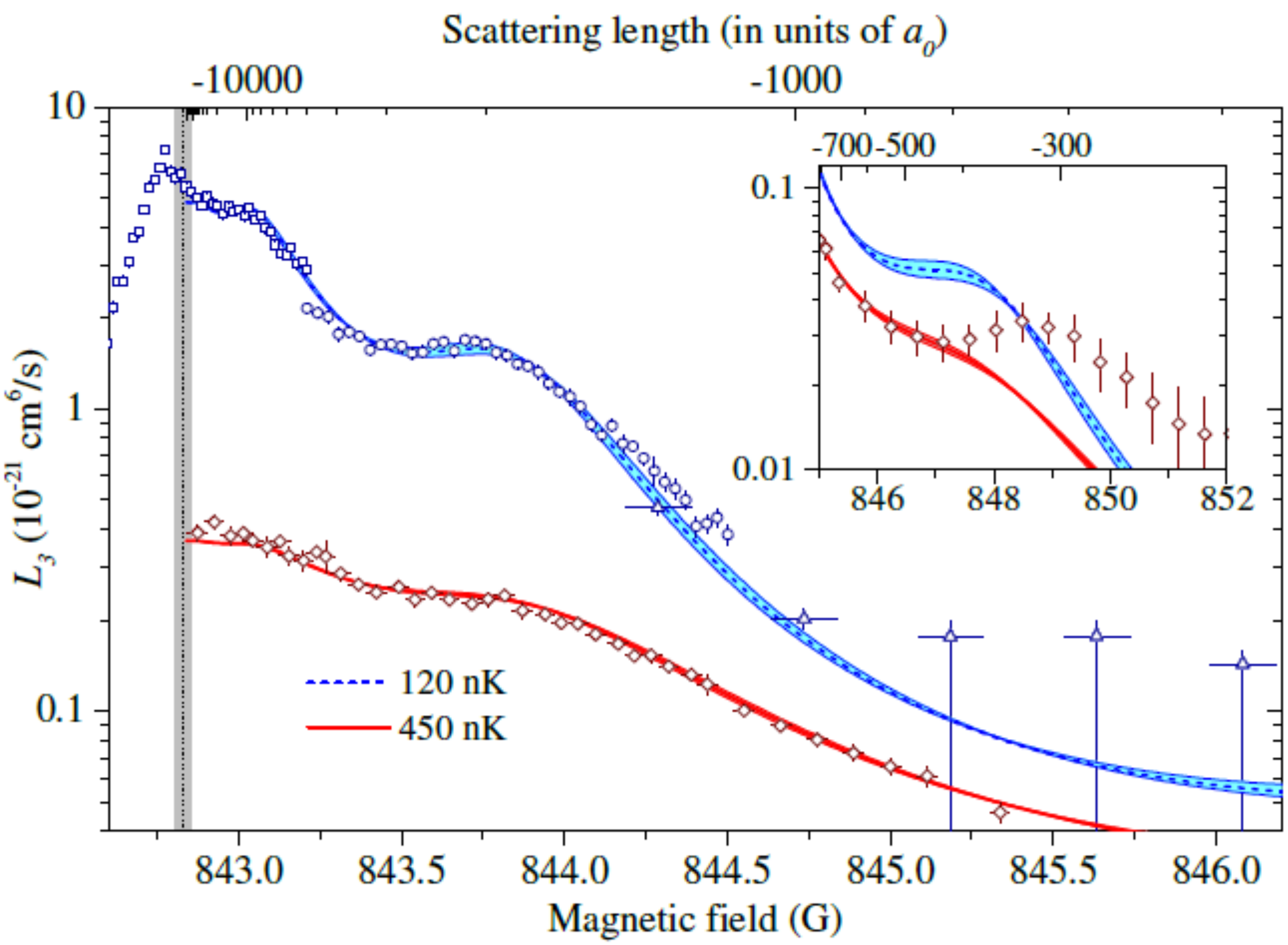
Finite T :
saturation when

$$|a| > \lambda_{dB}$$

$$L_3 \propto a^4 \rightarrow L_3 \propto \lambda_{dB}^4$$

Cs-Cs-Li loss rate constant

with Werner (theory), Ulmanis, Haefner, Pires, Kuhnle, Weidemueller (exp) (2015)

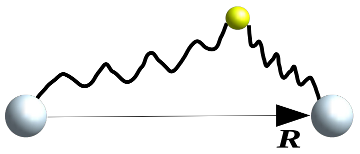


Tuning the three-body parameter(s)?

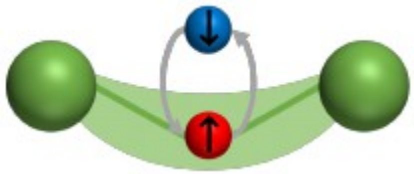
Ultra-cold atoms for modeling nuclei (need to switch off losses !)

Tune the three-body and (in)elasticity parameter (shielding?)

- Floquet drive of *a* [Sykes,Landa,Petrov'2017]



- RF drive [Zulli,Mulkerin,Parish,Levinsen'2025]



- vdW+dipole-dipole (Dy, Er with Li ?) [Oi,Naidon,Endo'2024]

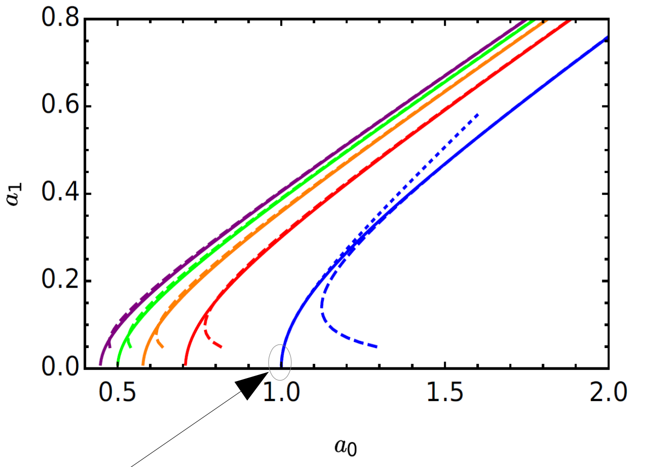
- ?

Single-frequency drive [Sykes,Landa,Petrov'17]

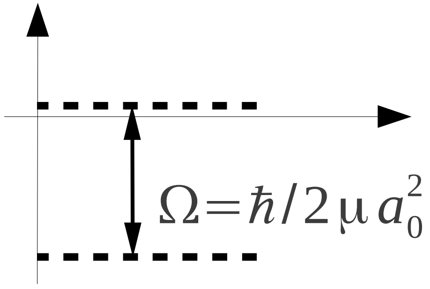
$$\lim_{r\to 0}\frac{\partial_r[r\psi]}{r\psi} = -\frac{1}{a(t)}$$

$$a(t) = a_0 + 2a_1\cos t$$

recurrence relations for f_n → find a_0 and a_1 where $a_{\mathrm{eff}}=\infty$ →



Small-amplitude drive resonant with the energy of the weakly-bound state (Hudson Smith'15)



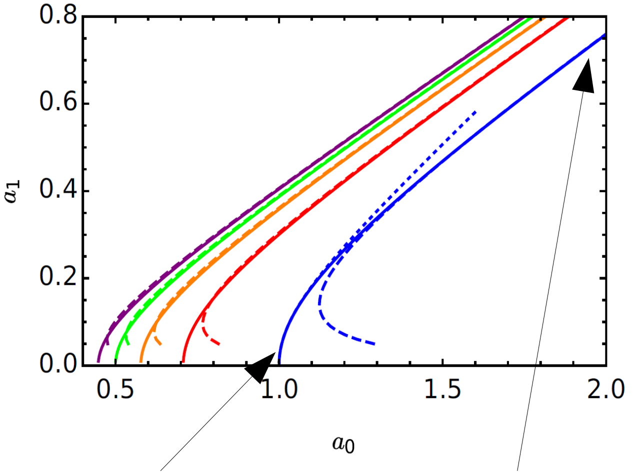
Behavior close to resonance [Sykes,Landa,Petrov'17]

i.e., for small collision energy $\omega = k^2$ and $\delta a_0 = a_0 - a_0^{res}$

$$f_0 = -\frac{1}{1/a_{\rm eff} + R^*\omega + i\sqrt{\omega}}$$

$$a_{\rm eff} = -a_1^2 \left|\phi_{-1}^{(\nu)}\right|^2 / \delta a_0$$

$$R^* = \frac{\sum_{n=1}^{\infty} n^{-3/2} \left|\phi_{-n}^{(\nu)}\right|^2}{2a_1^2 \left|\phi_{-1}^{(\nu)}\right|^2}$$



$$a_{\rm eff} \approx -a_1^2/\delta a_0 \qquad R^* \approx 1/2a_1^2$$

NARROW

$$a_{\rm eff} \sim -1/\delta a_0 \qquad R^* = 1/a_1$$

WIDE

Inelastic contribution appears in the second order

$$f_0 = -\frac{1}{1/a_{\rm eff} + R^*\omega + iC\left(1/a_{\rm eff} + R^*\omega\right)^2 + i\sqrt{\omega}}$$

$\delta = 2$ for $a_1 \ll 1$

$\delta = 5/3$ for $a_1 \gg 1$

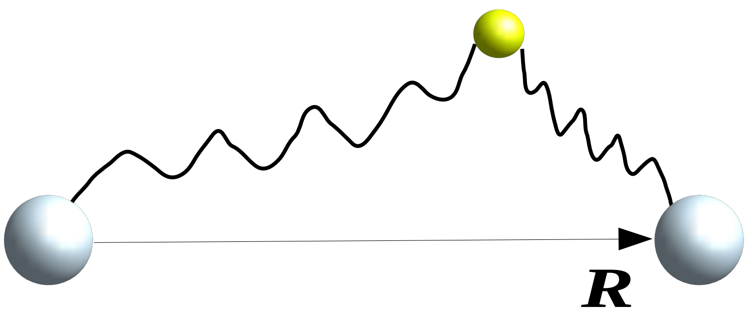
On resonance $a_{\rm eff} = \infty$ but at finite $\omega = k^2$ ⇨ $\sigma_r(k)/\sigma_e(k) \sim (k L_\Omega)^3 (L_\Omega / a_1)^\delta$

cf. (inelastic three-body rate)/(elastic two-body rate) $\propto \text{density}/k^3$

Two-body inelastic loss is not going to be the main issue!

Heavy-heavy-light problem [Sykes,Landa,Petrov'17]

Floquet-resonant heavy-light interaction, i.e., a_0 and a_1 on the two-body resonance



$$\psi(\mathbf{r},t) = \sum_{n\in\mathbb{Z}} f_n \sum_{\pm} \frac{\exp\left[ik_n|\mathbf{r}\pm\mathbf{R}/2| - i\omega_n t\right]}{|\mathbf{r}\pm\mathbf{R}/2|}$$

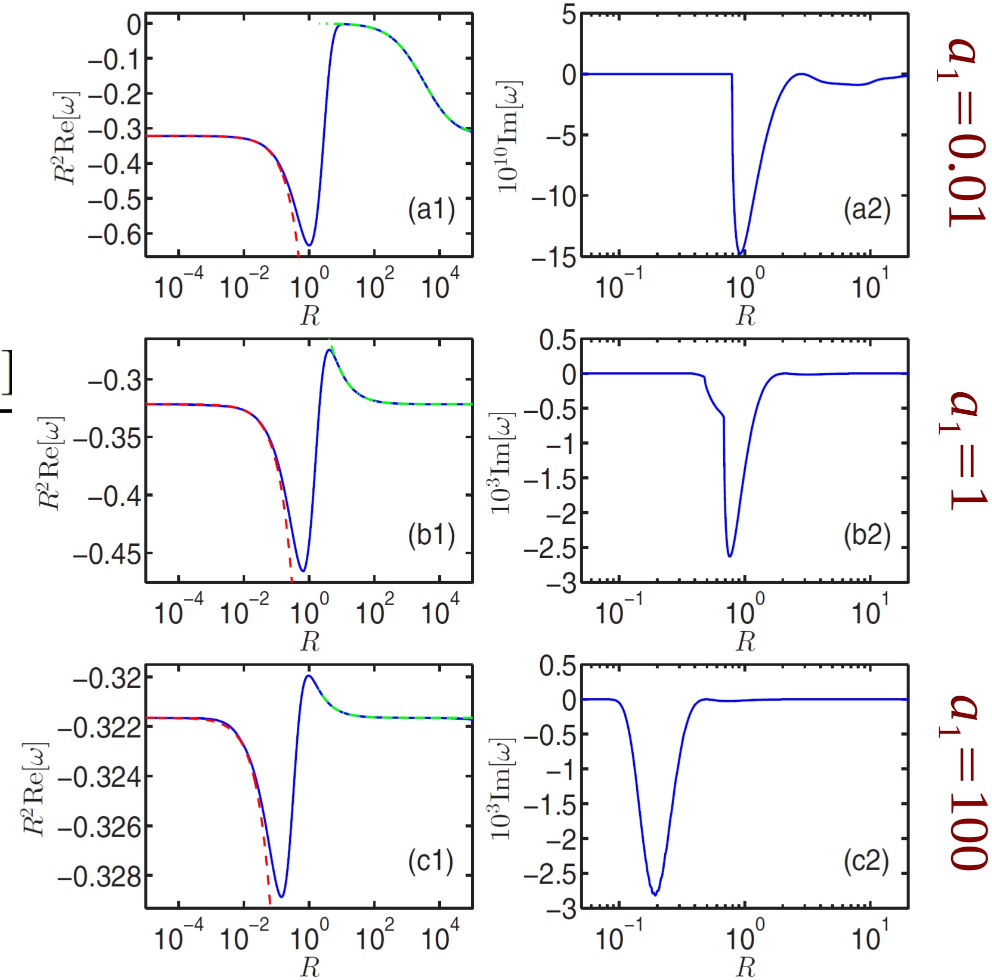
boundary conditions for $\boldsymbol{r}\pm\boldsymbol{R}/2\to 0$

↓

recurrence relations for f_n

↓

effective potential $\omega(R)$



Heavy-particle wave function [Sykes,Landa,Petrov'17]

$$\left[-\frac{\partial^2}{\partial R^2}+\frac{l(l+1)}{R^2}+\frac{M}{2m}\omega(R)\right]\chi_l(R)=\frac{M}{2m}E\chi_l(R)$$

$$z=\ln R$$
$$\chi_l(R)=\exp(z/2)\Phi_l(z)$$

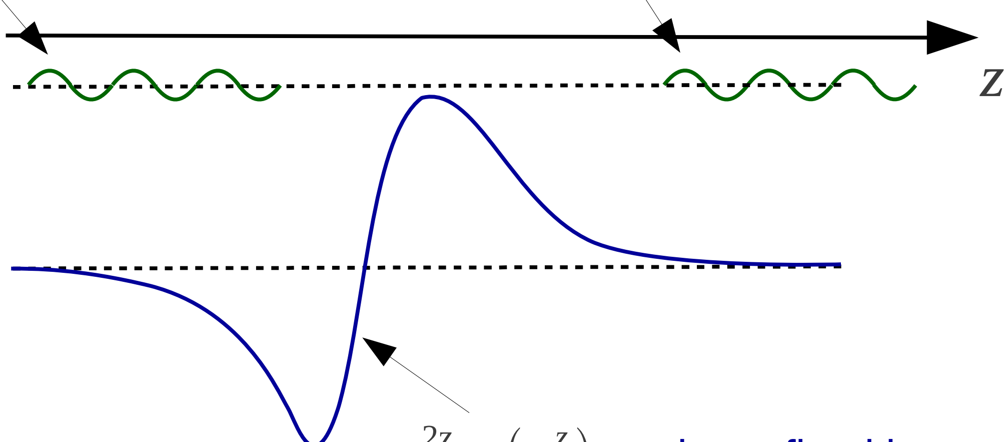
$$-\frac{1}{M/2m}\Phi_l''(z)+e^{2z}\omega(e^z)\Phi_l(z)=-\frac{(l+1/2)^2}{M/2m}\Phi_l(z)$$

$$\Phi_l\propto e^{-\eta+is_0z+i\phi}-e^{\eta-is_0z-i\phi} \qquad \Phi_l\propto e^{-\eta_d+is_0z+i\phi_d}-e^{\eta_d-is_0z-i\phi_d}$$

$$s_0=[C^2M/2m-(l+1/2)^2]^{1/2}$$

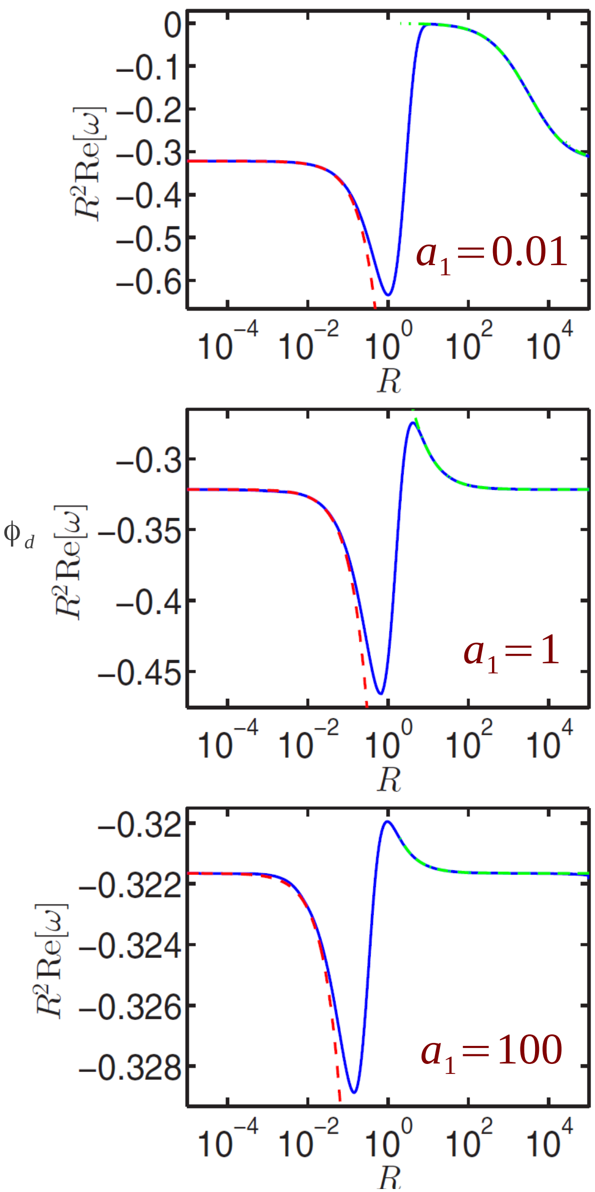
0

$-(l+1/2)^2/(M/2m)$



$-C^2\approx-0.32$
$(C=e^{-C})$

$e^{2z}\omega(e^z)$ – shape fixed by a_1



Next

Universal non-Efimovian (N+1)-clusters

Scaling invariance : from 3D Efimov to 2D Pitaevskii-Rosch