Three-dimensional Tensor-Network Renormalization Group Enhanced by Entanglement Filtering

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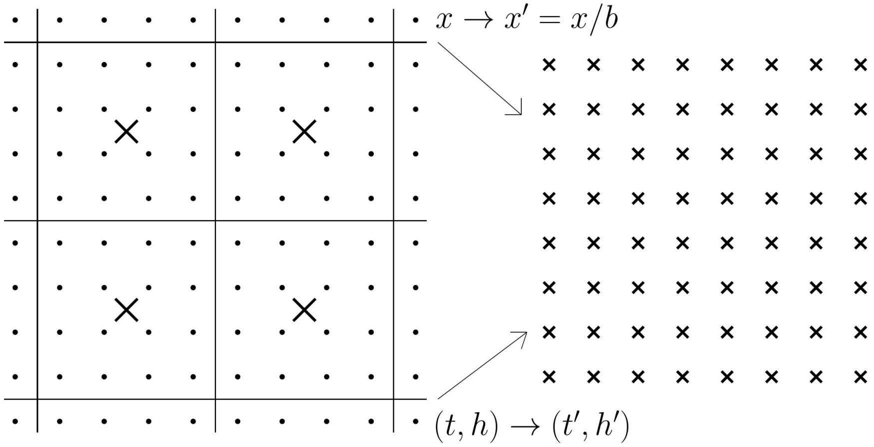
@ Benasque, 4 March 2025

XL and Kawashima, arXiv:2412.13758

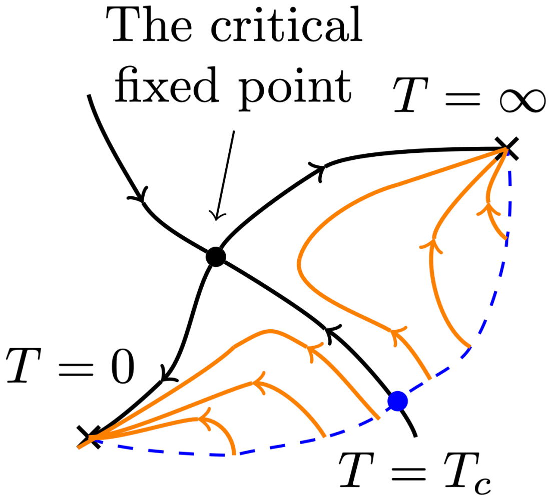
https://github.com/brucelyu/efrg3D

Criticality and renormalization group

Wilsonian renormalization group (RG) is a way to estimate critical exponents



Real-space RG in a statistical physics context



IF you have an well-behaved RG transformation

Real-space RG in spin language

Real-space RG using spin-based representation: *uncontrolled approximation*

- A single RG step entails all possible interactions
- Truncation schemes are based on the conjecture that long-range one is less important
- Decimation, majority rule, …?

John Cardy's textbook

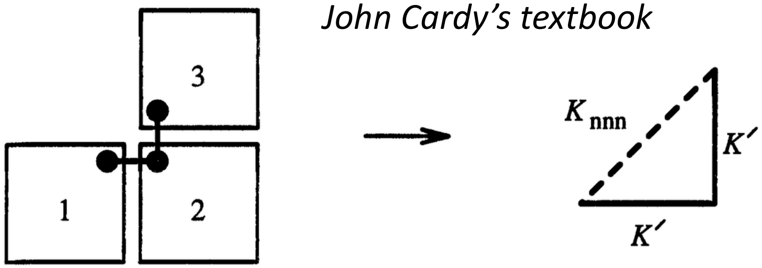


Figure 3.8. Generation of next-nearest neighbour coupling.

Here are two examples:

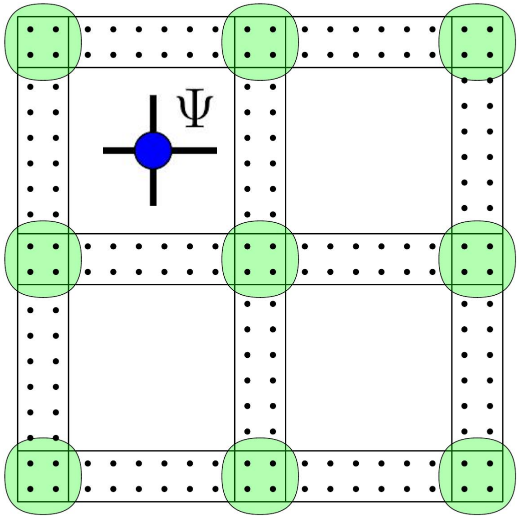
Wilson (1975) implemented a numerical 3x3 block-spin map by keeping 217 couplings of 2D Ising:

- High accuracy—1% or even 0.1% for first two exponents

"Difficult for 3D Ising… since 3x3x3 block contains about 30 spins, corresponding to 10^9 configurations"

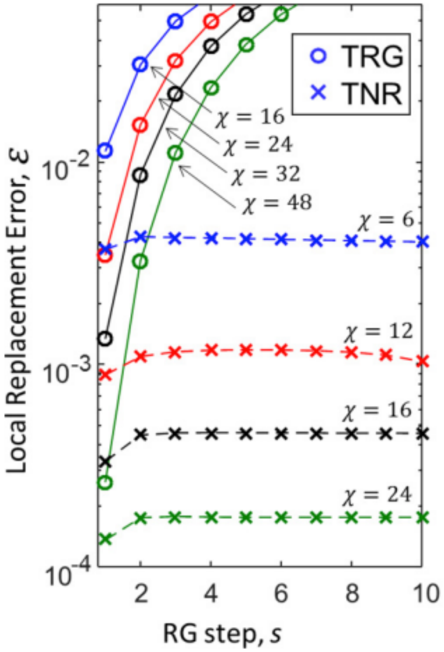
Migdal-Kadanoff bond moving (1976) gives $x_\epsilon = 2.1$ (accept value is 1.41) for 3D Ising; the relative error is about 50%…

Real-space RG in TN language



2D classical → 1D quantum chain (radial quantization) → Entanglement-entropy area law: $S(L) \sim S_0$ [due to Levin and Nave, *PRL* **99**, 120601 (2007)]

Constant S_0 can justify the practice of keeping constant number of couplings!



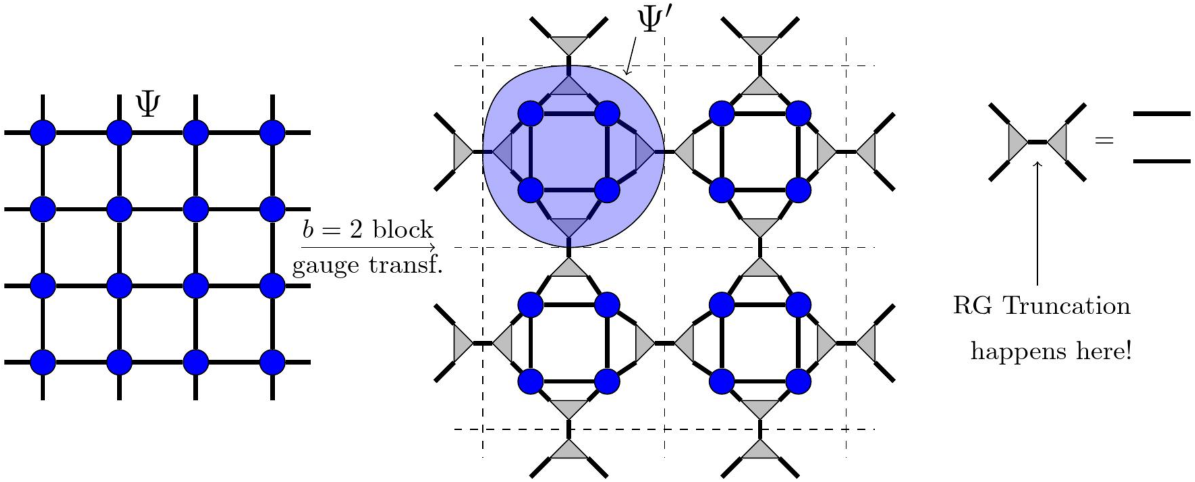
Systematically improvable 2D real-space RG!

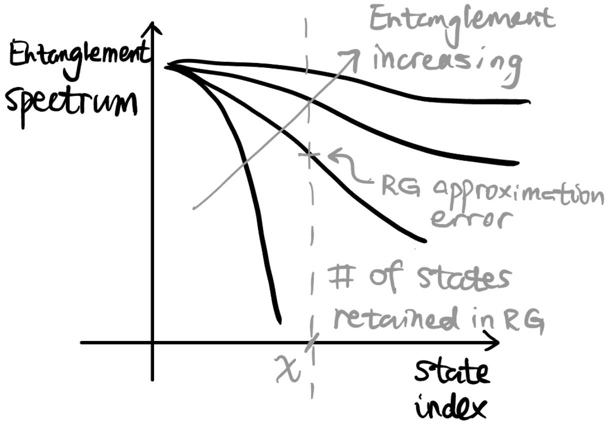
exact	TNR(6)	TNR(16)	TNR(24)
0.125	0.125679	0.124941	0.124997
1	1.001499	1.000071	1.000009
1.125	1.125552	1.125011	1.124991
1.125	1.127024	1.125201	1.125027
max err.	0.83%	0.046%	0.0069%

Evenbly and Vidal, *PRL* **115**, 180405 (2015)

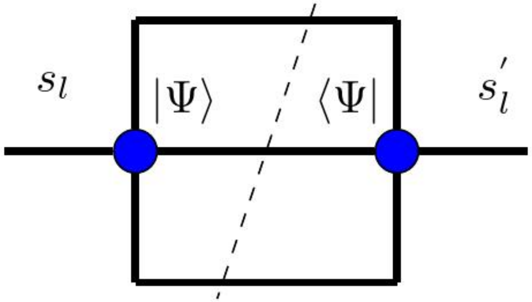
EE and TNRG: block-tensor map

Block idea in tensor-network language: *block-tensor transformation*





An RG flow in tensor space: $\Psi^{(0)} \to \Psi^{(1)} \to \Psi^{(2)} \to \cdots$



$$\rho_{s_l,s_l'} \equiv \mathrm{Tr}_{s_r,s_u,s_d} |\Psi\rangle\langle\Psi|$$

EE and block-tensor map (Levin and Nave, *PRL*, 2007):

- Entanglement entropy ↗ indicates RG error ↗
- Changing entanglement entropy indicates your tensor isn't fixed (but we *wish* to have a fixed-point tensor).

EE and TNRG: block-tensor in 3D

(2+1)D EE area law goes like:

UV physics

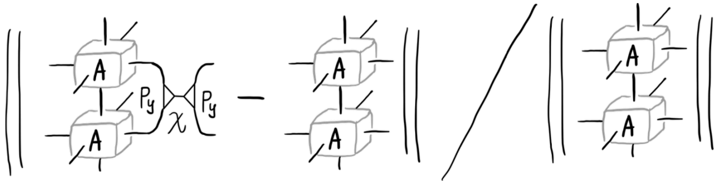
Universal physics

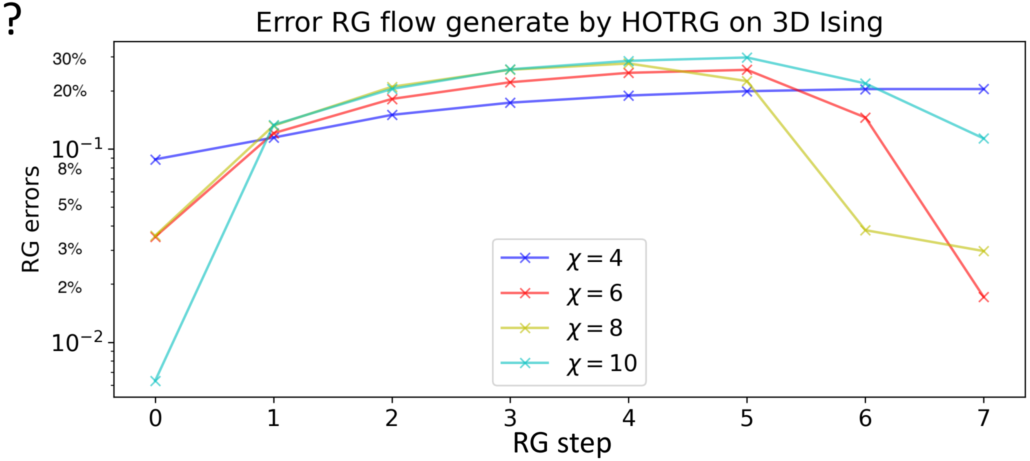
$$S(L) = \alpha L - F$$

Linear growth of S marks a *qualitative* difference between 3D and 2D for block-tensor RG!

Consequences on the numerical side?

- Large RG truncation errors
- Increase states doesn't help

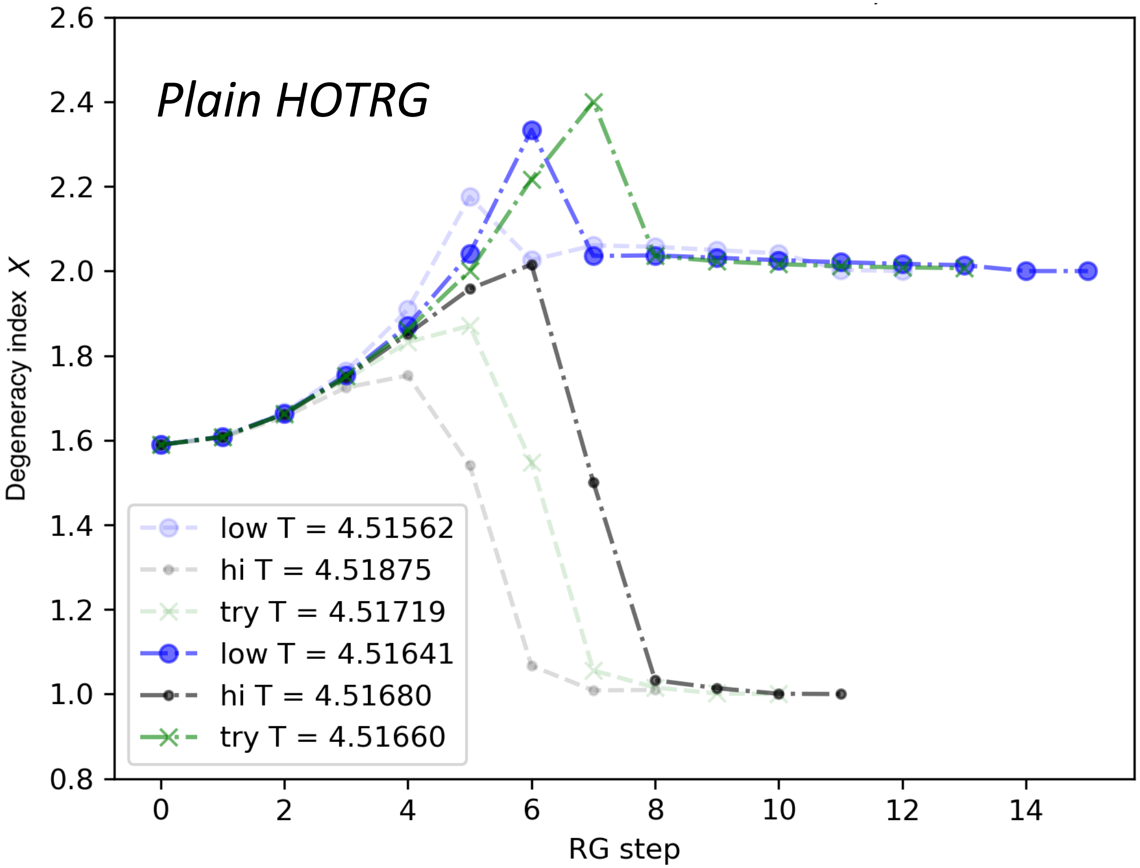




***XL** and Kawashima, arXiv:2311.05891*

EE and TNRG: block-tensor in 3D

Estimates fail to convergence w.r.t RG step!



EE and TNRG: block-tensor in 3D

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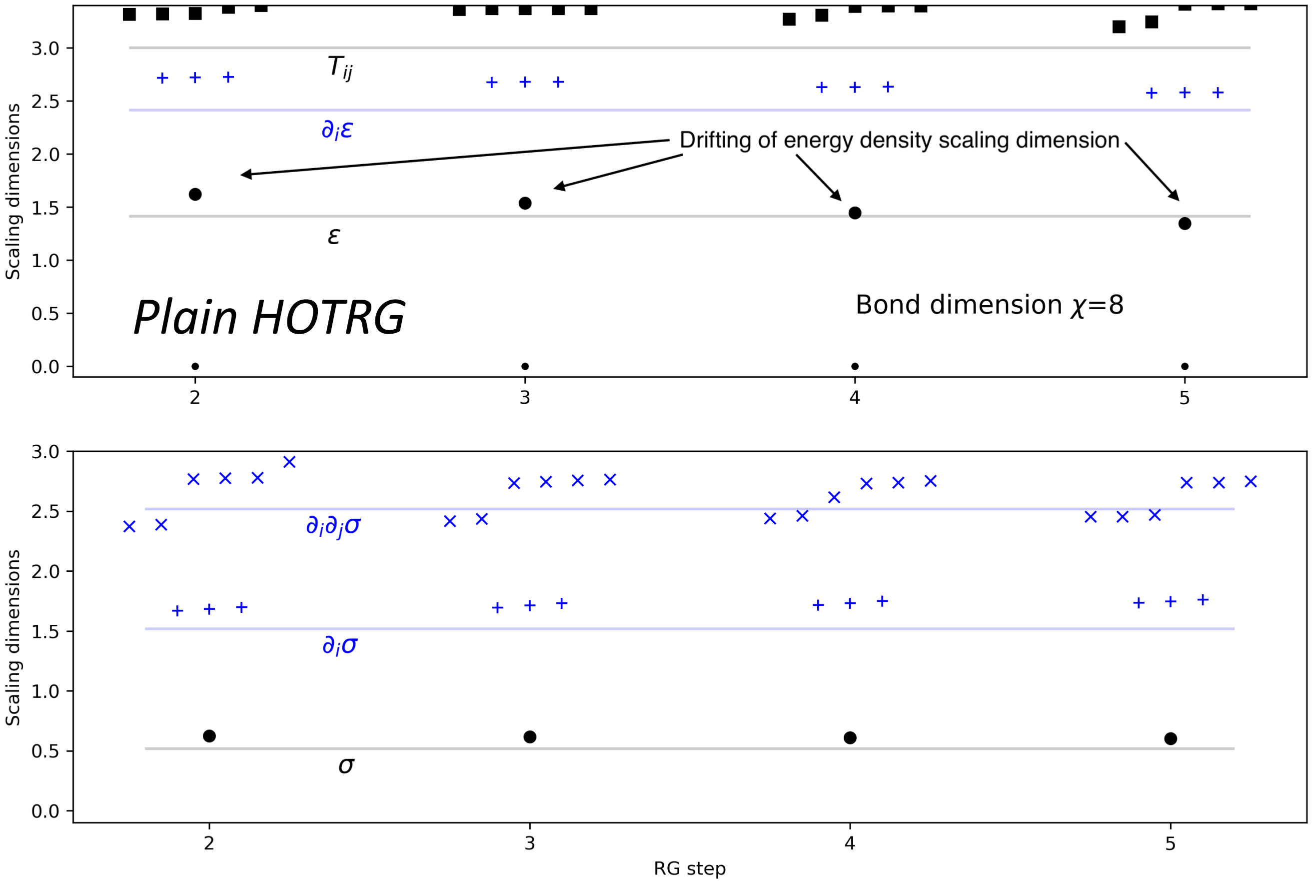
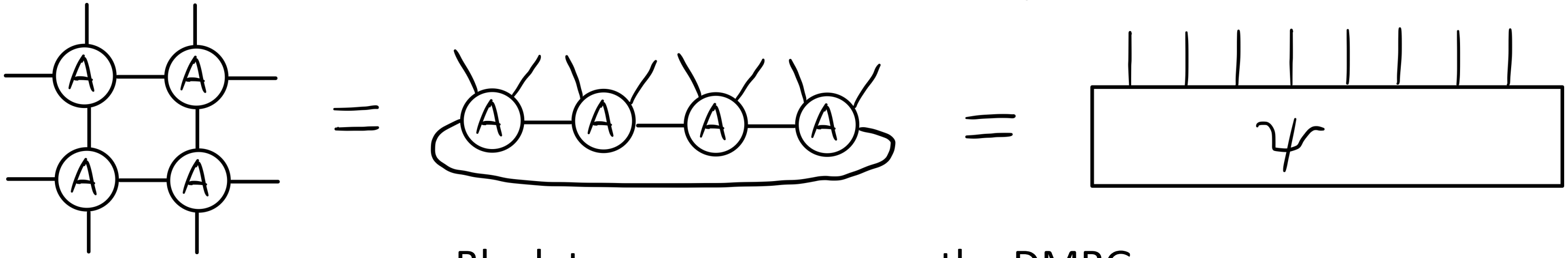


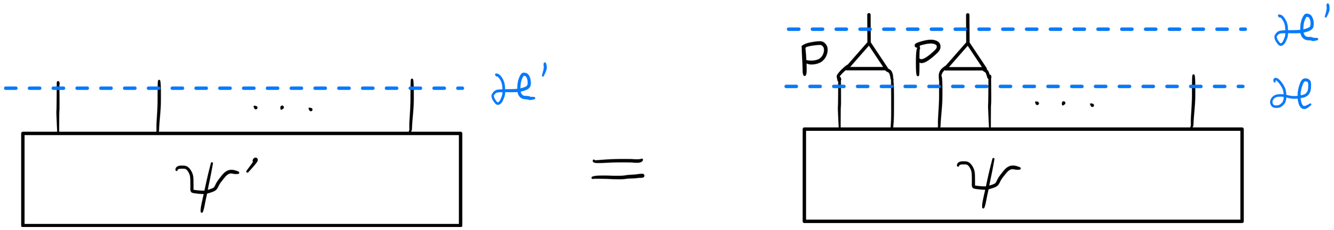
Entanglement filtering: basic idea

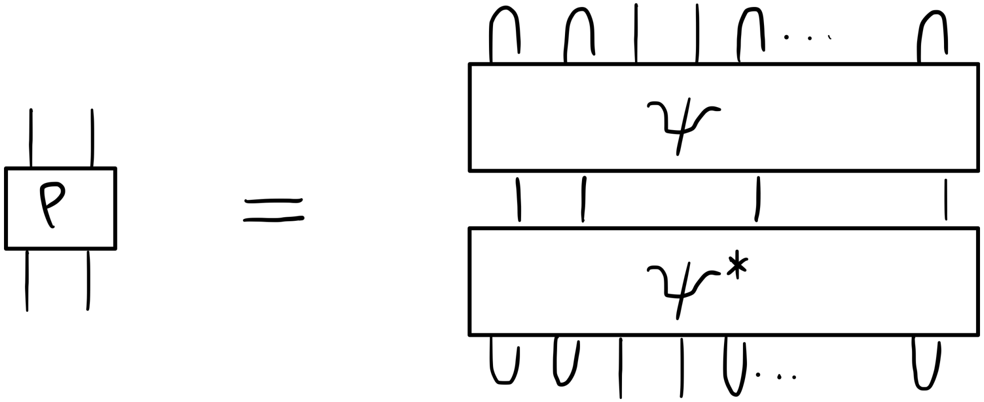
Area law can be circumvented in coarse-grained description if the boundary of the block is ”dissolved”

Invoke the wave function interpretation



Block-tensor map seen as the DMRG





Entanglement filtering: basic idea

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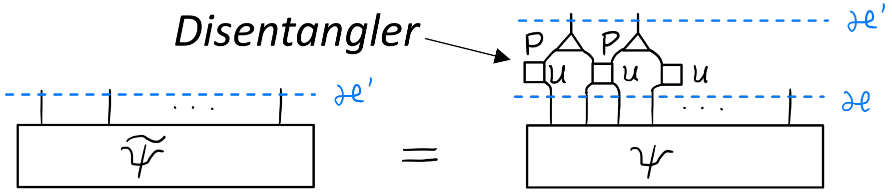
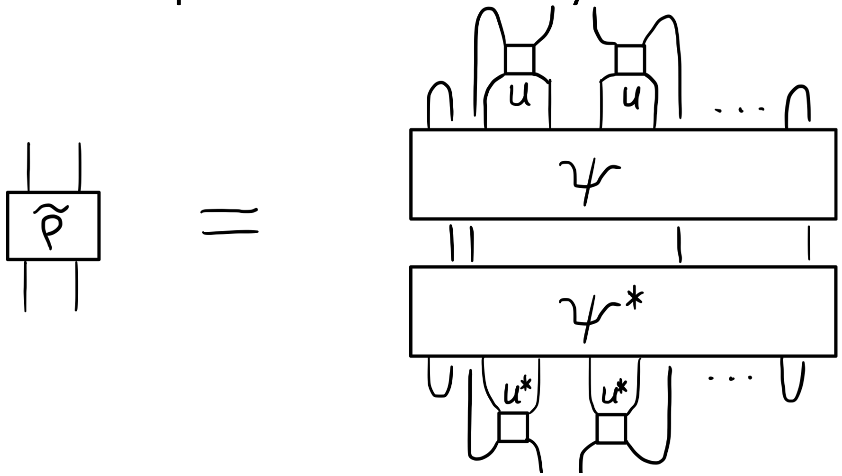
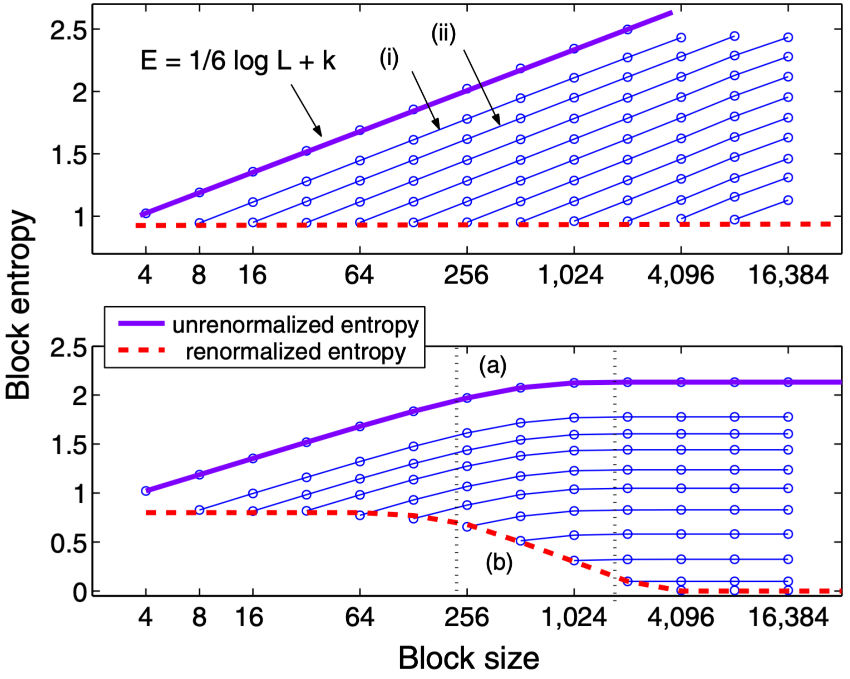
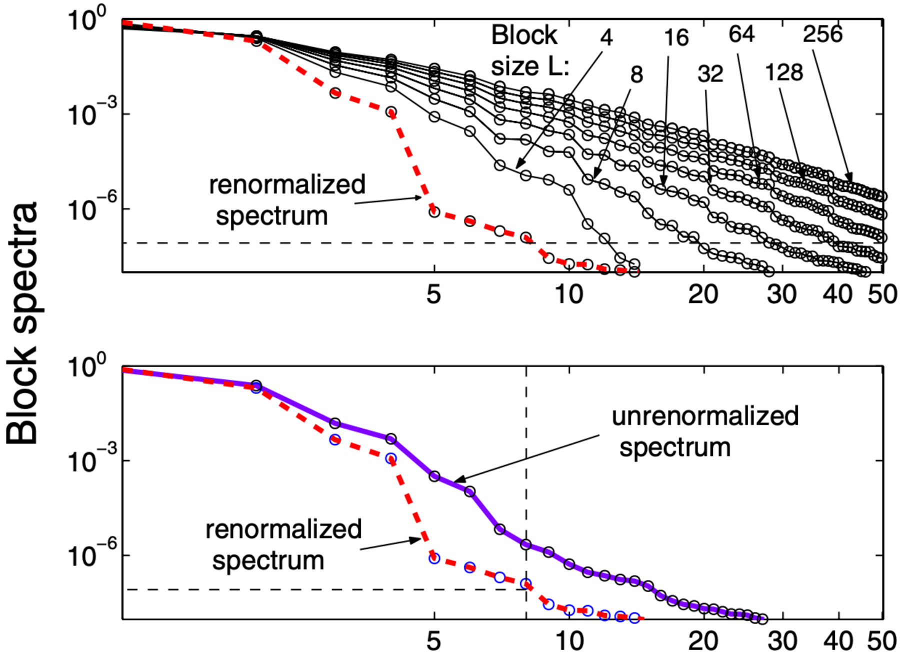


Disentangler



Vidal, *PRL* **99**, 220405 (2007)

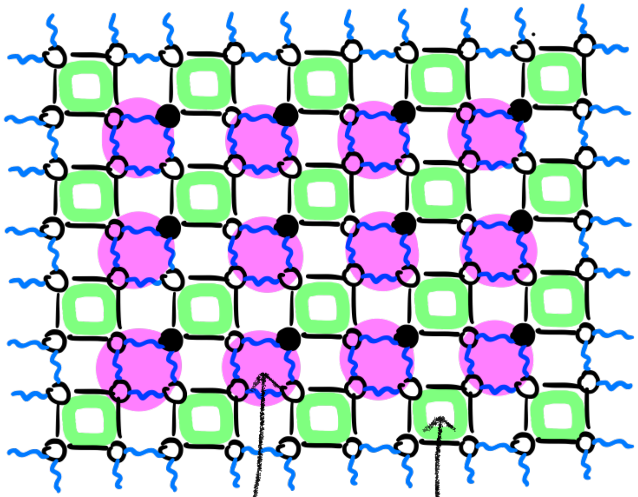




Proposed filtering scheme

Demonstrated in the 2D square lattice, here is how to *integrate Entanglement Filtering into a block-tensor transformation*:

(a) Panorama

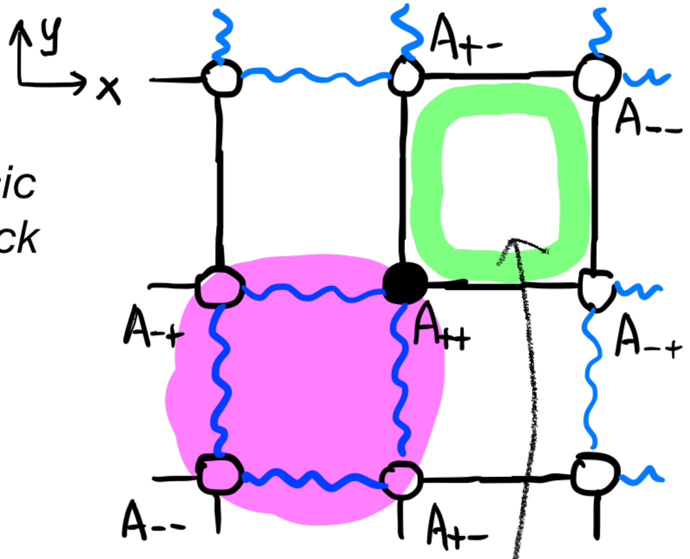


The block-tensor patch

Entanglement that contributes to the area law

with the basic building block patch as

(b) Zoom-in view



Target patch of the entanglement filtering

***XL** and Kawashima,* arXiv:2412.13758

Proposed filtering scheme

We adopt the graph-independence idea in Gilt

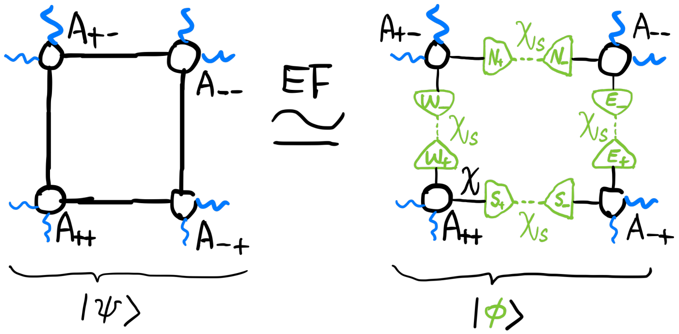
+

Use another way to find the filtering matrix: full environment truncation

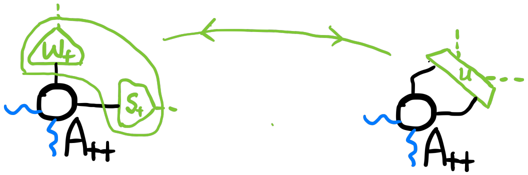
Hauru, Delcamp, and Mizera, *PRB* **97**, 045111 (2018)

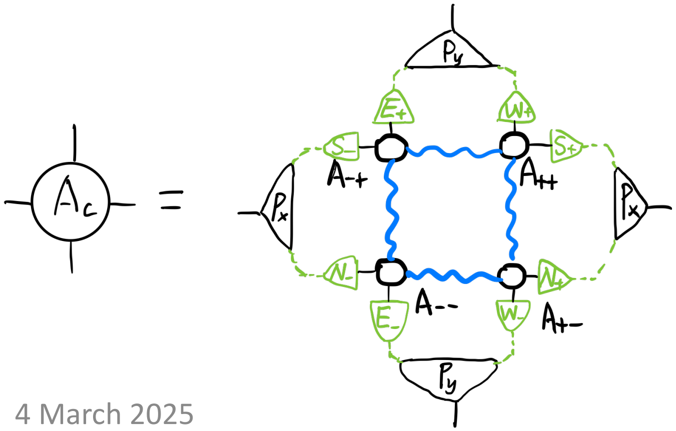
Evenbly, *PRB* **98**, 085155 (2018)

Demonstrated in the 2D square lattice, we propose:



Disentangler interpretation:





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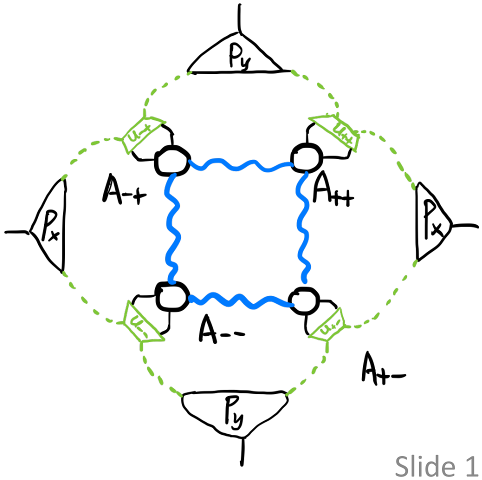
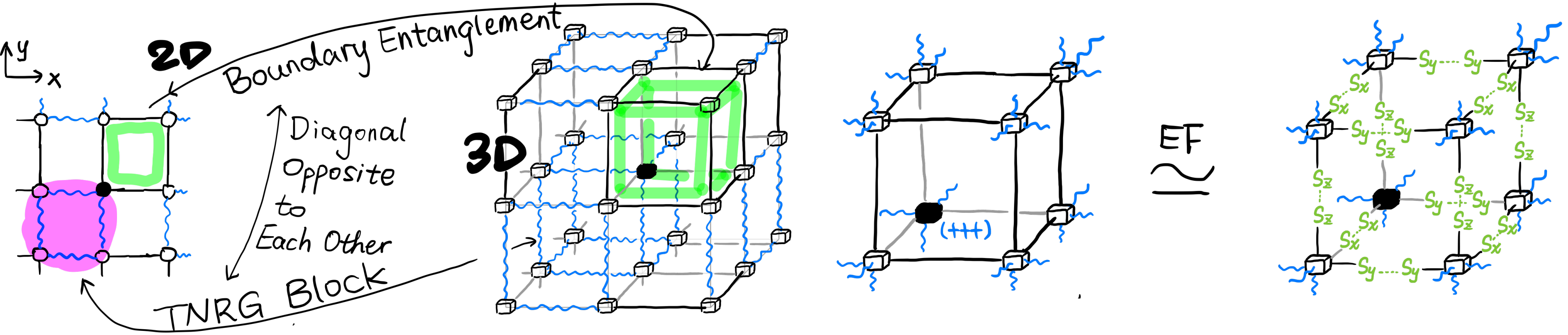
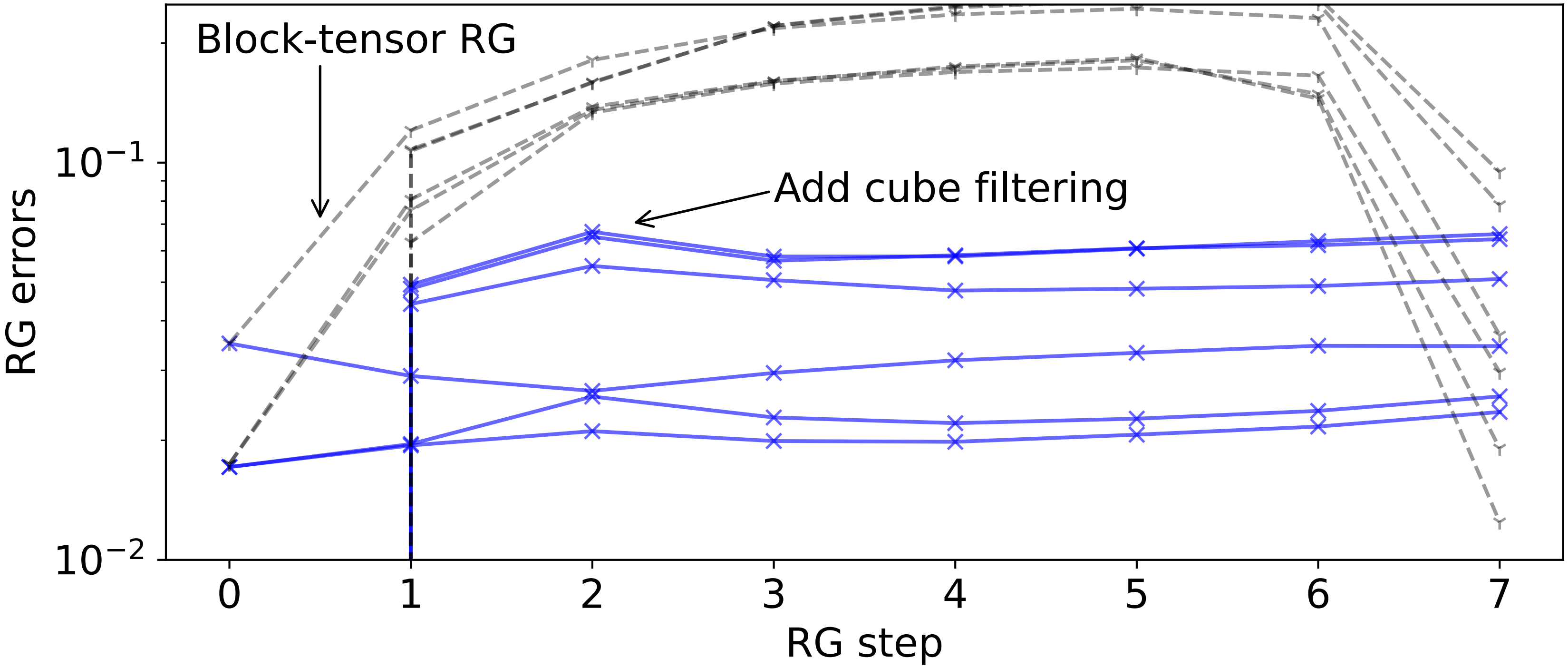


Entanglement filtering in 3D





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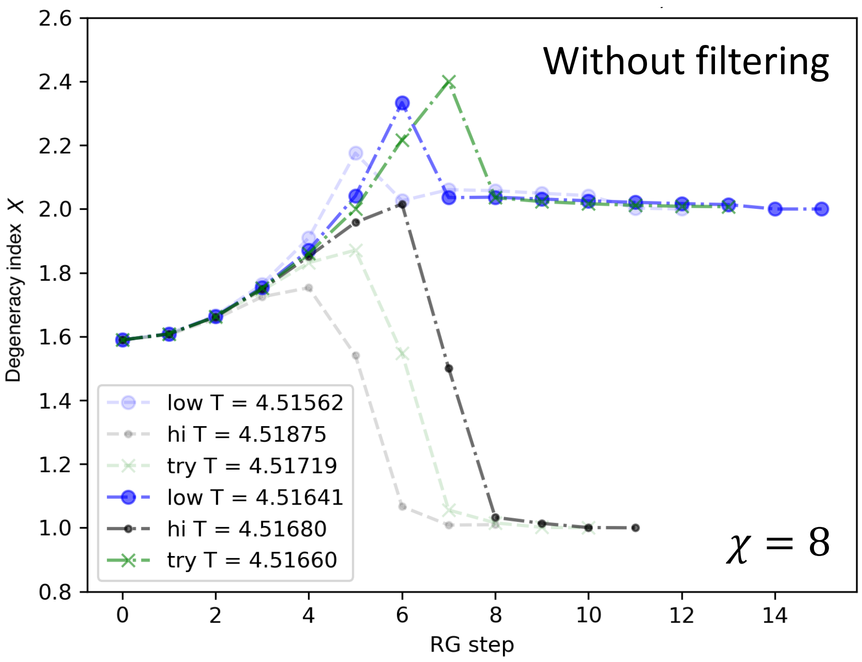

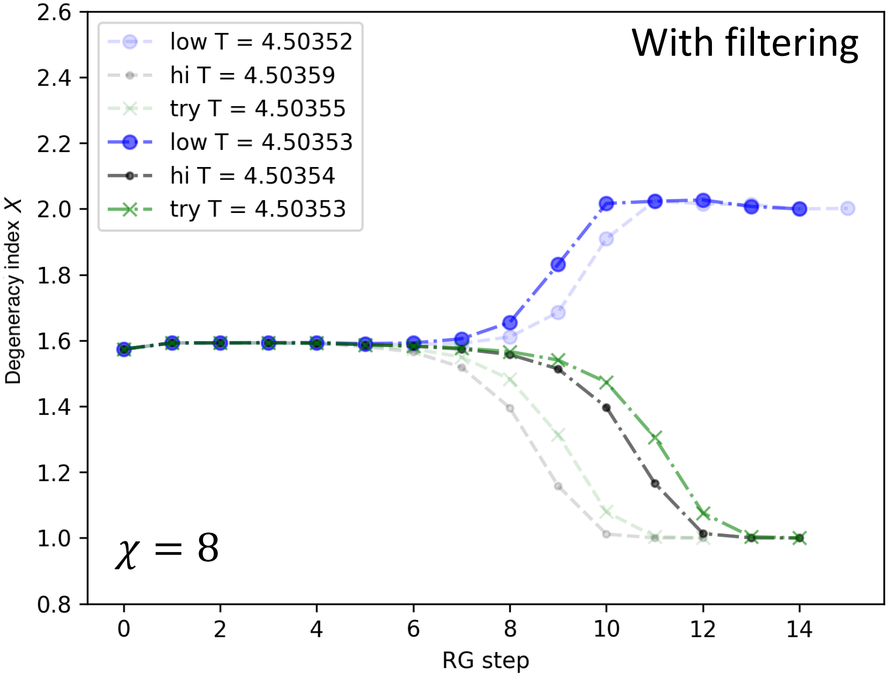

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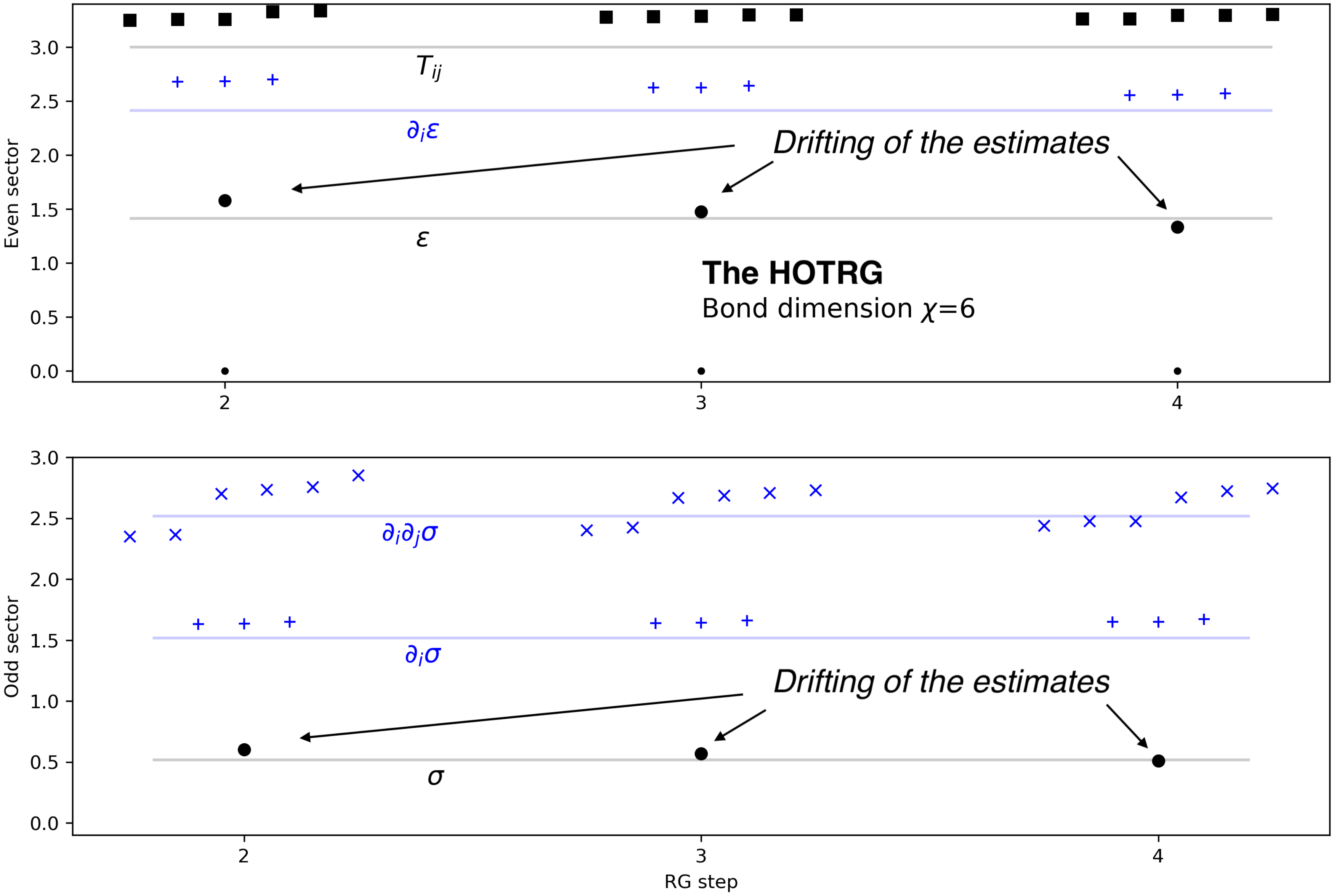
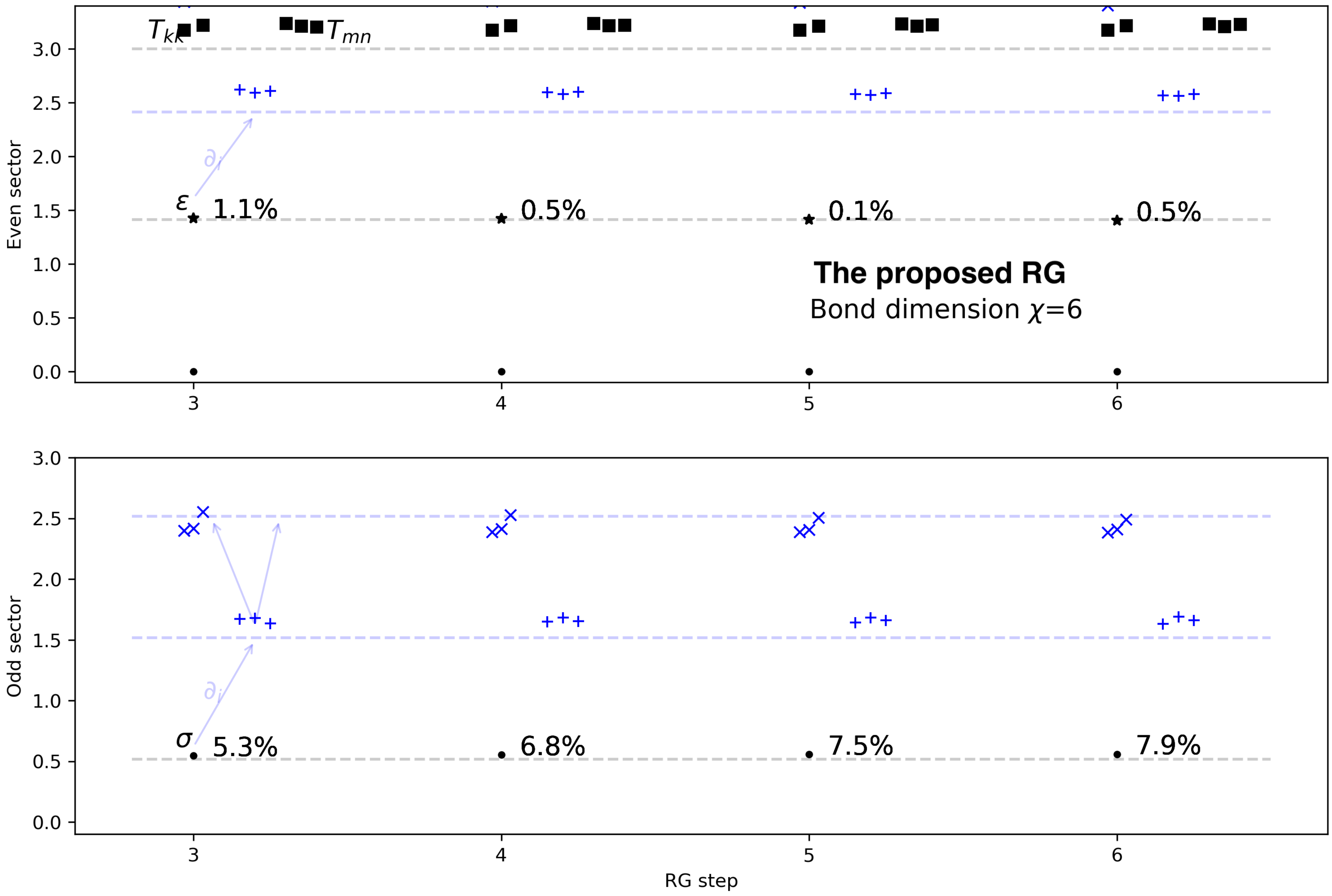


The HOTRG

Bond dimension χ=6

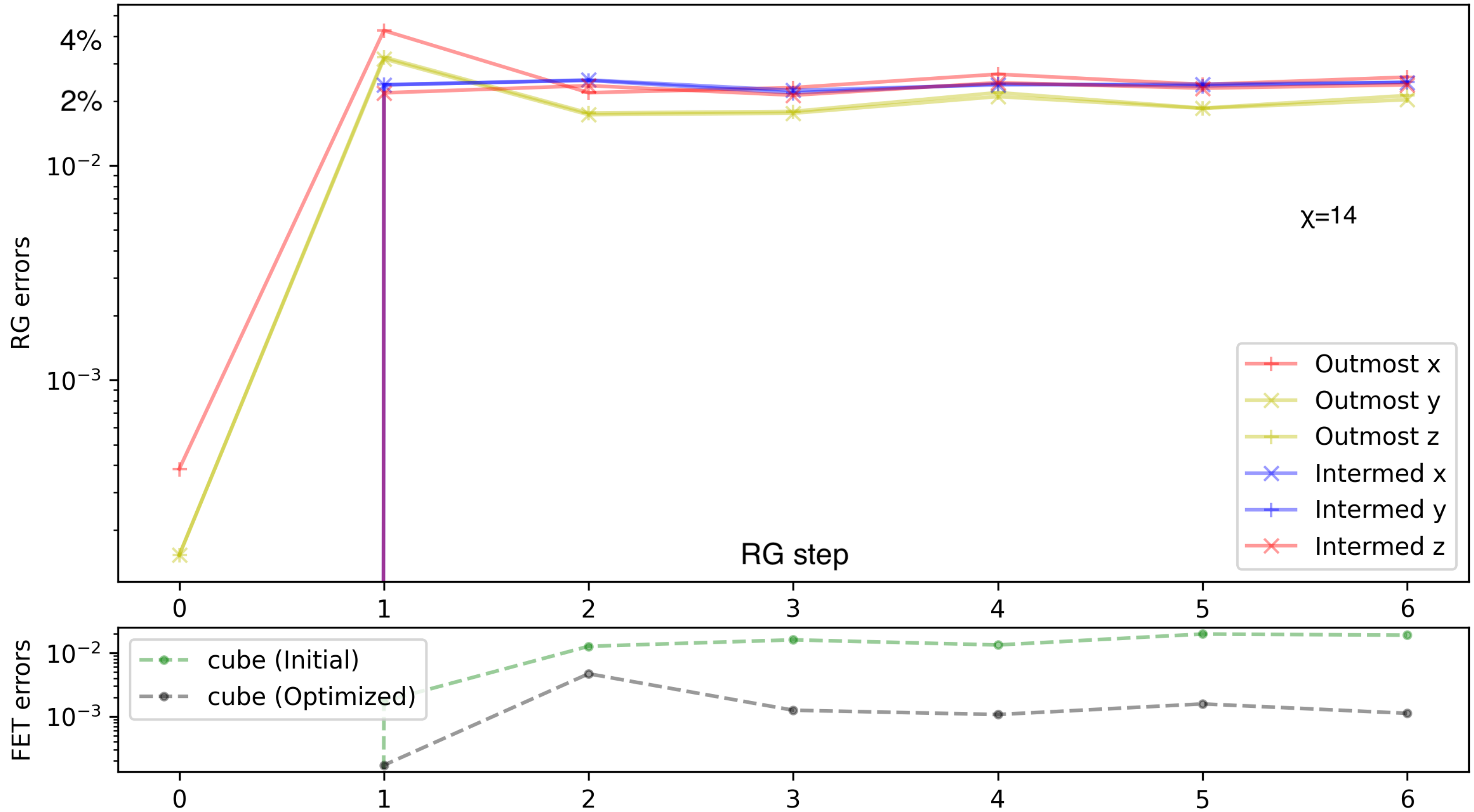
Entanglement filtering in 3D



Entanglement filtering in 3D

RG truncation errors versus the **bond dimension** χ

χ	6	8	11	14
RG error	6%	7%	4%	2%



***XL** and Kawashima,* arXiv:2412.13758

Entanglement filtering in 3D

Scaling dimensions versus the **bond dimension** χ

χ	6	8	11	14
min error	5%	4%	3%	0.4%
max error	8%	6%	6%	0.5%

Table 8.1: Estimation errors for x_σ versus bond dimension

χ	6	8	11	14
min error	0.1%	4%	1%	2%
max error	1%	5%	6%	4%

Table 8.2: Estimation errors for x_ϵ versus bond dimension

For spin field x_σ
- ✓ Mild decay of error with increasing bond dimension
- ✓ The magic bond dimension is $\chi = 14$

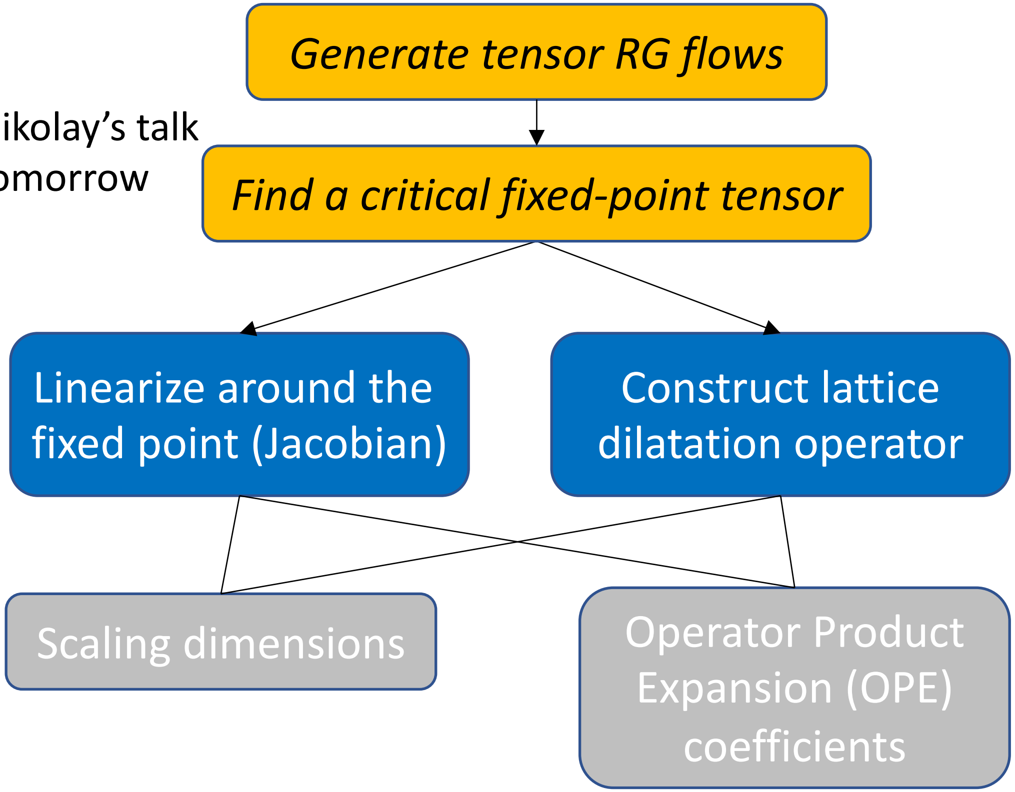
For energy density field x_ϵ
- ✓ Decay of error isn’t clear; but there is no apparent increase either.
- ✓ The magic bond dimension is $\chi = 6$

Remark: in 2D TNR, the systematical improvement is demonstrated by increasing the bond dimension $\chi = 6 \rightarrow 16 \rightarrow 24$

XL *and Kawashima,* arXiv:2412.13758

TNRG and other methods

Embedded in Wilsonian RG framework,
the TNRG reveals more universal data!



Generate tensor RG flows

Find a critical fixed-point tensor

Linearize around the fixed point (Jacobian)

Construct lattice dilatation operator

Scaling dimensions

Operator Product Expansion (OPE) coefficients

Nikolay's talk tomorrow

Other methods: Physical observable versus parameter + finite-size scaling or fitting

- ✓ Monte Carlo simulation
- ✓ Methods that use tensor networks as ground state ansatz: MPS, PEPS,…

Summary

XL and Kawashima, arXiv:2412.13758

https://github.com/brucelyu/efrg3D

- The Kadanoff's block idea has been upgraded to become a *reliable* 3D real space RG
- In its best scenario, the error of x_σ is 0.4% and that of x_ϵ is 0.1%

x_σ, x_ϵ $m \sim (\lambda - \lambda_c)^\beta$

TN Methods	Proposed	HOTRG	2D MERA	iPEPS
Smallest error	0.1%, 0.4%	0.9%	1.0%	1.7%
Computional cost	$\chi^{12.5}$	χ^{11}	χ^{16}	$D^{10\sim14}$

- The *conformal tower structure* is unique among all well-established numerical techniques
- It is a solid step towards a systematically improvable numerical RG