Renormalization group using tensor networks

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Renormalization Group from 1970s until today

- **Perturbative RG in field theory**

  4-ϵ expansion Wilson, Fisher 1972

  Still a valuable tool

- **Numerical RG for quantum Hamiltonians**

  Kondo model Wilson RMP 1975

  Flawed for lattice models => DMRG/MPS White 1993

- **Real-space RG for stat-phys spin models**

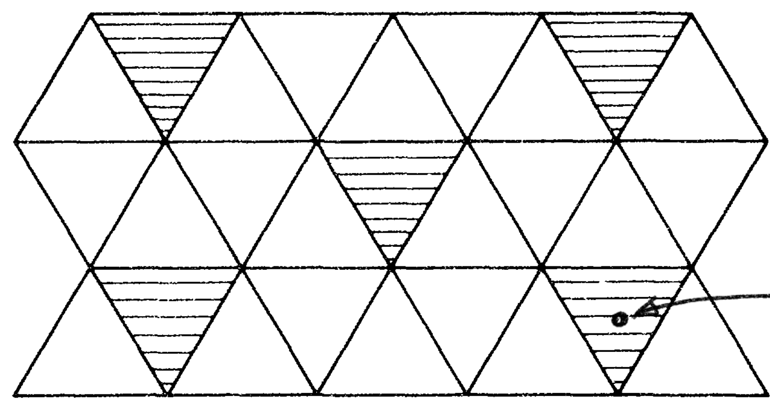
  Niemeijer, van Leeuwen 1973, Wilson RMP 1975

  Nothing fundamentally wrong, but challenging to implement

- **...**

Real-space RG

Niemeijer, van Leeuwen 1973



$s_i = \pm 1$

$s_i' = \mathrm{sign}(s_1 + s_2 + s_3)$ "BLOCK SPIN"

$$\sum_{\{s\}} e^{H[\{s\}]} = \sum_{\{s'\}} \underbrace{\sum_{\{s\}\ \text{compatible with}\ \{s'\}} e^{H[\{s\}]}}_{=:\ e^{H'[\{s'\}]}}$$

RG

Z is preserved

<u>Conjecture:</u> For any **fixed** {s'}, the system of {s} spins has finite correlation length **even @ T=Tc**

(e.g. Kennedy 1992)

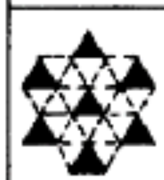
=> H'[{s'}] exists and is short-range (with exponential tails)

But to compute it one must perform an infinite sum over {s} :(

Results for the 2D Ising critical exponents, triangular lattice

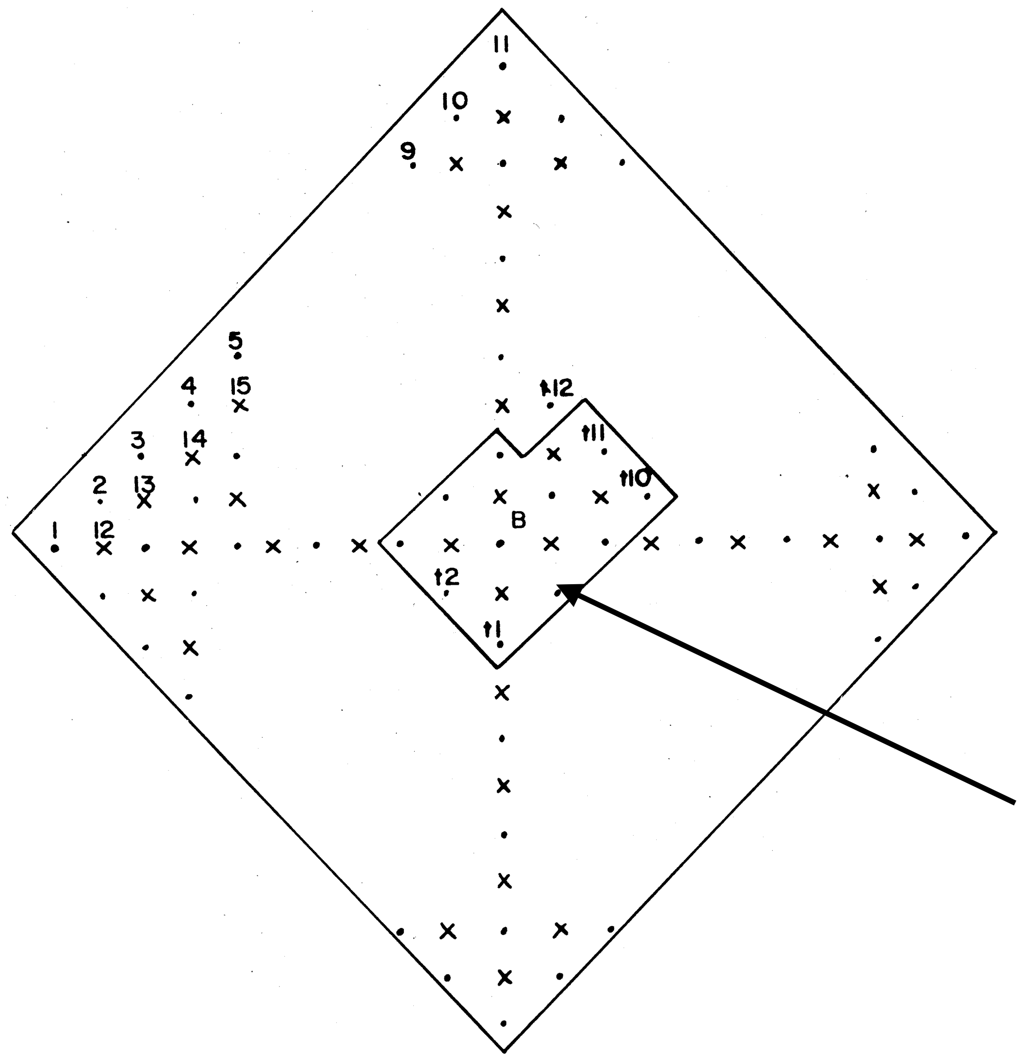
Domb et al, vol.6

TABLE IV. The eigenvalues and critical temperatures in the successive approximations as obtained by Niemeijer and van Leeuwen (1973). The results of the six-cell cluster have been obtained by Subbarao (1975).

CLUSTER	Y_T	Y_H	K_C
	.7908	2.0217	0.365
	.7394	1.6688	0.255
	.8177	1.7562	0.253
	1.0518	1.9232	0.281
	1.0518	1.9219	0.281
	1.0281	1.87375	0.27416
EXACT	1.000	1.87500	0.27465

11 couplings

Results for the 2D Ising critical exponents, square lattice   Wilson RMP 1975



217 couplings fitting into this region

Fixed-point Hamiltonian:

TABLE III. Dominant spin couplings in the fixed point Hamiltonian for $K^* = 0.2817$. The spin numbering shown in Fig. 7.

Spin product	Coefficient	Spin product	Coefficient
s_1s_2	0.281758	$s_1s_2s_3s_5$	0.001762
s_1s_5	0.095562	$s_1s_2s_5s_6$	−0.001615
$s_1s_2s_4s_5$	−0.017242	$s_2s_4s_6s_7$	−0.001045
s_1s_3	0.008422	$s_1s_3s_4s_5$	−0.001023
s_1s_6	0.004704	$s_1s_2s_3s_4$	0.000736
$s_2s_4s_6s_8$	−0.004008	$s_1s_3s_4s_6$	−0.000612
$s_2s_5s_6s_7$	0.001803	$s_1s_5s_6s_7$	0.000575

Critical exponents:

K^*	ρ^*	η	ν	h_1
0.2384	1.030	0.17	0.95	0.0287
0.2618	1.041	0.23	0.984	0.0085
0.2897	1.044	0.2486	1.002	0.0013
0.2992	1.04421	0.2497	0.998	0.0012
0.3104	1.04422	0.2497	0.988	0.0021
0.3306	1.04417	0.2494	0.97	0.0059

Wilson RMP 1975

Open problem 0

Reproduce/improve Wilson's results on modern computers

"Calculations were performed on a CDC 7600 computer:
one iteration of the transformation required 3.3 sec"

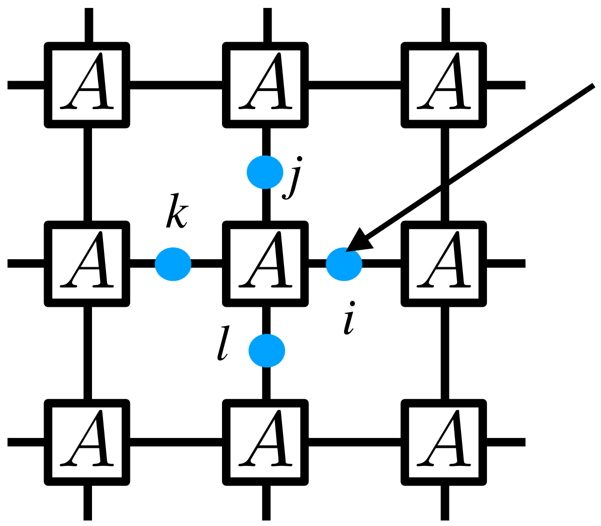
"The author found many ways to increase the efficiency
of the program<...>
Further details of simplifications like this will
not be reported here."

RG using tensor networks

Levin-Nave PRL 2006
talk Levin 2007

Rewrite Z as a tensor network contraction

$$\sum_{\{s\}} e^{H[\{s\}]} =$$



<u>Exercise:</u> Do this for NN Ising model, with $\chi = 2$.
At least 2 inequivalent solutions, one rotating the lattice by 45 degrees

RG map: $\mathscr{R} : A \mapsto A'$

so that $$Z(A, N \times M) = Z(A', \frac{N}{b}, \frac{M}{b})$$

b - scale factor

Real space RG	Tensor network RG
$Z' = Z$	
$H \to H'$	$A \to A'$
H becomes nonlocal => truncate	$\chi' > \chi$ => truncate

Measure of error of free energy

sum of truncated couplings

$$H = \sum_X K_X \prod_{x \in X} s_x$$

$$\|H\| = \sum_{X \ni x_0} \frac{|K_X|}{|X|}$$

change in Hilbert-Schmidt norm (?)

$$\varepsilon = \| A' - A'_{trunc} \|$$

See Griffiths in Domb et al vol. 1

Open problem 1

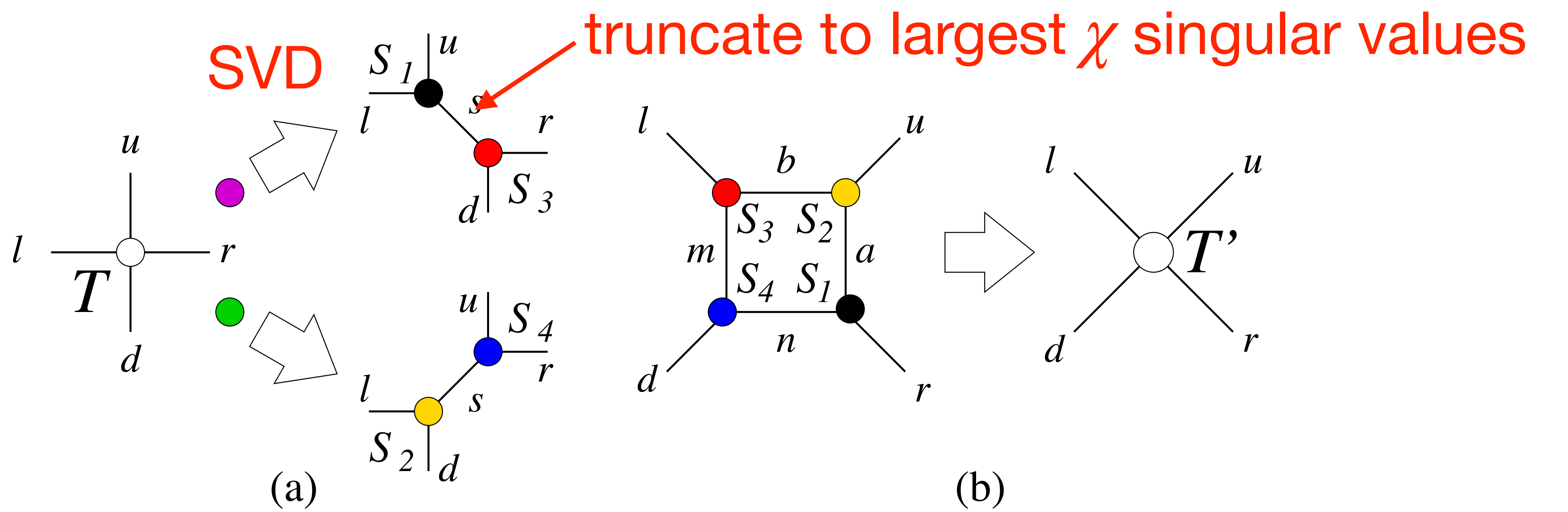
Conjecture: Free energy estimate for tensor networks

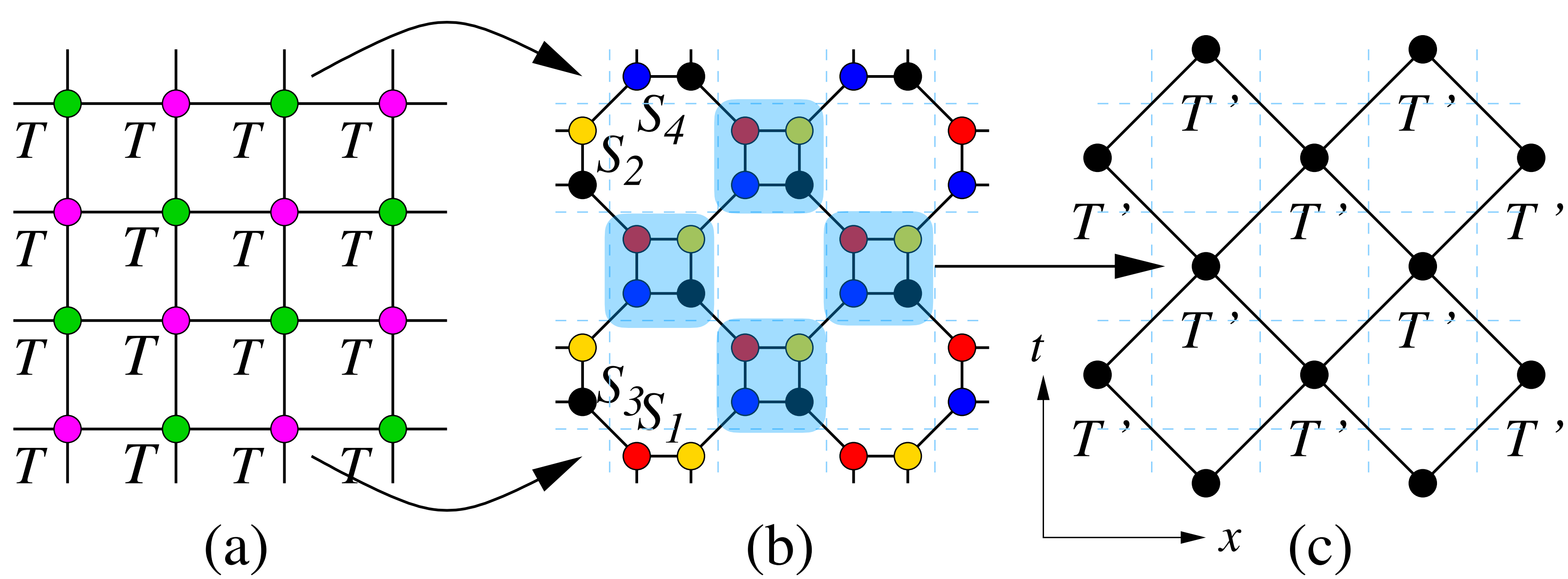
Under some natural conditions on A, free energy per site:

$$\lim_{L\to\infty} \frac{1}{L^2} \log Z[A, L\times L]$$

exists and varies continuously in A in Hilbert-Schmidt norm

TRG algorithm of Levin-Nave





Figs from Gu, Wen 2009

TRG illustrates features of all tensor RG algorithms:

- Perform local changes in the network structure without modifying Z

- Truncate (in numerical studies)

Hope: local truncation error small => global will also be small

- **Reconnection** - tensors are regrouped/contracted in a different order

- The number of tensors is reduced

Free energy computation using TRG

2D NN Ising @ β => transform to TN; write initial tensor as

$$A_{NN}(\beta) = e^{f_0} A_0\,, \qquad \|A_0\| = 1$$

$$Z(e^{f_0} A_0, L\times L)$$
$$= e^{f_0 \mathrm{Vol}}\, Z(A_0, L\times L)$$

↓TRG

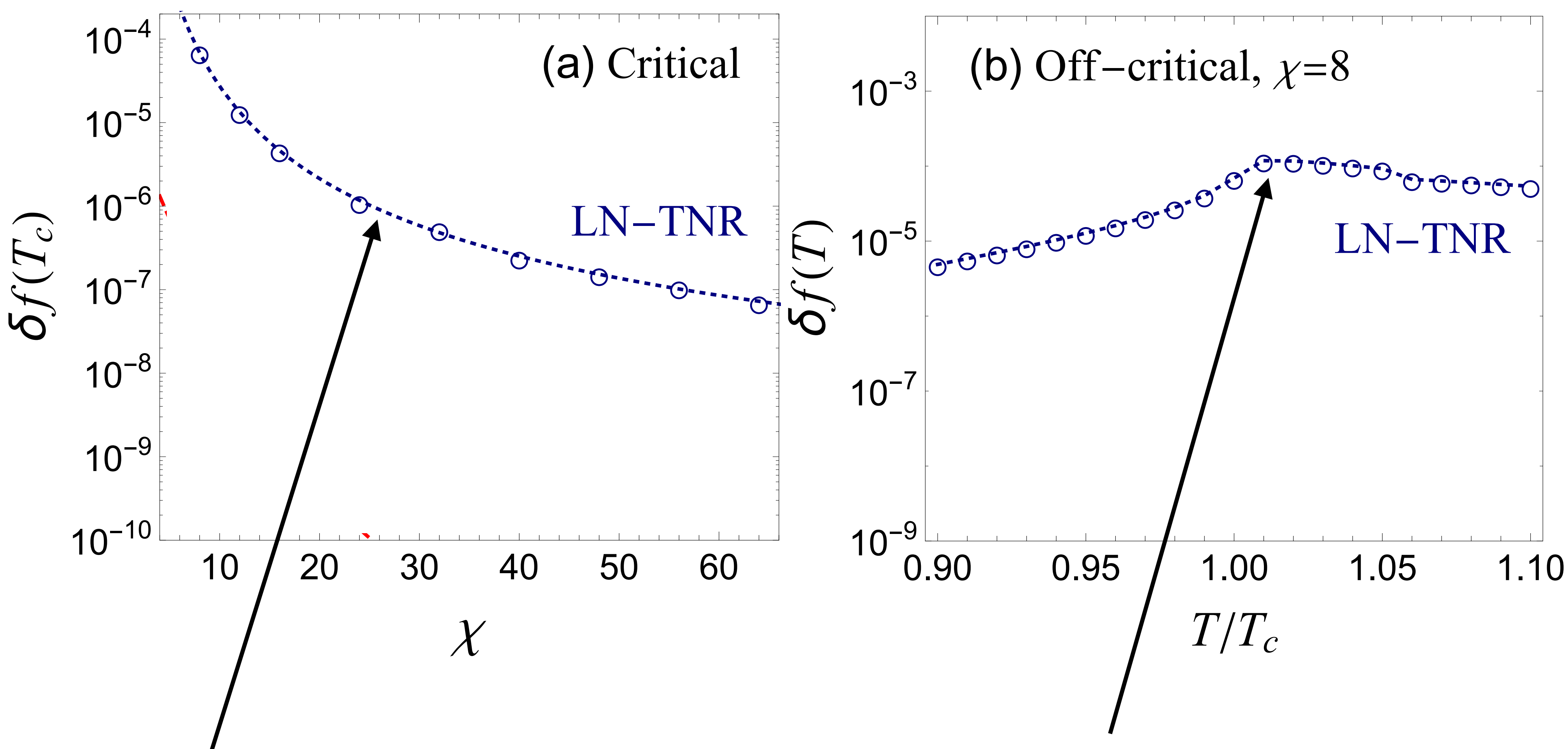
$$= e^{f_0 \mathrm{Vol}}\, Z(A_0', \tfrac{L}{b}\times\tfrac{L}{b})$$
$$= e^{(f_0 + \frac{f_1}{b^2})\mathrm{Vol}}\, Z(A_1, \tfrac{L}{b}\times\tfrac{L}{b})$$
$$= \ldots$$

$$A_0' = e^{f_1} A_1, \qquad \|A_1\| = 1$$

$$\Rightarrow \quad f = f_0 + \frac{f_1}{b^2} + \frac{f_2}{b^4} + \ldots$$

Error in free energy of TRG

Fig from Yang, Gu, Wen 2015



Error decreases with χ

T=Tc is more challenging

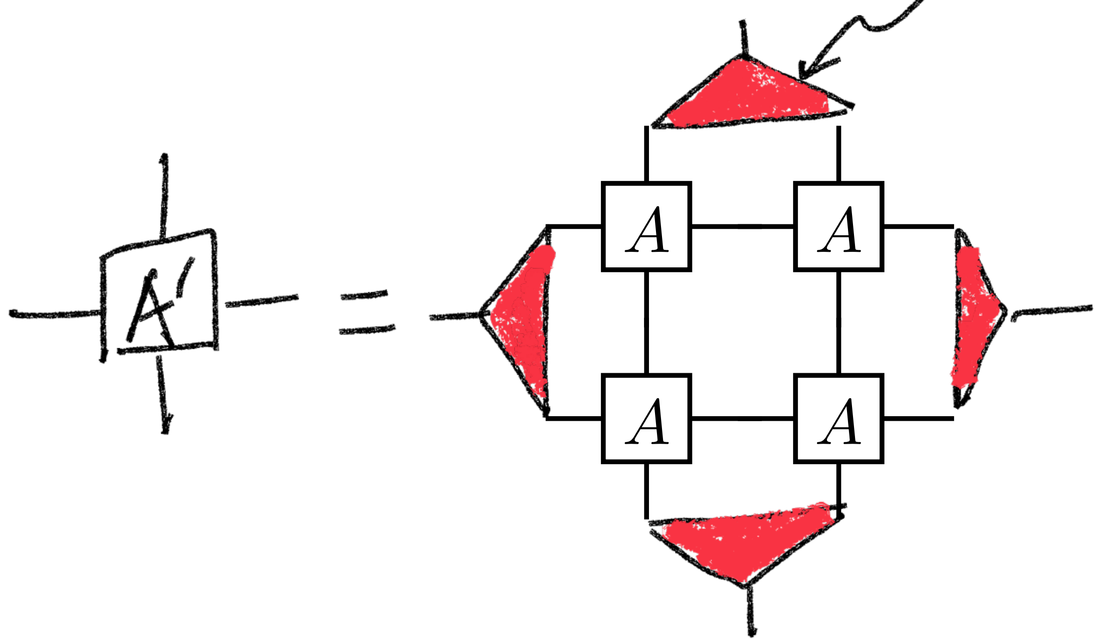
Entanglement structure of TRG

talk by Levin, 2007

TRG belongs to a group of algorithms <u>without disentangling</u> (like SRG, HOTRG…)

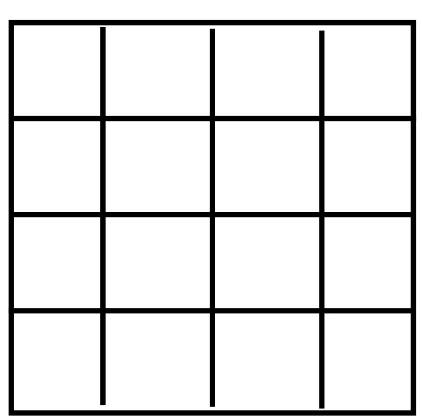
In essence, they all carry out <u>simple coarse-graining step</u>:

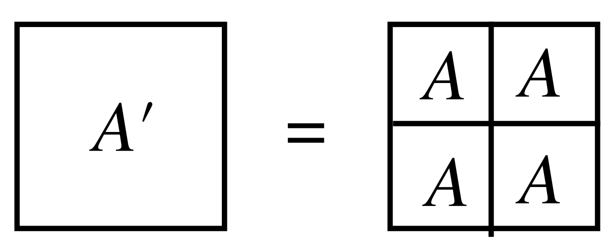
isometries, implementing truncation



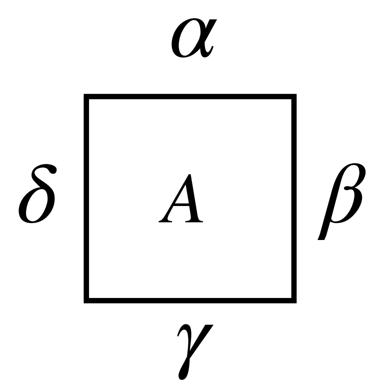
Without truncation:

Z =



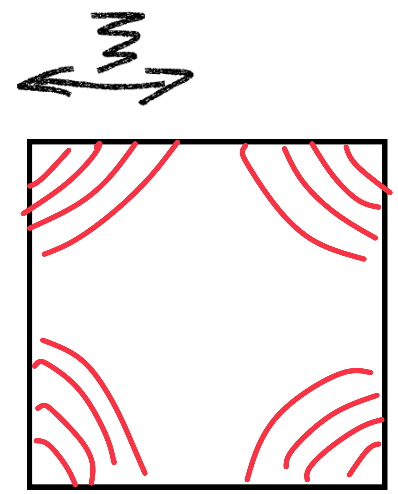


View the tensor obtained after many RG steps as a “ground state wavefunction”

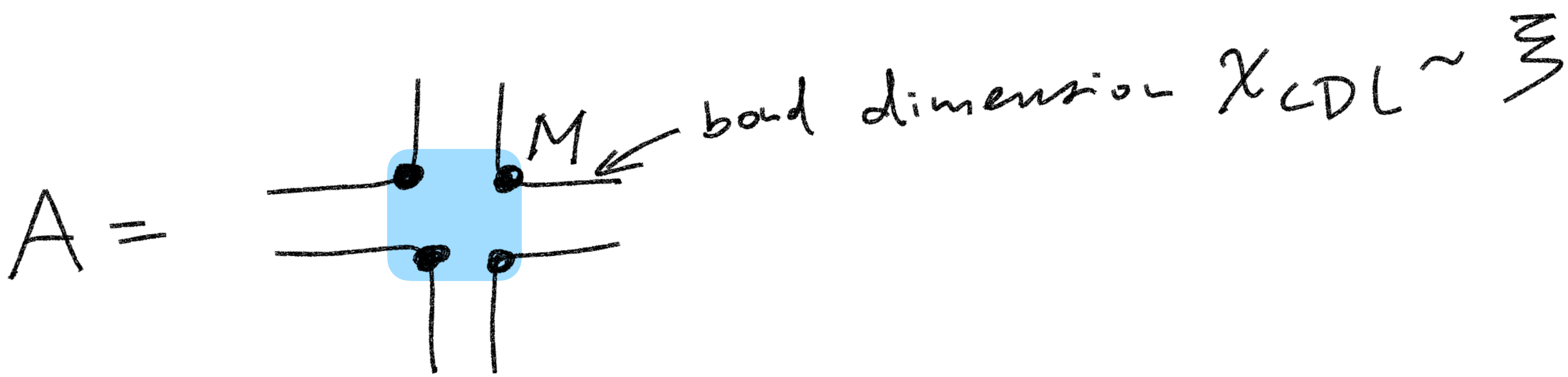


$=: \Psi_{\alpha\beta\gamma\delta}$

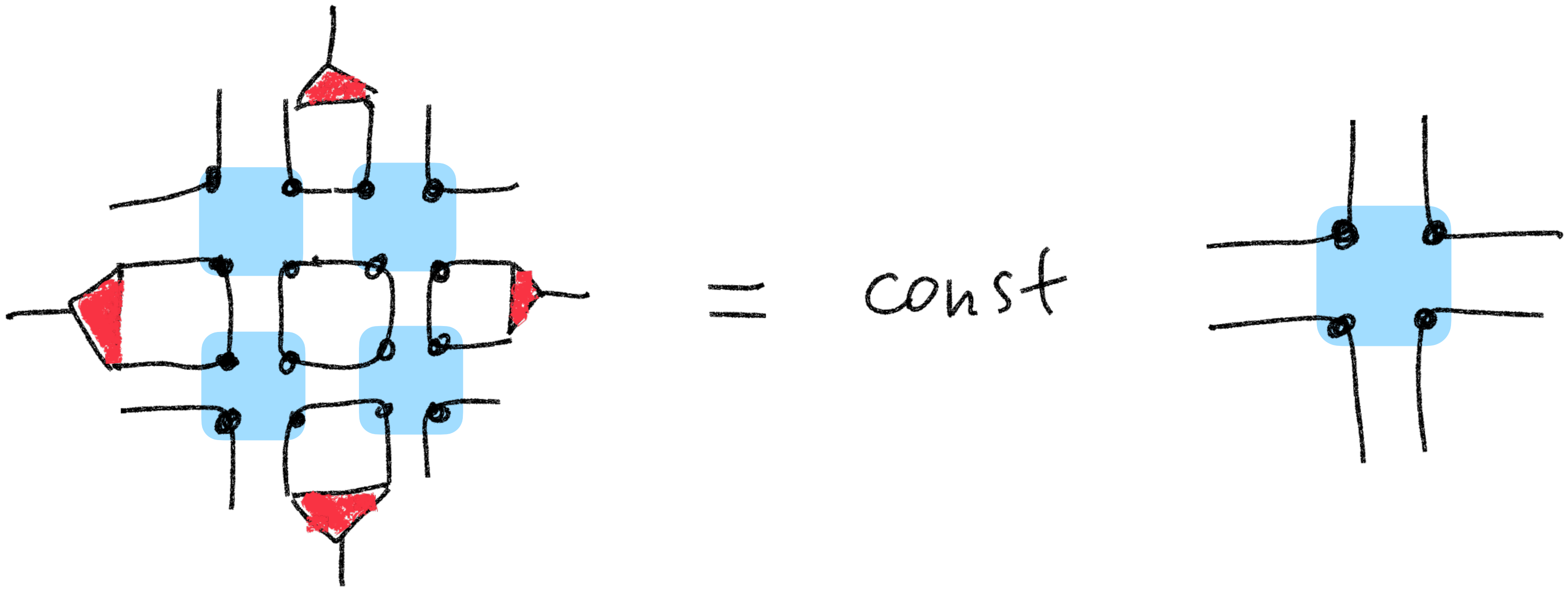
In the gapped phase we expect to be able to decompose Ψ as



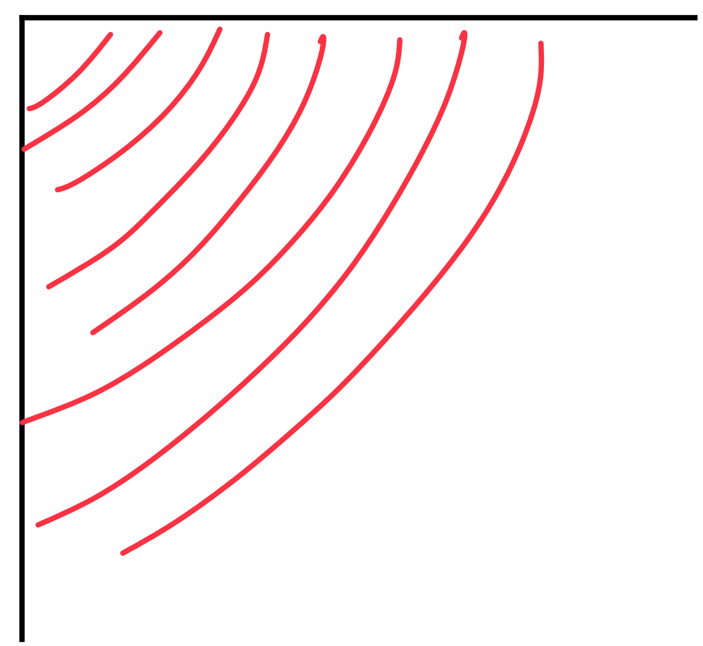
This is called **Corner-Double-Line** (CDL) tensor:



CDL tensors are fixed points of the coarse-graining step:

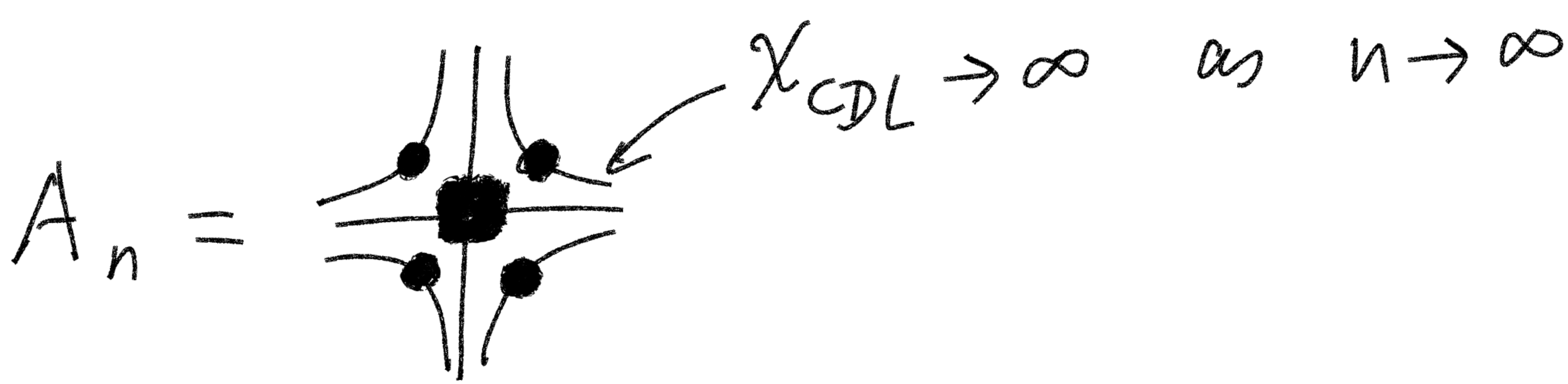


At the critical point $\xi = \infty$



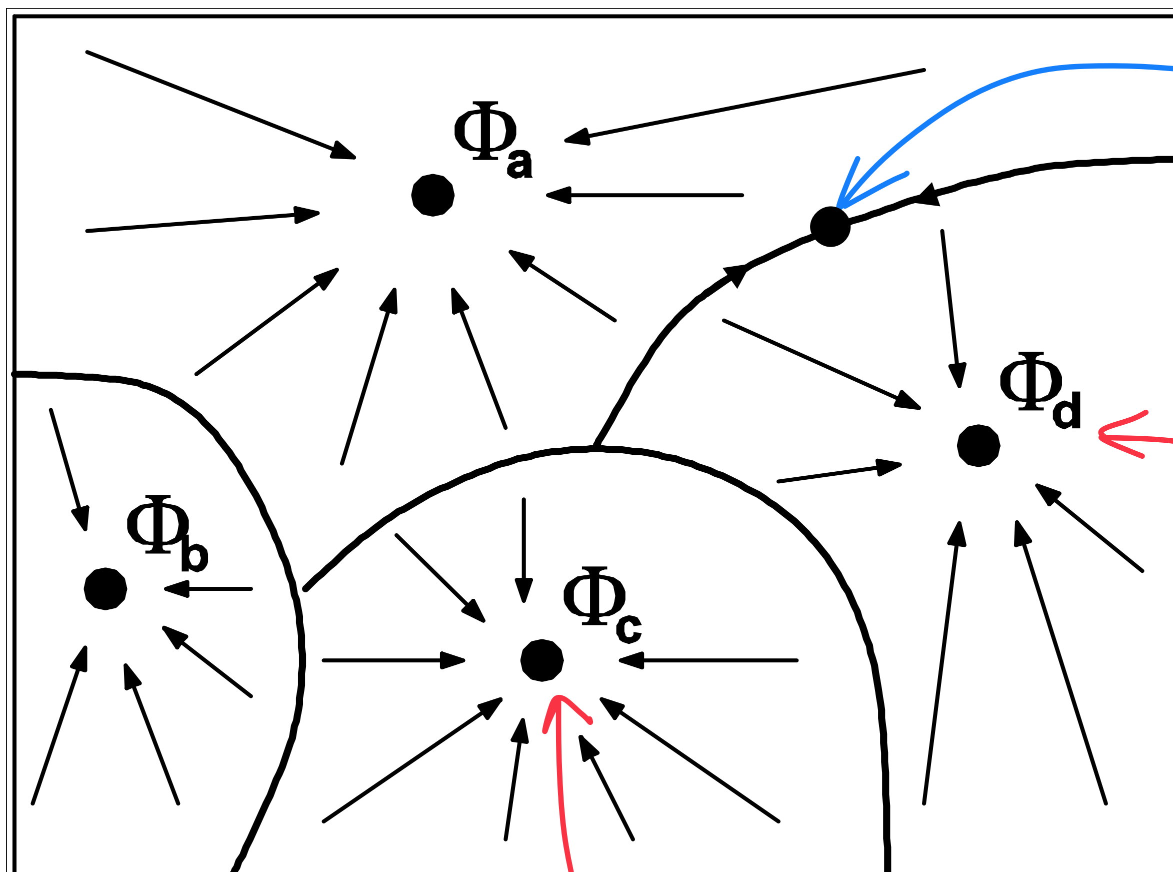
By conformal invariance, Corner Transfer Matrix has spectrum quantized in units of $1/\log L \to 0$ Peschl, Truong 1987

We expect after many RG steps:



Fixed point structure (desired)

Want to use (tensor) RG fixed points to classify phases:

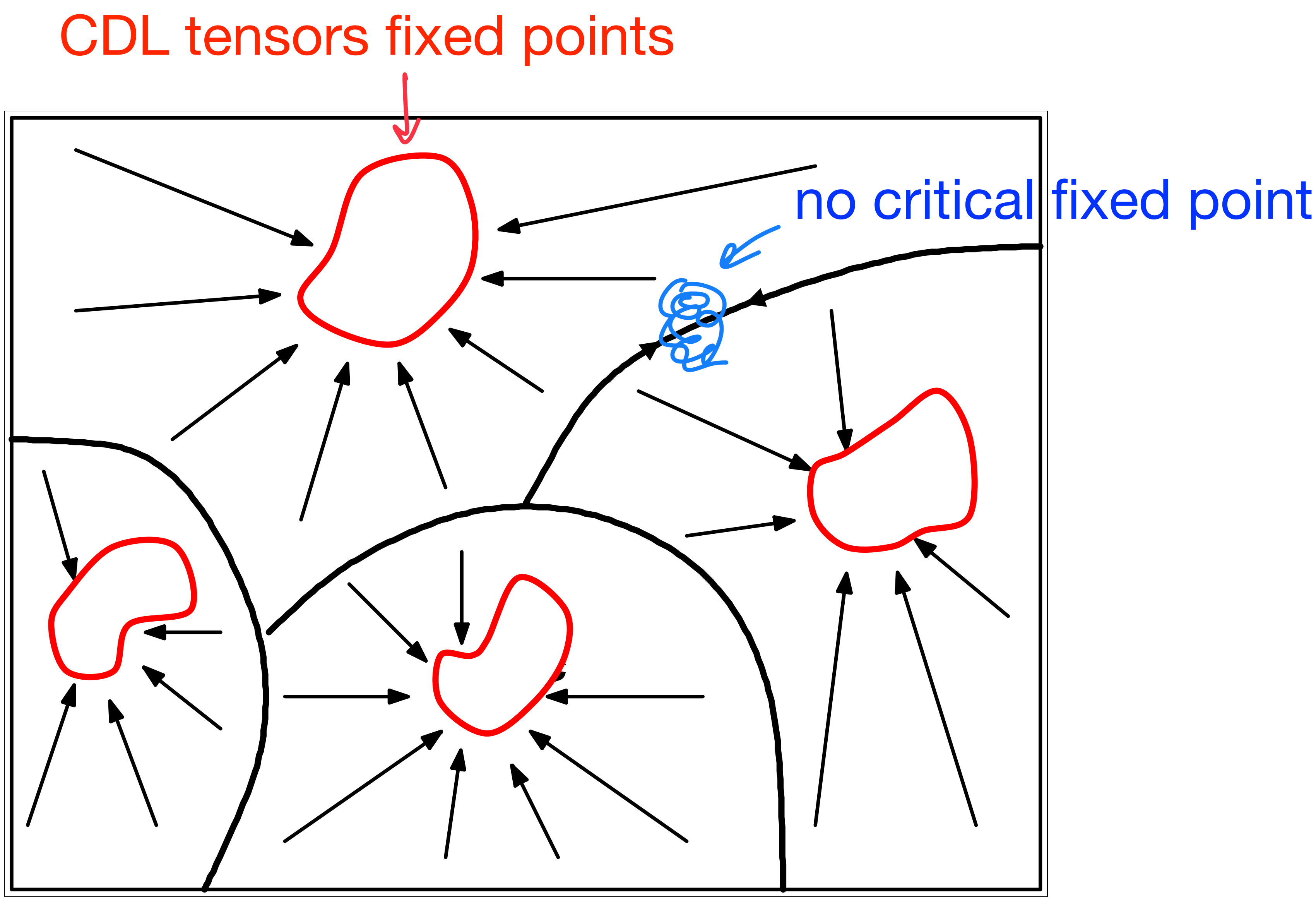


critical f.p.

noncritical f.p.
inside gapped phases

fig from talk Levin 2007

Instead, Fixed point structure for TRG

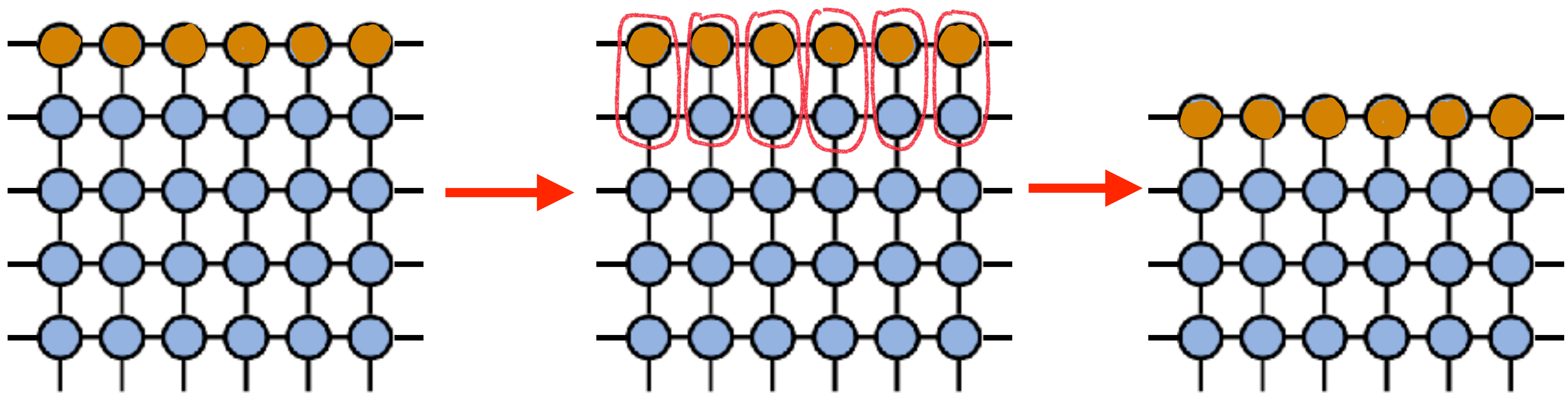


To resolve these issues, need disentangling (L2)

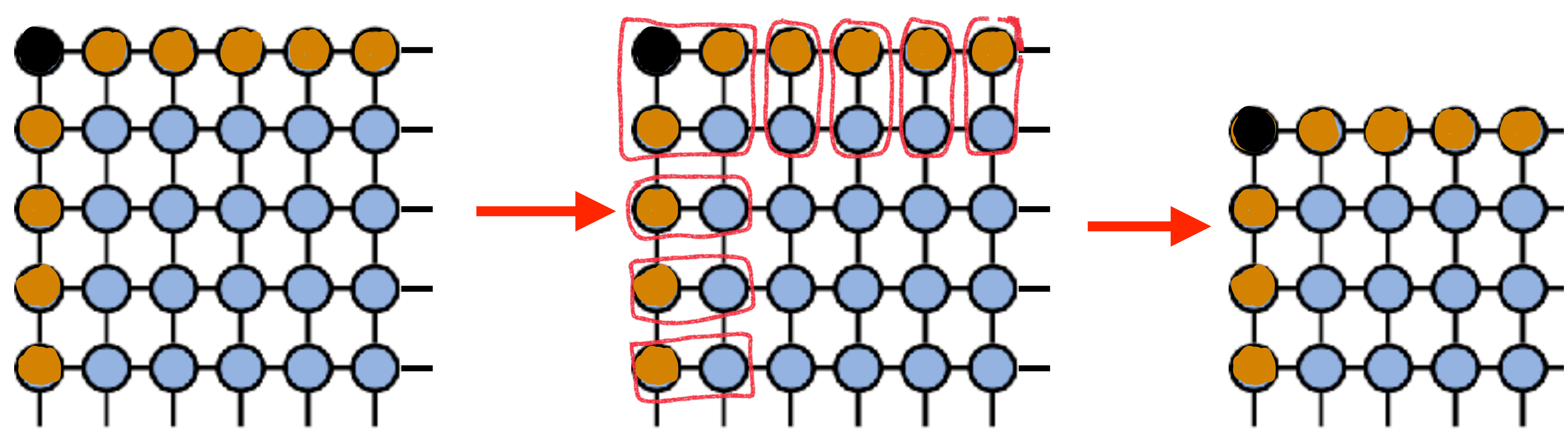
fig from talk Levin 2007

Other techniques for calculating partition functions

Row-by-row:



Corner transfer matrix:



May also benefit from disentanglement