THE CHIRAL PHASE TRANSITION & THE AXIAL ANOMALY

Fabian Rennecke

CRC-TR 211
Strong-interaction matter under extreme conditions

JUSTUS-LIEBIG-UNIVERSITÄT GIESSEN

HFHF

CONFINEMENT AND SYMMETRY FROM VACUUM TO QCD PHASE DIAGRAM
BENASQUE - 13/02/2025

QCD PHASE DIAGRAM

- chiral crossover at $(T, \mu_B) \approx (156{,}0)\,\text{MeV}$ [Bazavov et al., 1812.08235] [Borsanyi et al., 2002.02821]
- explicit chiral symmetry breaking due to finite quark masses + axial anomaly

$$U(3)_L \times U(3)_R \approx SU(3)_V \times SU(3)_A \times U(1)_V \times U(1)_A$$

↓ axial anomaly

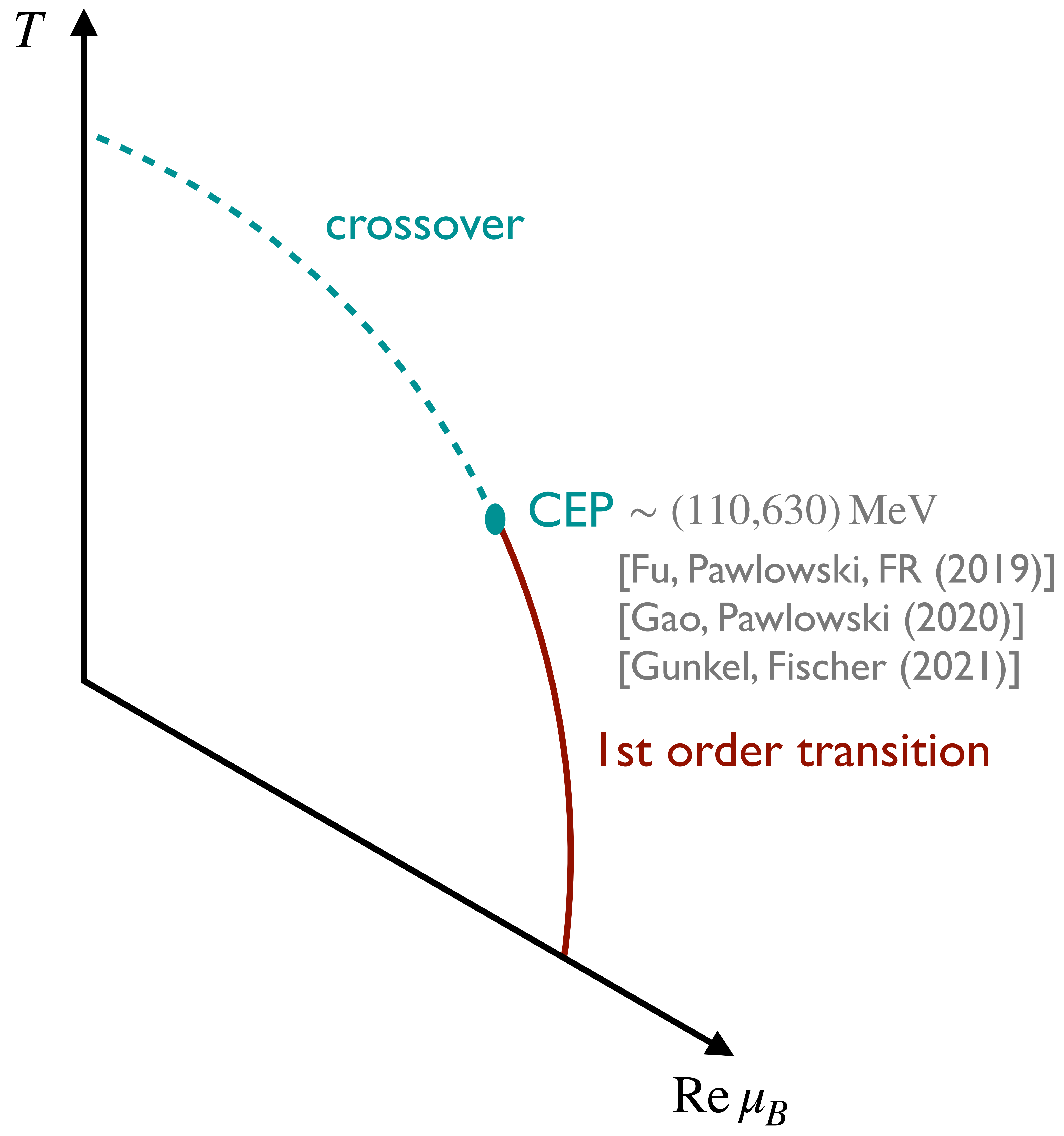
$$SU(3)_V \times SU(3)_A \times U(1)_V$$

↓ chubby s-quark

$$\underbrace{SU(2)_V \times SU(2)_A}_{\sim O(4)} \times U(1)_V$$

↓ quark masses

$$SU(2)_V \times U(1)_V$$

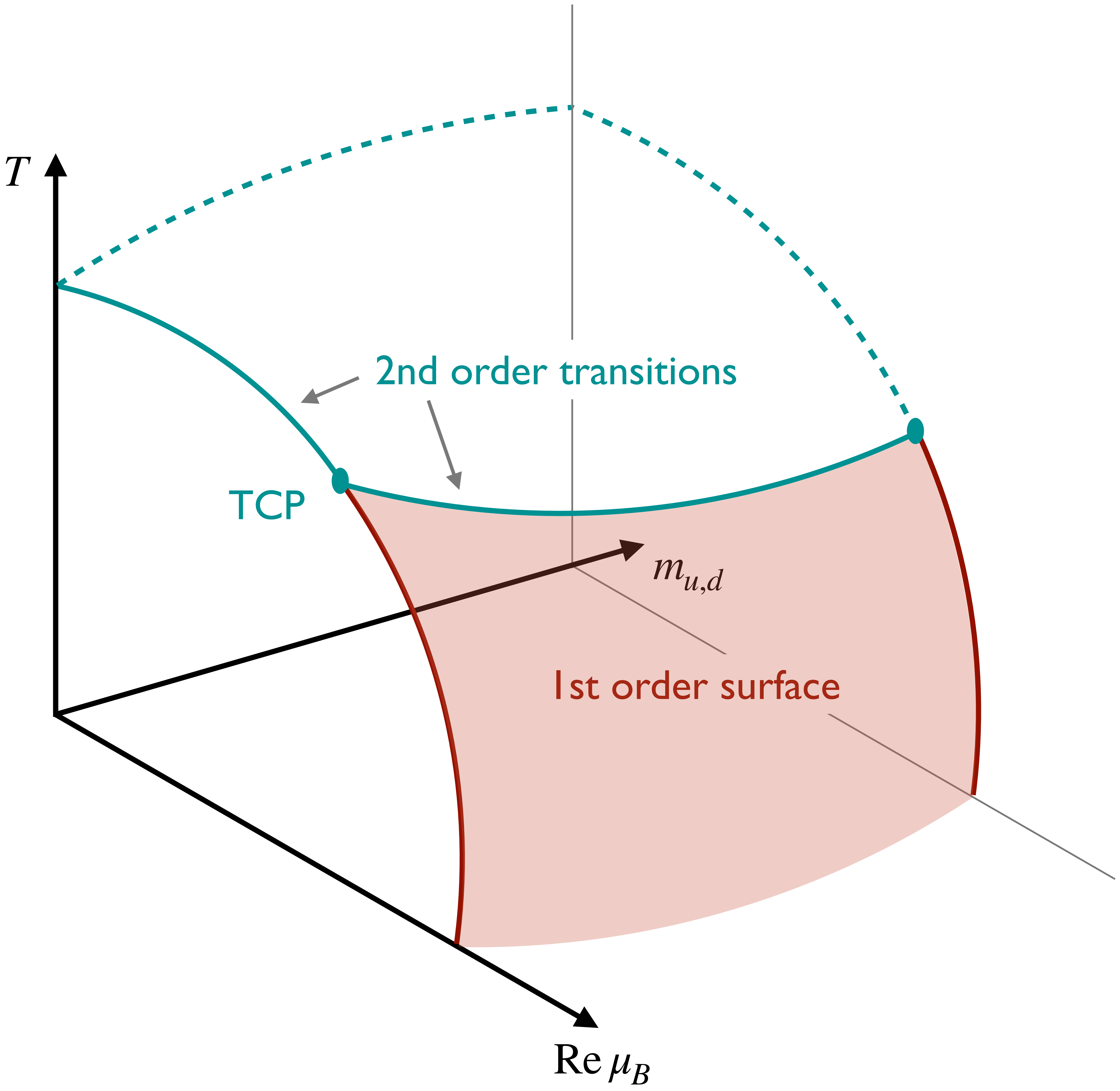


QCD PHASE DIAGRAM

reduce/remove explicit symmetry breaking

↓

actual phase transition



- 2nd order transition for some $0 \leq m_q < m_{q,\,\mathrm{physical}}$ at $\mu_B = 0$
- depends on symmetries, i.e. masses of the different quark flavors and the fate of the axial anomaly

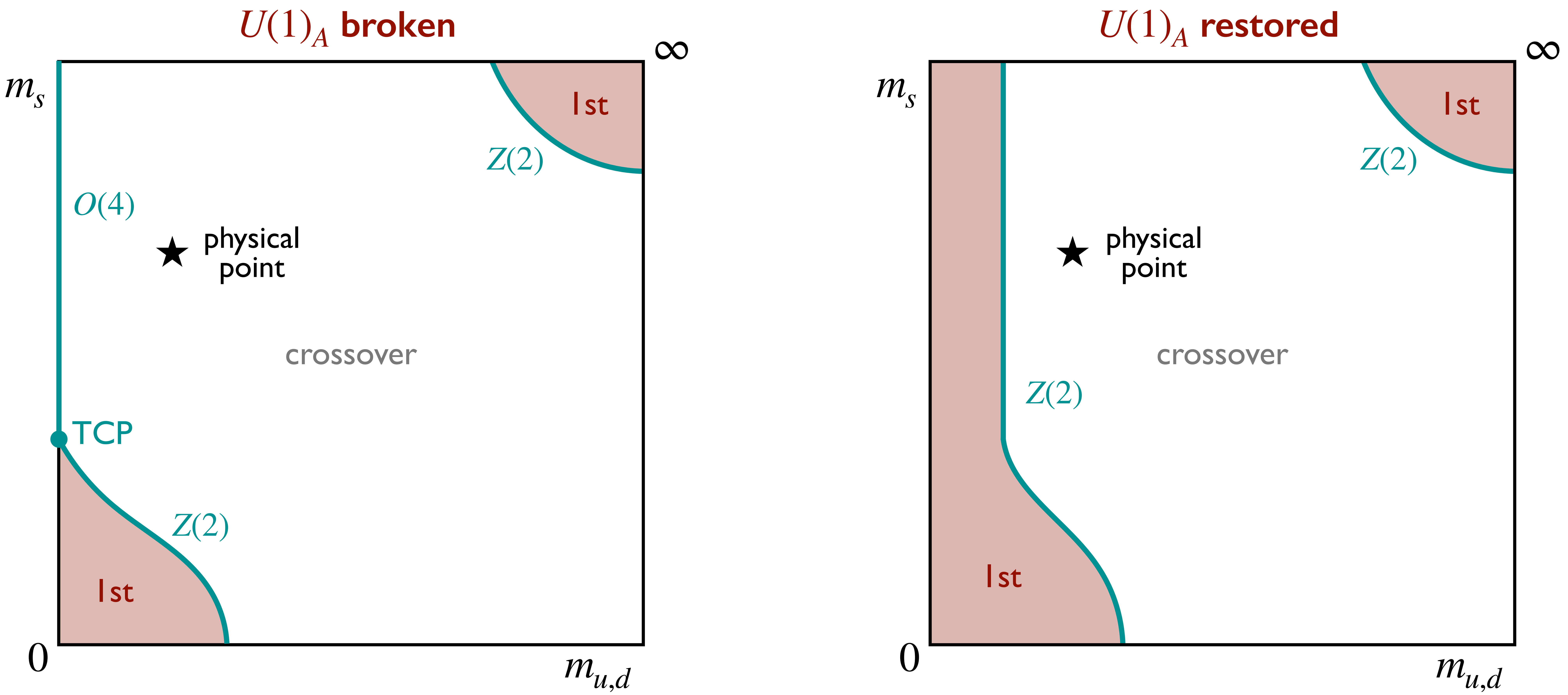
THE COLUMBIA PLOT

"Classic" Pisarski-Wilczek scenario:

[Pisarski, Wilczek, PRD 29 (1984)]

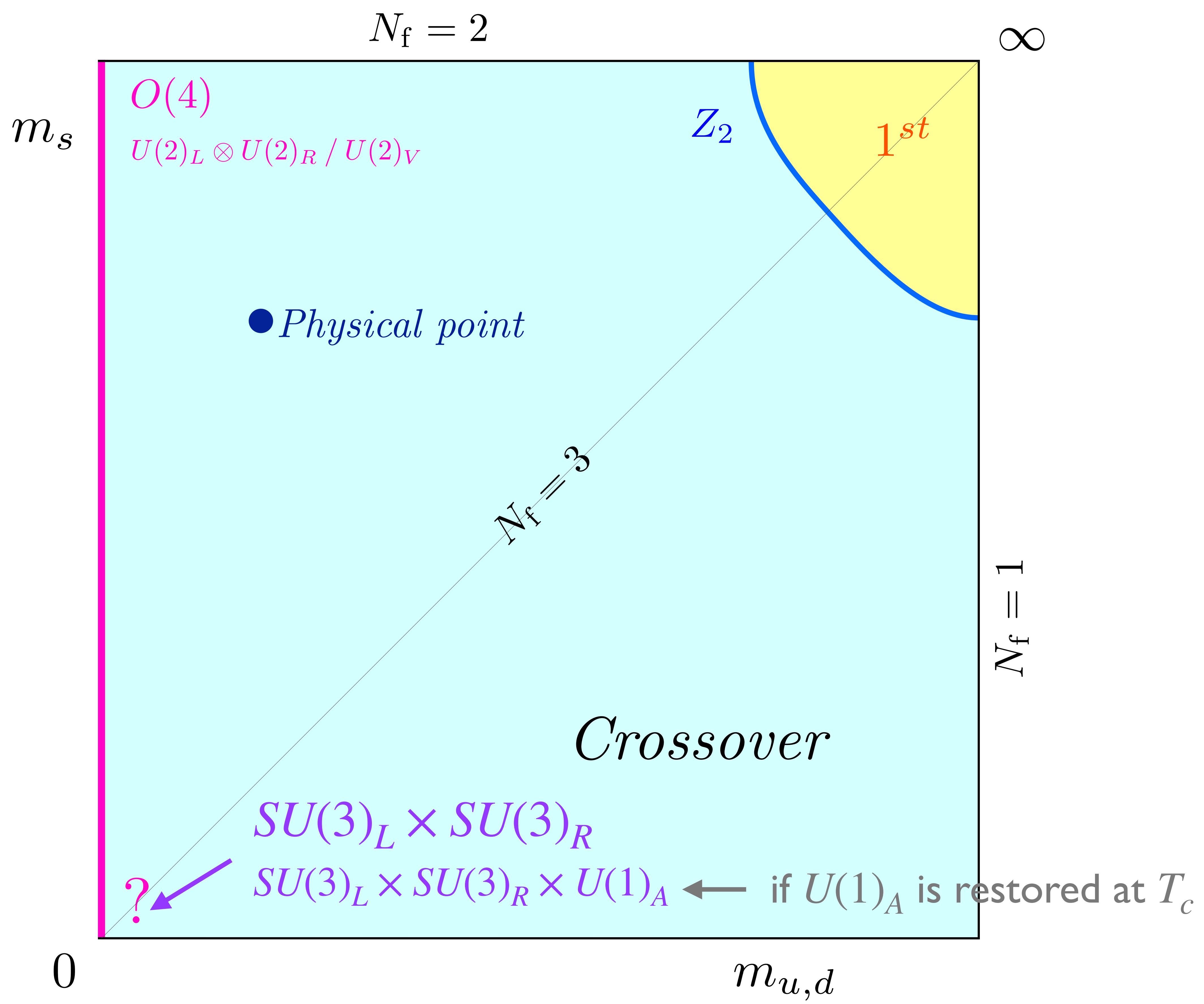
- 2nd order transition in the light chiral limit ($m_{u,d} = 0$) if $U(1)_A$ remains broken, otherwise 1st order
- 1st order transition in the chiral limit ($m_{u,d,s} = 0$), irrespective of the fate of the axial anomaly

Conjecture based on NLO ϵ-expansion (perturbative RG) of a linear sigma model:



THE COLUMBIA PLOT

- fixed point analysis has been improved significantly since 1984
- classic scenario has recently been challenged by direct calculation [Cuteri, Philipsen, Sciarra (2021)]



(cf. Aoki's and Philipsen's talks)

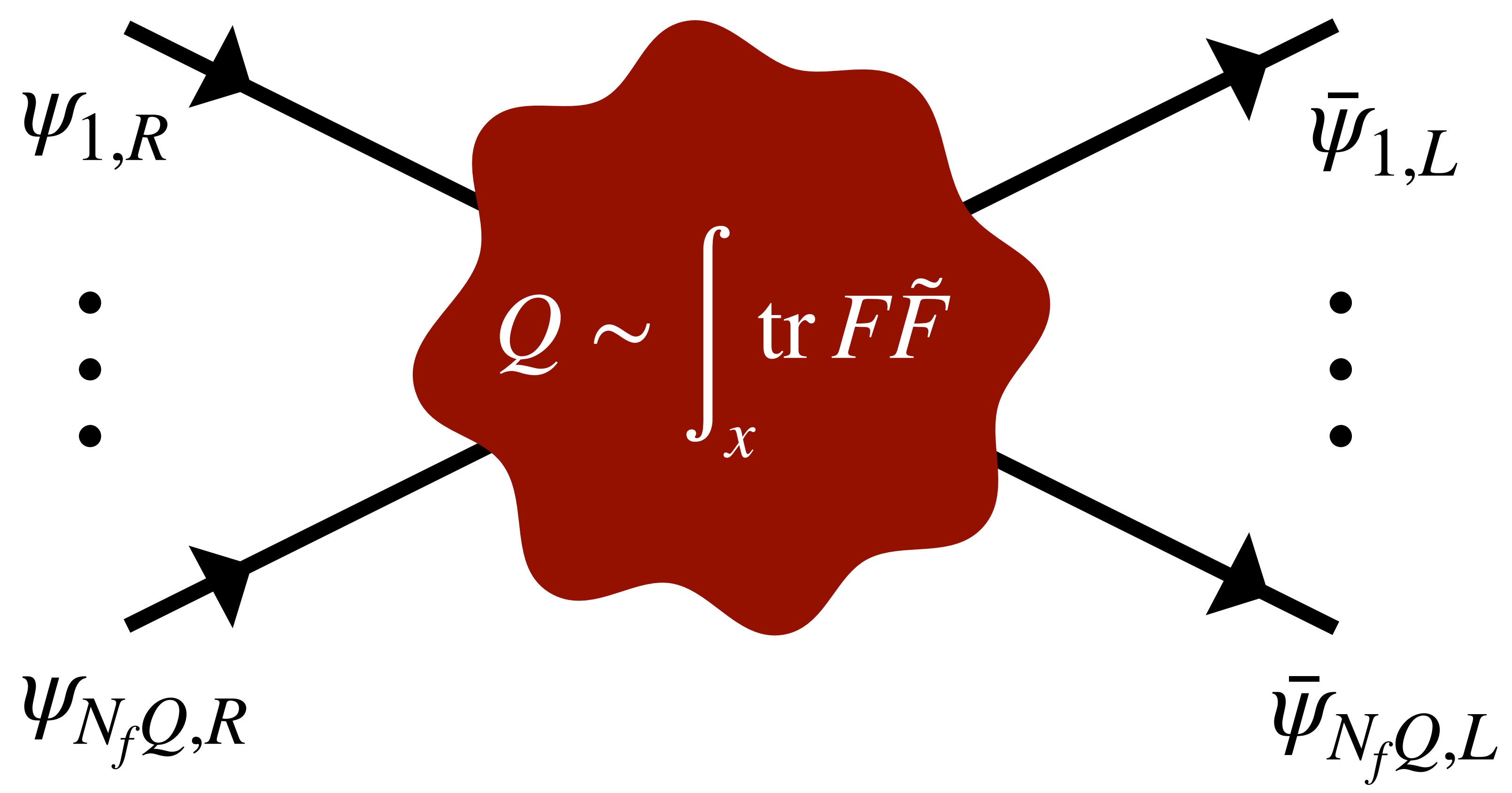
No evidence of 1st order transition in bottom left corner from lattice QCD. Maybe a very small region? Maybe no 1st order at all?

upper bound/"small" region:
[Bazavov et al., 1701.03548]
[Kuramashi et al., 2001.04398]
[Dini et al., 2111.12599]

no 1st order transition
[Cuteri, Philipsen, Sciarra, 2107.12739]
[Bernhardt, Fischer, 2309.06737]

⟶ can we understand what's going on?

HOW TO UNDERSTAND THE CHIRAL PHASE TRANSITION?

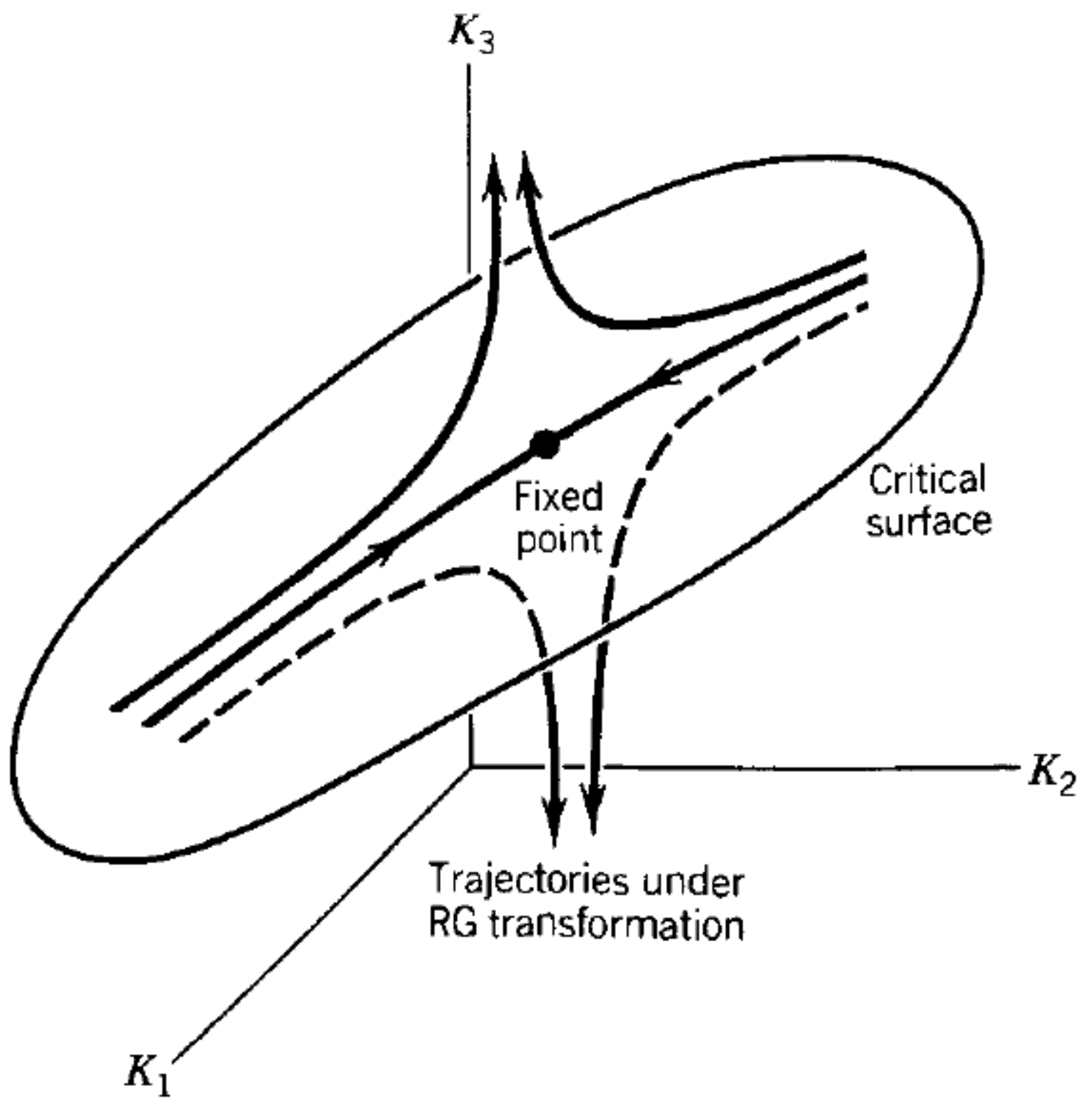


(1) What are relevant symmetries?

→ role of the axial anomaly at T_c

(2) Does the underlying critical theory have stable fixed points?

→ what do we know from critical phenomena?



(3) Are the fixed points reached with physical parameters

→ can the system be tuned to criticality with available control parameters?

NECESSARY CONDITIONS FOR 2ND ORDER TRANSITIONS

necessary ingredient from QCD

symmetry at transition

- If $U(1)_A$ remains broken at T_c, possibilities are $SU(3)_L \times SU(3)_R$, $O(4)$ and $Z(2)$
- If $U(1)_A$ is restored at T_c, possibilities are $SU(N)_L \times SU(N)_R \times U(1)_A$ for $N = 2, 3$ and $Z(2)$

we must be able to tune the system to the fixed point

- CEP: tune T and μ
- Columbia plot: tune T and m

→ stable fixed point for us: two relevant directions

necessary ingredients from critical physics

relevant critical theory must have a stable fixed point
(no more relevant parameters than we can control)

- all $3d$-$O(N)$ theories have fixed points with two relevant directions (t and h)
- no evidence for stable fixed point for $SU(3)_L \times SU(3)_R$ [Butti, Pelissetto, Vicari (2003); Fejos (2022)]
- evidence for stable fixed points for $SU(N)_L \times SU(N)_R \times U(1)_A$ for $N = 2, 3$ [Pelissetto, Vicari (2013); Nakayama, Ohtsuki (2014); Fejos (2022); Kousvos, Stergiou (2022)]

→ not all relevant systems seem to allow for 2nd-order transitions, 1st-oder expected then

Note: these are not sufficient conditions!

STABLE FIXED POINT ≠ 2ND ORDER TRANSITION

Even if there is a stable critical point, the physical transition can still be 1st order

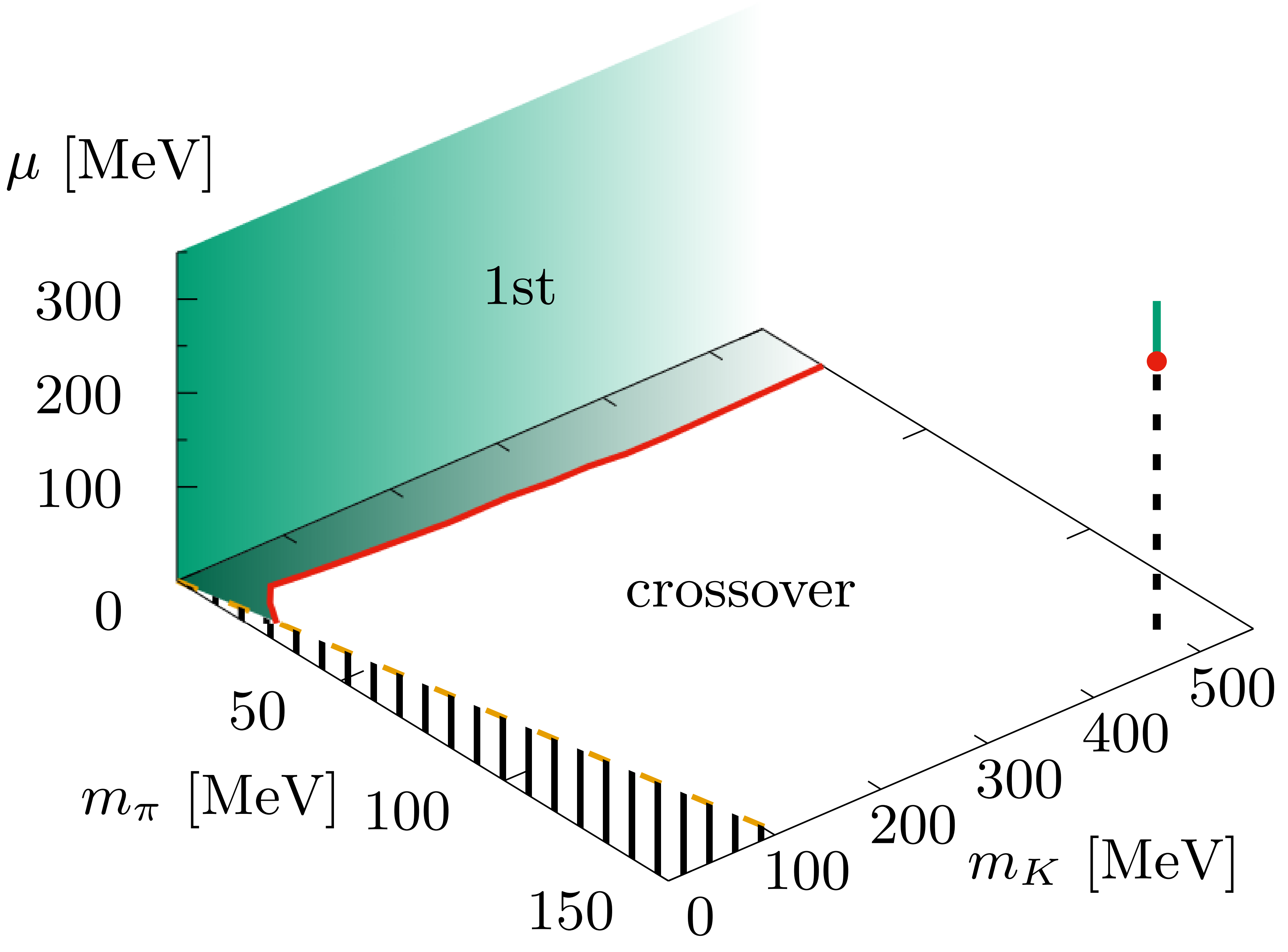
Example: functional RG (nonperturbative RG, LO derivative expansion) analysis of quark-meson model

- fixed point analysis: stable fixed point for $SU(N)_L \times SU(N)_R \times U(1)_A$ for $N = 2,3$

[Fejos, 2201.07909]
[Fejos, Hatsuda, 2404.00554]

- direct computation of phase diagram (QM model): 1st order transition for "physical" parameters in same universality class in (light) chiral limit

[Resch, FR, Schaefer, 1712.07961]



→ **fixed point is there, but system doesn't seem to reach it**

Note: same method and very similar approximation is used in both studies!

UNIVERSALITY AND THE CHIRAL TRANSITION

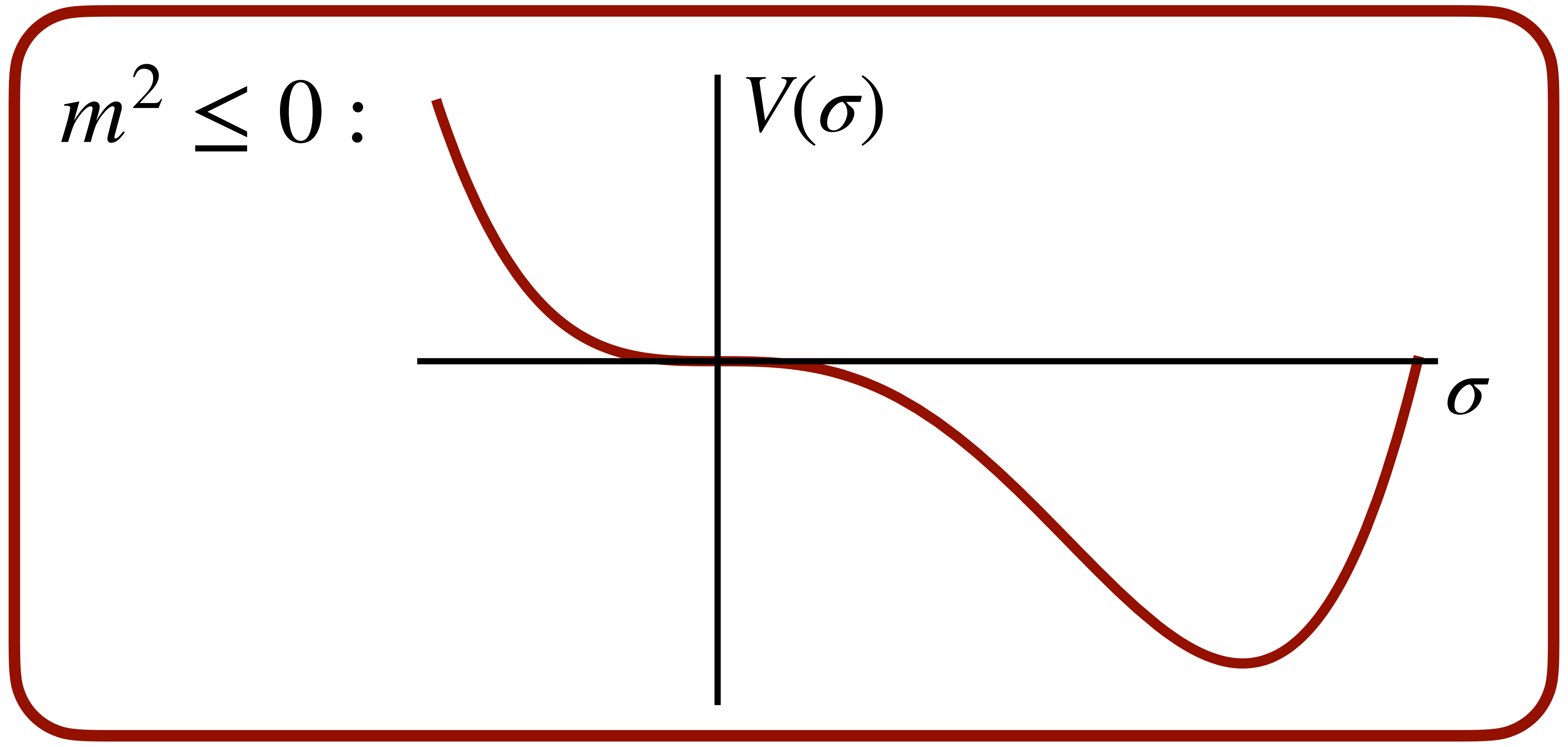
Critical theory is, in general, given by a matrix model, $\Phi \sim \bar{q}_L q_R \xrightarrow{SU(N_f)_L \times SU(N_f)_R \times U(1)_A} e^{i\theta_A} U_R \Phi U_L^\dagger$

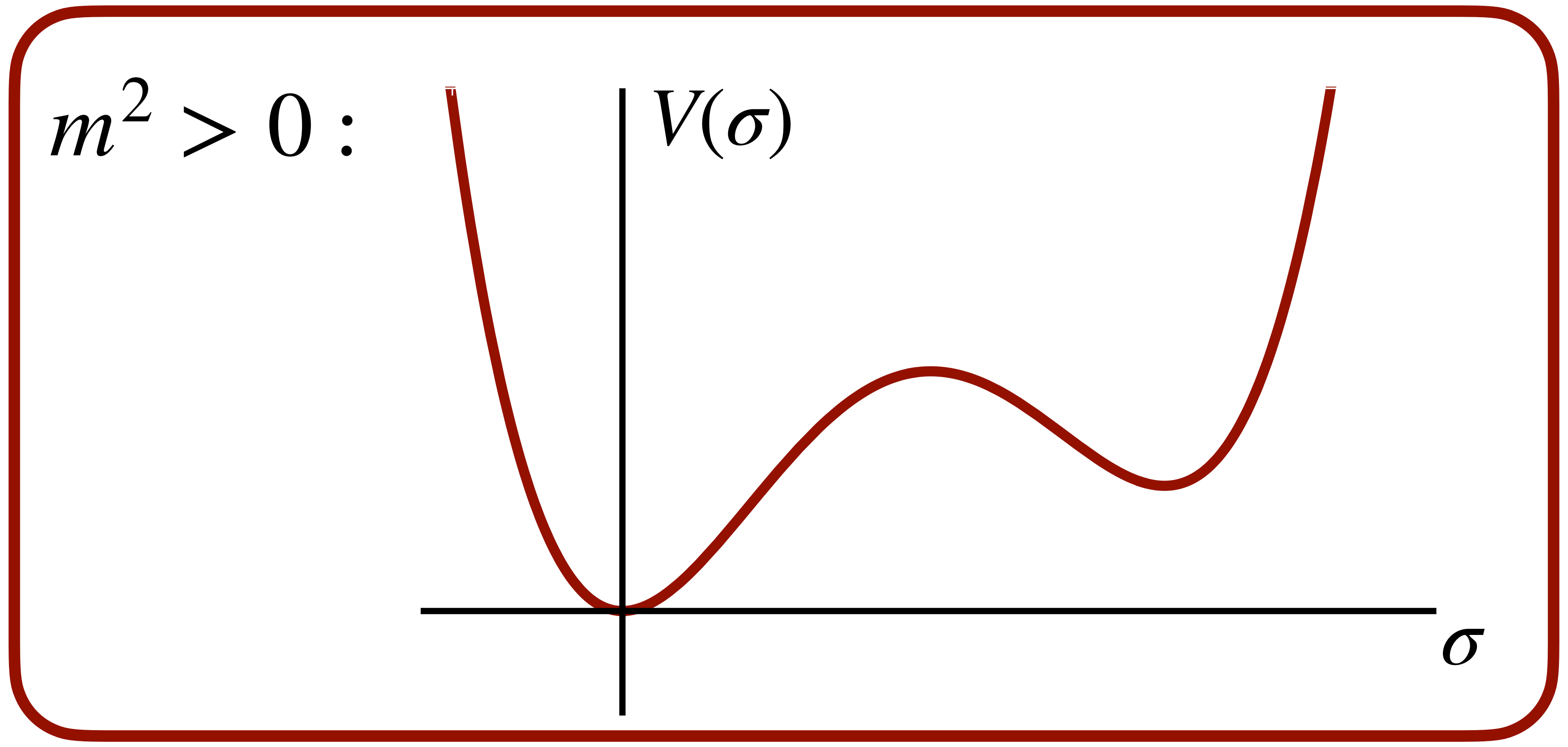
Universal effective Lagrangian can be constructed systematically from (chiral) invariants,

- $SU(N_f)_L \times SU(N_f)_R \times U(1)_A$ - invariants: $\mathrm{tr}\,(\Phi^\dagger \Phi)^{1,\ldots,N_f}$ ($\phi^{2,\ldots,2N_f}$ - terms)
- $SU(N_f)_L \times SU(N_f)_R$ - invariant: $\det \Phi$ ('t Hooft determinant, ϕ^{N_f} - term)

E.g., for $N_f = 3$ the $SU(3)_L \times SU(3)_R$ phase transition is described by

$$\mathscr{L}_{\mathrm{eff}} = \mathrm{tr}\left(\partial_\mu \Phi^\dagger\right)\left(\partial_\mu \Phi\right) + m^2\,\mathrm{tr}\,\Phi^\dagger\Phi - \xi_1^{(\mathrm{eff})}\left(\det\Phi + \det\Phi^\dagger\right) + \lambda_1\left(\mathrm{tr}\,\Phi^\dagger\Phi\right)^2 + \lambda_2\,\mathrm{tr}\left(\Phi^\dagger\Phi\right)^2 + \mathscr{O}\left(\phi^6\right)$$





→ 1st order transition seems inevitable for any $\xi_1^{(\mathrm{eff})} > 0$ (broken $U(1)_A$)

(consistent with apparent absence of stable fixed point for $SU(N_f)_L \times SU(N_f)_R$)

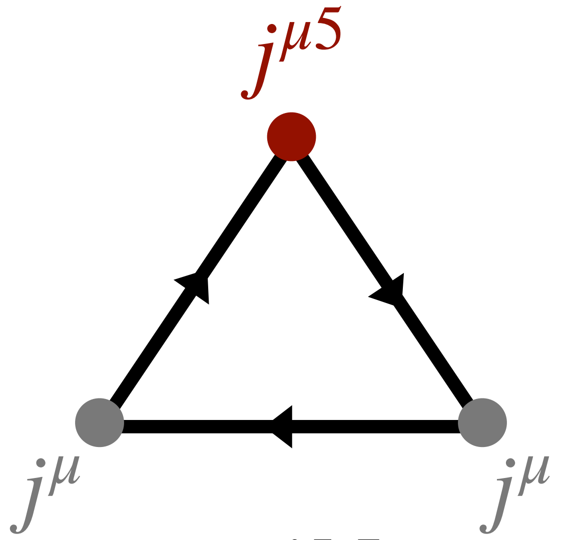
AXIAL ANOMALY

How could 1st-order transition be avoided?

Revisit microscopic origin of the anomaly, at weak coupling:

ABJ anomaly

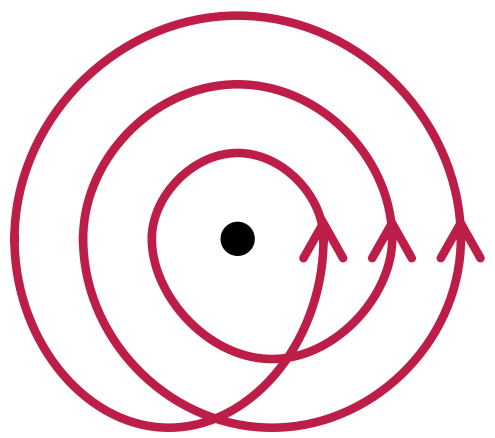
[Adler, Bell & Jackiw (1969)]



$$\partial_\mu j^{\mu 5} = \frac{iN_f}{8\pi^2}\,\mathrm{tr}\,F\tilde{F}$$

topological gluons

instantons [BPST (1975)]



$$Q = -\frac{1}{16\pi^2}\int d^4x\,\mathrm{tr}\,F\tilde{F}$$

axial charge changes + quark zero modes

[Atiyah, Singer (1963)]

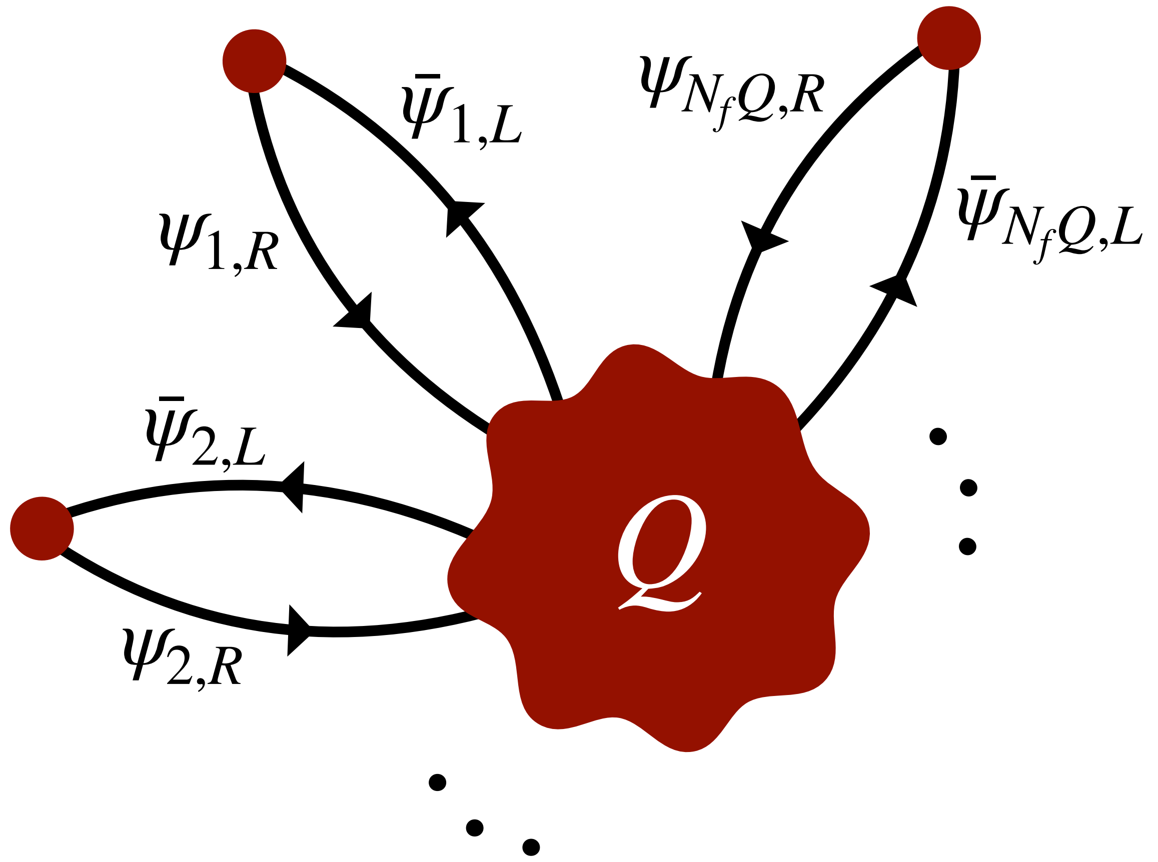
['t Hooft (1976)]

$$\Delta Q_5 = 2N_f\,Q$$

$$n_L - n_R = N_f\,Q$$

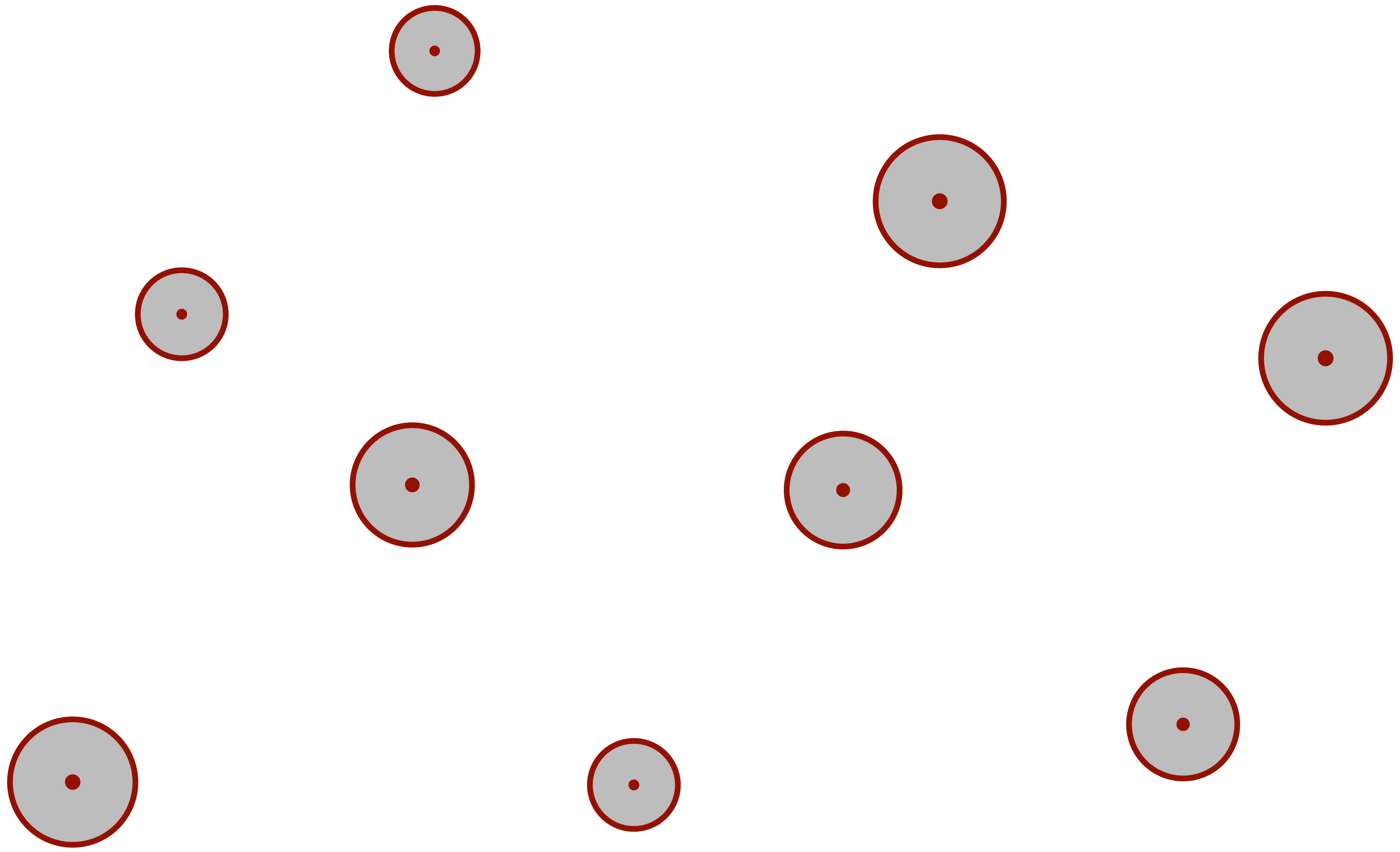
Functional determinant of quark zero modes accounts for change in axial charge:

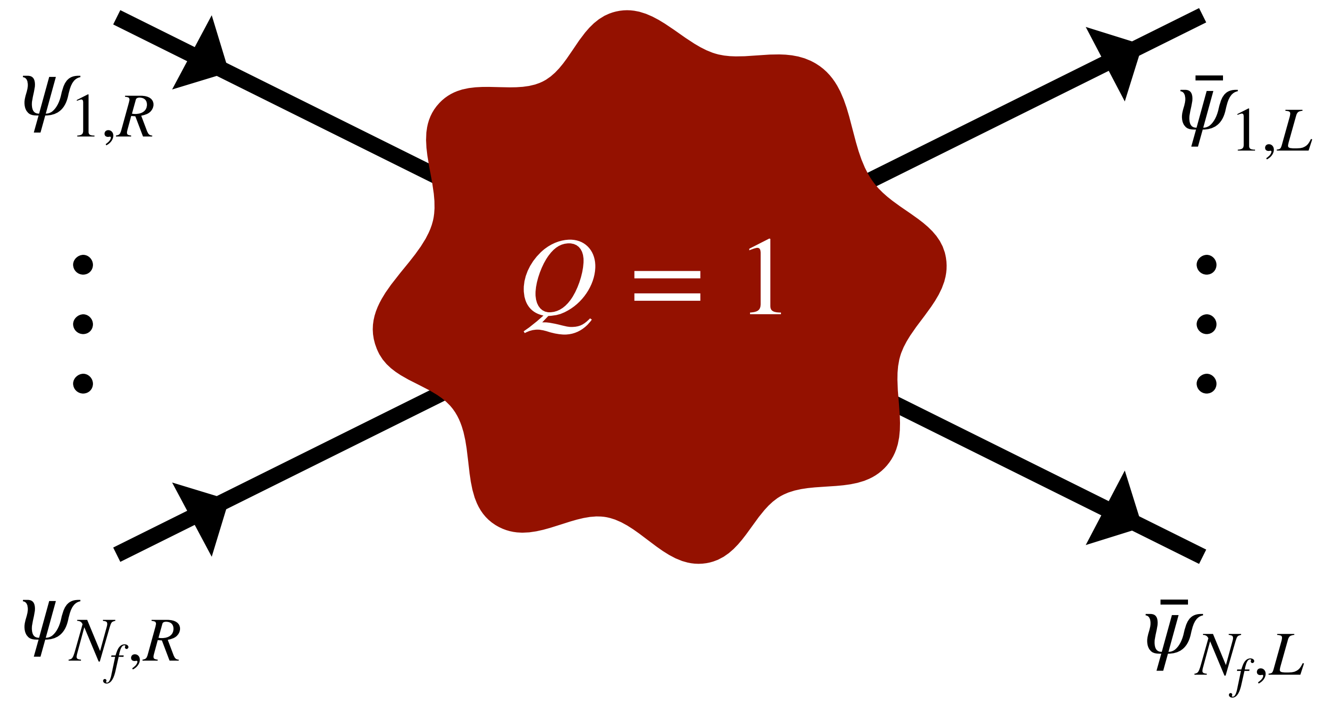
(otherwise partition function is zero)



ANOMALOUS CORRELATIONS

Dilute gas of $Q = \pm 1$ instantons: $U(1)_A$-breaking effective interaction ['t Hooft (1976)]





$$\det_f \bar{\psi}_L \psi_R + \det_f \bar{\psi}_R \psi_L \sim \det \Phi + \det \Phi^\dagger$$

→ anomalous $2N_f$ - quark (N_f - meson) correlation

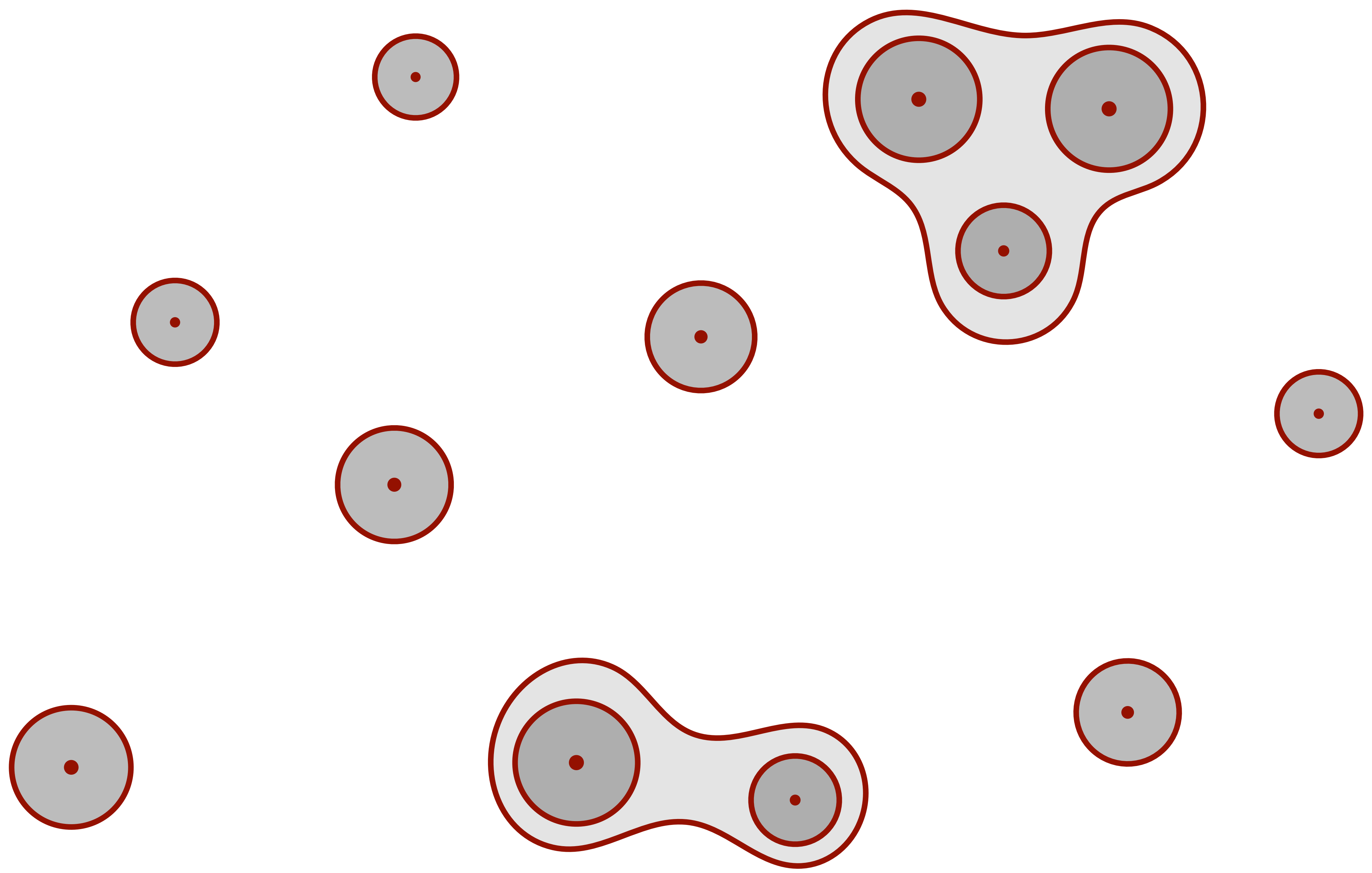
- dilute approximation reasonable at large T due to thermal screening of instantons: $\bar{\rho} \ll \frac{1}{\pi T}$ (average instanton size: $\bar{\rho}$) [Pisarski, Yaffe (1980)] [Gross, Pisarski, Yaffe (1981)]
- larger-Q instantons are suppressed in the semi-classical weak-coupling limit, $\sim \exp\left(-8\pi^2 |Q|/g^2\right)$

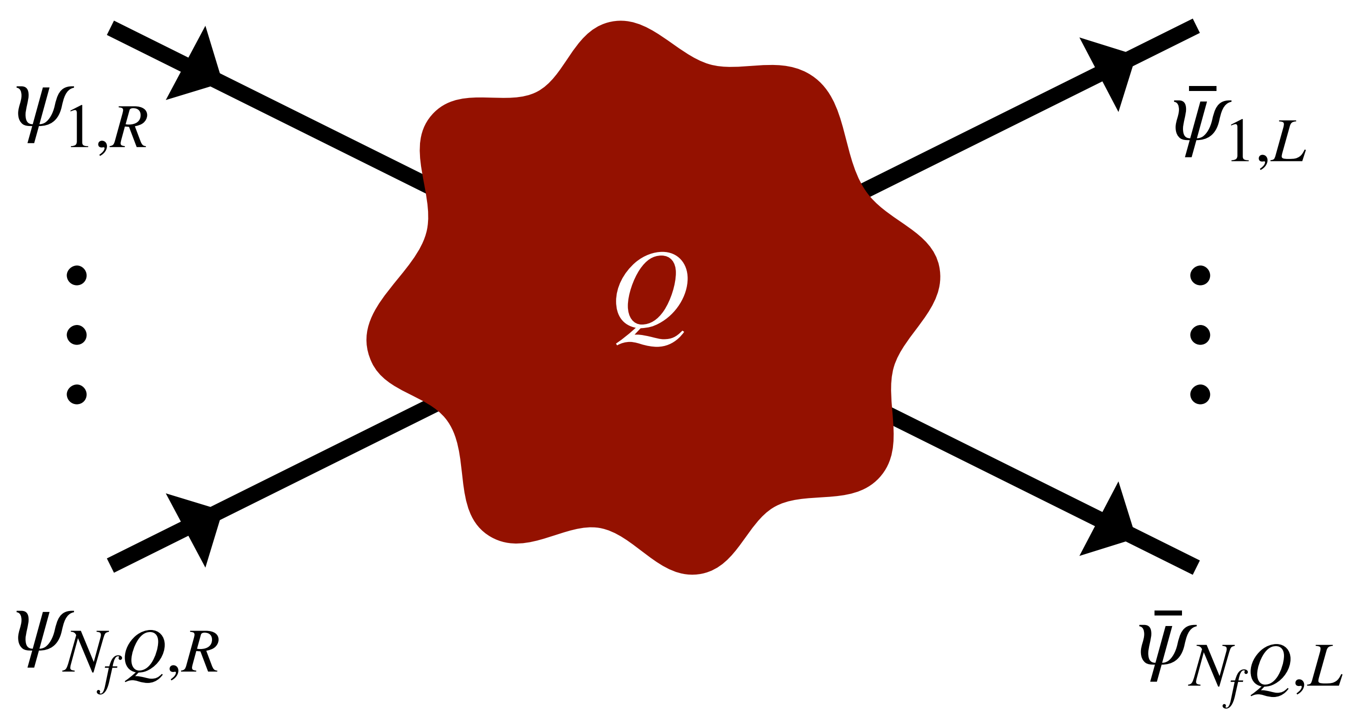
ANOMALOUS CORRELATIONS

Lower T/larger g: corrections to the dilute gas become relevant.

Dilute gas of instantons with multi-instanton $(|Q| > 1)$ corrections: more $U(1)_A$-breaking effective interactions

[Pisarski, FR, 1910.14052]
[FR, 2003.13876]





$$\left(\det_f \bar{\psi}_L \psi_R\right)^{|Q|} + \left(\det_f \bar{\psi}_R \psi_L\right)^{|Q|} \sim \left(\det \Phi\right)^{|Q|} + \left(\det \Phi^\dagger\right)^{|Q|}$$

→ anomalous $2N_f Q$ - quark ($N_f Q$ - meson) correlations

ANOMALOUS CORRELATIONS & THE CHIRAL TRANSITION

Topological quark zero modes give rise to tower of fundamental anomalous interactions,

$$\xi_1 \left[\det\Phi + \det\Phi^\dagger\right] + \xi_2 \left[(\det\Phi)^2 + (\det\Phi^\dagger)^2\right] + \xi_3 \left[(\det\Phi)^3 + (\det\Phi^\dagger)^3\right] + \ldots$$

They feed into anomalous effective couplings at $T \lesssim T_c$,

$$\mathscr{L}_{\text{anom}} = \xi_1^{(\text{eff})} \left[\det\Phi + \det\Phi^\dagger\right] + \xi_2^{(\text{eff})} \left[(\det\Phi)^2 + (\det\Phi^\dagger)^2\right] + \xi_3^{(\text{eff})} \left[(\det\Phi)^3 + (\det\Phi^\dagger)^3\right] + \ldots$$

$$\xi_n^{(\text{eff})} = \xi_n + \sigma_0^{N_f}\,\xi_{n+1} + \sigma_0^{2N_f}\,\xi_{n+2} + \ldots$$

$\sigma_0 \sim \langle \bar{q} q \rangle$: chiral condensate

Explain small/no 1st-order region in lower-left corner of the Columbia plot by:

conjecture:

higher Q effects dominate for $T \lesssim T_c$, e.g., $\langle Q \geq 0 \rangle_{T \lesssim T_c} > 1$

[Pisarski, FR, 2401.06130]

⟶ 2nd order transition in the chiral limit: $\xi_1(T = T_c) = 0 \;\Rightarrow\; \xi_1^{(\text{eff})}(T = T_c) = 0$

EXTENDED LINEAR SIGMA MODEL

[Giacosa, G. Kovacs, P. Kovacs, Pisarski, FR, 2410.08185]

Test consistency with vacuum phenomenology using a low-energy model in mean-field approximation

$$\mathscr{L} = \mathscr{L}_{U(3)\times U(3)} + \mathscr{L}_{\text{anom}}$$

$\mathscr{L}_{U(3)\times U(3)} =$
- LSM containing all possible relevant and marginal terms in $d = 4$ involving mesons in the scalar, pseudoscalar, vector and axialvector nonets
- coupled to quarks, A_0 background field + Polyakov loop potential (lattice input)

$$\mathscr{L}_{\text{anom}} = \xi_1^{(\text{eff})} \left(\det\Phi + \det\Phi^\dagger \right) + \xi_1^{1,(\text{eff})} \operatorname{tr}(\Phi^\dagger\Phi)\left(\det\Phi + \det\Phi^\dagger \right) + \xi_2^{(\text{eff})} \left[(\det\Phi)^2 + (\det\Phi^\dagger)^2 \right]$$

- χ^2 fit to 29 physical quantities (meson masses, decay constants, decay widths, T_c at the physical point; 14-16 free parameters)
- test viability of different realizations of the anomaly (average over steepest descent minimizations from 10^6 random starting points)

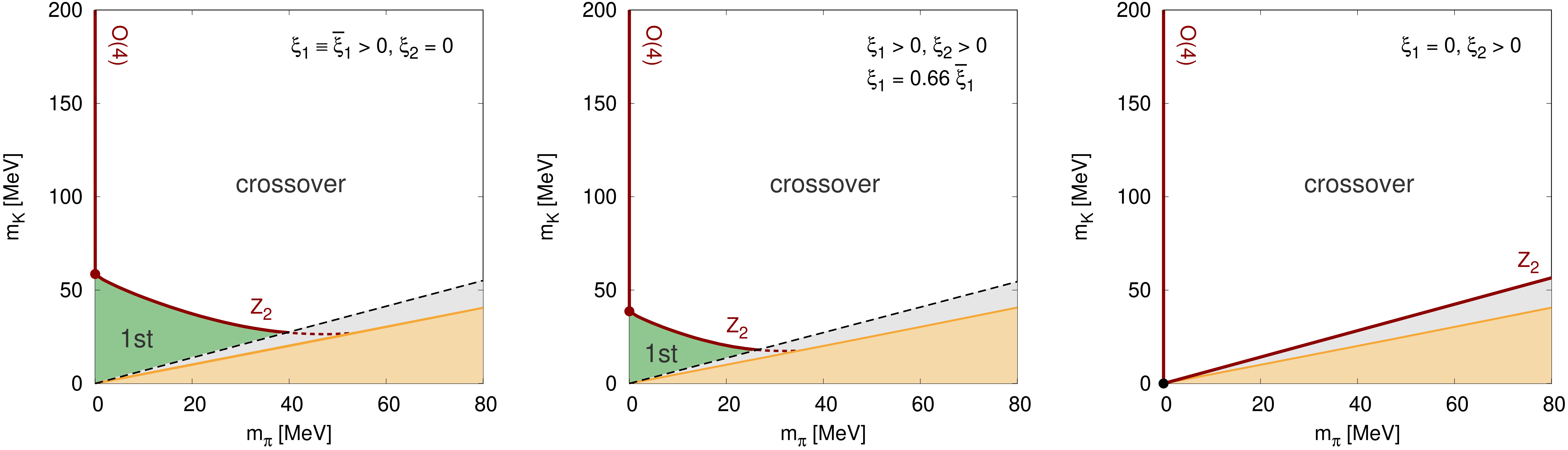
→ $\xi_1^{(\text{eff})} = 0$ consistent with vacuum phenomenology; in fact, any realization of the anomaly is

nonzero params.	$\bar{\chi}^2$	$\bar{\chi}^2_{red}$
ξ_1	31.31	2.09
ξ_1^1	29.70	1.98
ξ_2	33.50	2.23
$\xi_1,\ \xi_1^1$	29.51	2.11
$\xi_1,\ \xi_2$	30.90	2.21
$\xi_1,\ \xi_1^1,\ \xi_2$	30.81	2.37

EXTENDED LINEAR SIGMA MODEL

[Giacosa, G. Kovacs, P. Kovacs, Pisarski, FR, 2410.08185]

Once good parameters for vacuum phenomena are identified, we compute the Columbia plot:



→ **the smaller $\xi_1^{(\text{eff})}$, the smaller the first order region; vanishes for $\xi_1^{(\text{eff})} = 0$**

- in all these cases $U(1)_A$ remains broken, $\xi_n^{(\text{eff})} = \text{const}\,.$
- gray region can only be reached for $m_s < 0$, so like $\theta = \pi$: we find spontaneous CP violation (η' condensate)
  → Dashen's phenomenon [Dashen (1971), Witten (1980)] (to be investigated further)

Note: **mean-field analysis cannot be the final answer**

SUMMARY/OUTLOOK

The order of the chiral phase transition remains an open question.
It is related to various subtle phenomena.

- no evidence for 1st order transition from the lattice: either small region or not there at all
- critical phenomena suggest that $U(1)_A$ needs to be restored at T_c for a second order transition in the chiral limit, but existing calculations may still have large systematic errors
- axial anomaly is encoded in a tower of anomalous correlations, are directly linked to higher topological charges in the semi-classical limit
- **conjecture:** small/no 1st order region because of **dominance of higher topological charge effects at $T \lesssim T_c$**
- 2nd order transition for vanishing $Q = 1$ contribution, $\xi_1 = 0$. But why would only one coupling vanish? Maybe all of them do, $\xi_n = 0\ \forall n \rightarrow U(1)_A$ **restoration right at T_c**
- how to test this? One suggestion: **look for actual phase transition in $N_f = 1$ QCD**

BACKUP

BEYOND MEAN-FIELD

[Resch, FR, Schaefer, 1712.07961]

Columbia plot of quark-meson model - mean-field vs FRG-LPA

~~$U(1)_A$~~ | $U(1)_A$

mean-field

FRG