Universal critical dynamics in QCD

Johannes Roth

Institut für Theoretische Physik, Justus-Liebig-University Gießen

Based on
JR, Ye, Schlichting, von Smekal, JHEP **01**, 118 (2025)
JR, Ye, Schlichting, von Smekal, arXiv:2409.14470

Confinement and symmetry from vacuum to QCD phase diagram,
Benasque Science Center, Spain, 12 February 2025

Motivation

QCD phase diagram

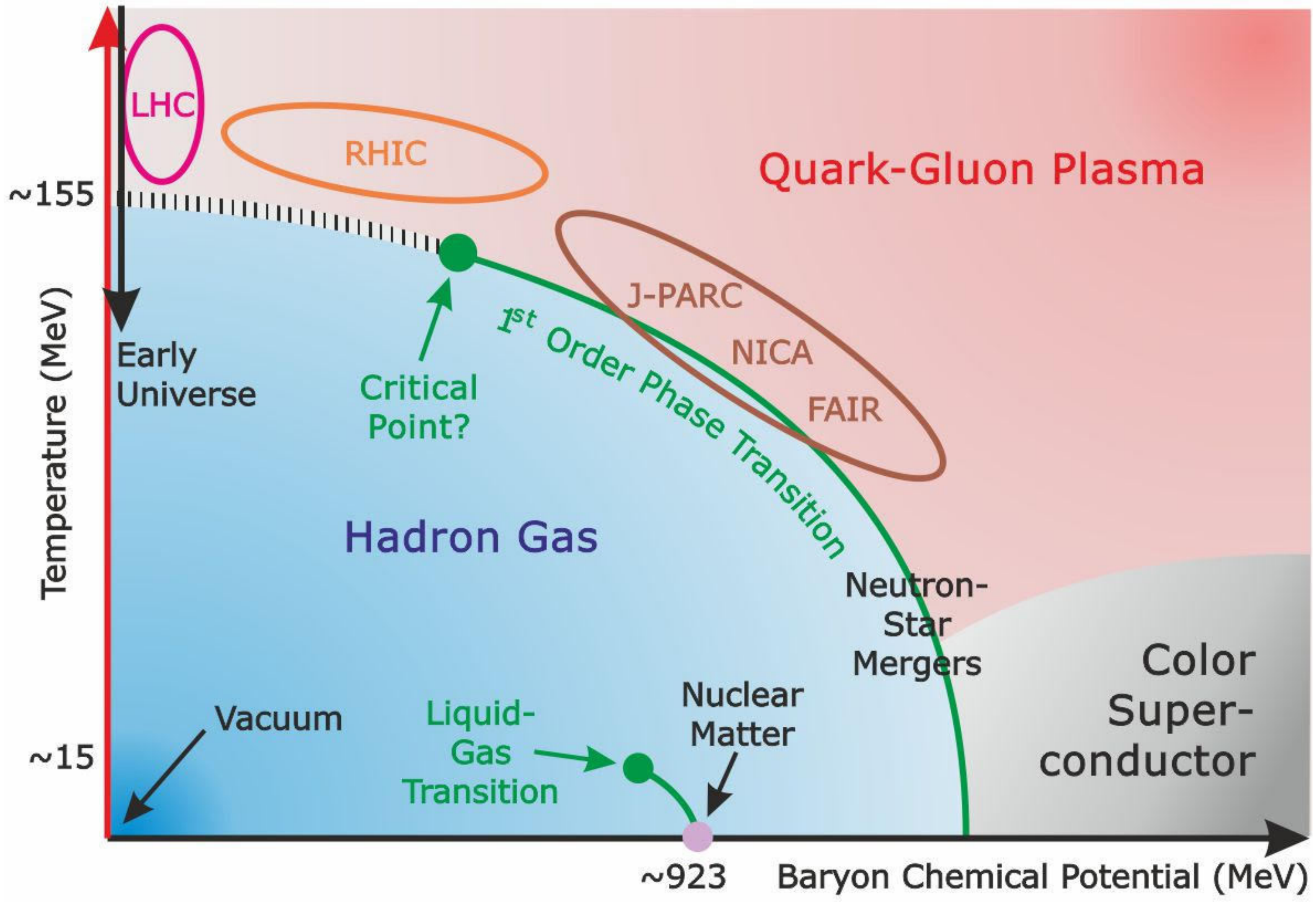

Figure: A. Steidl, CRC-TR 211

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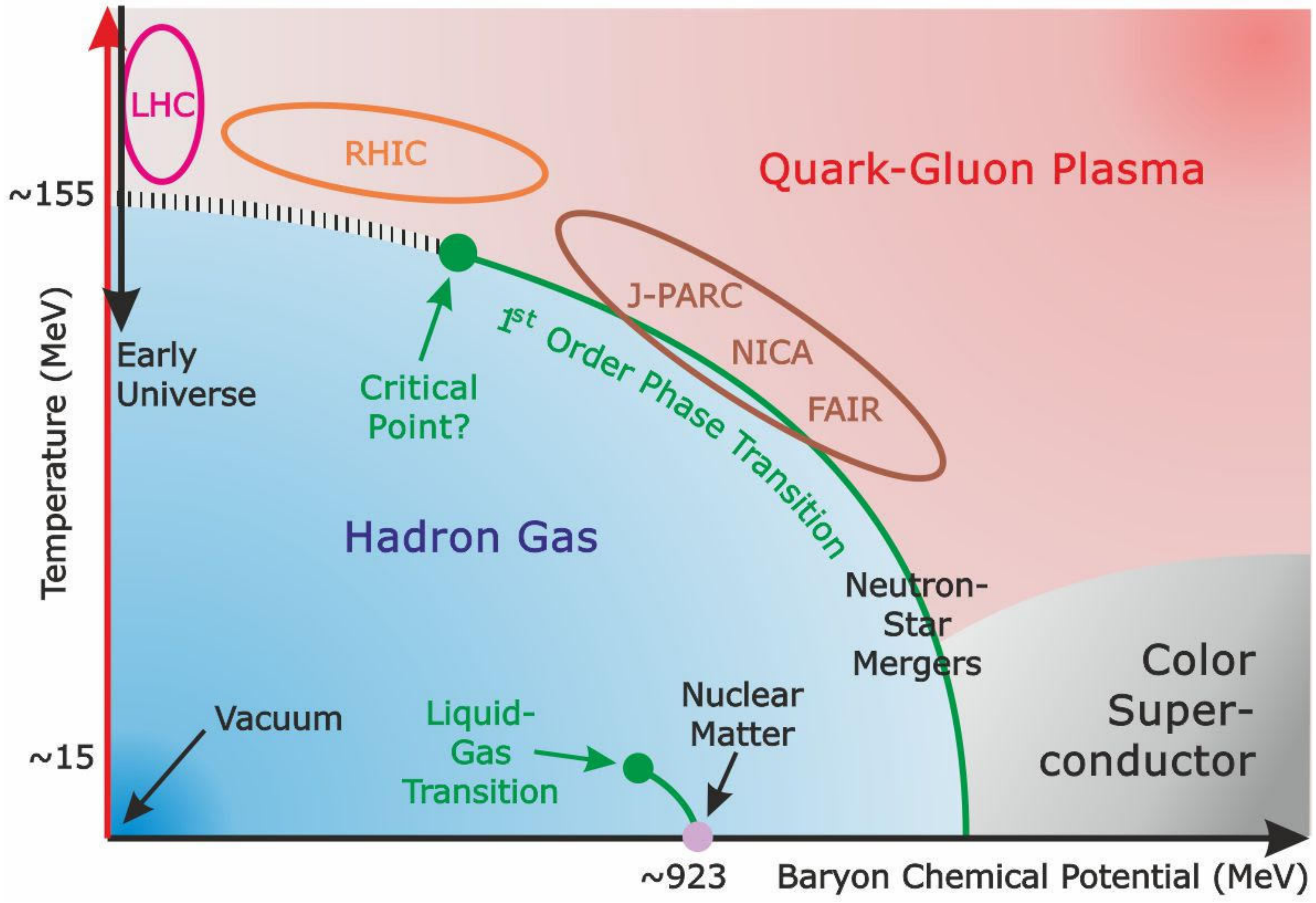

Figure: A. Steidl, CRC-TR 211

- Long-term goal: find **critical point**

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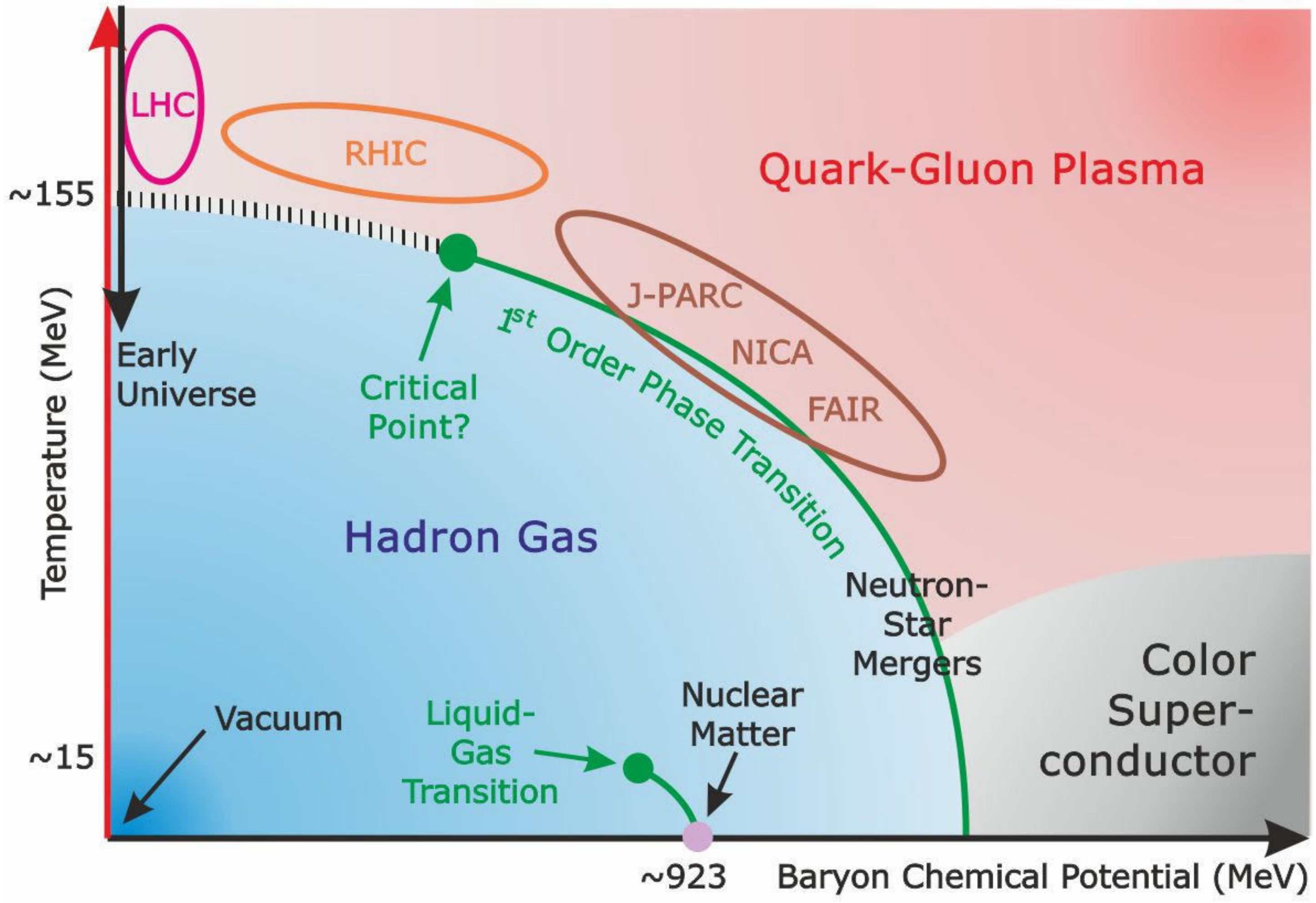

Figure: A. Steidl, CRC-TR 211

- Long-term goal: find **critical point**
- Possible signature in heavy-ion collisions: **critical fluctuations**

Stephanov, Rajagopal, Shuryak, PRD **60** (1999) 114028

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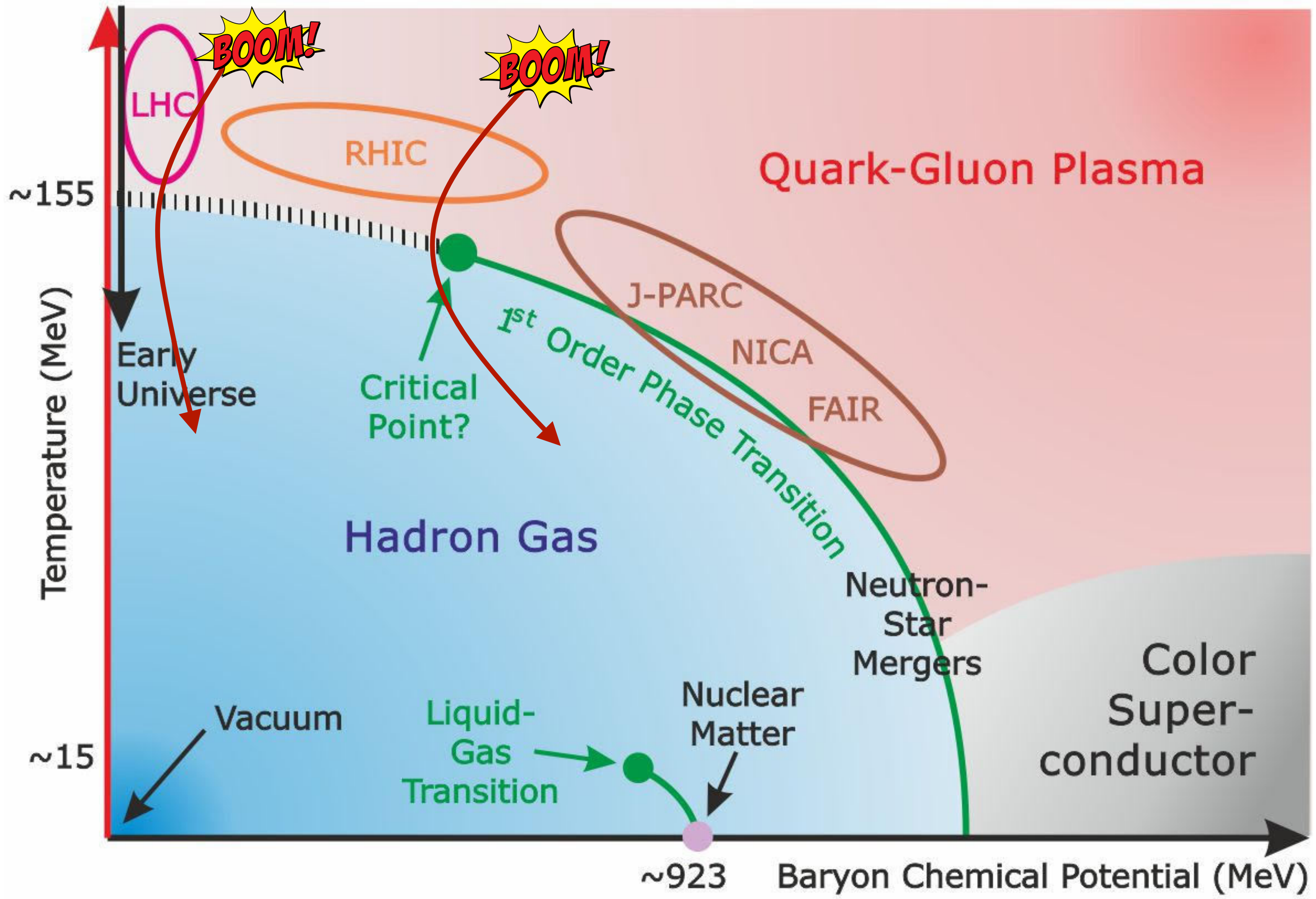

Figure: A. Steidl, CRC-TR 211

- Long-term goal: find **critical point**
- Possible signature in heavy-ion collisions: **critical fluctuations**

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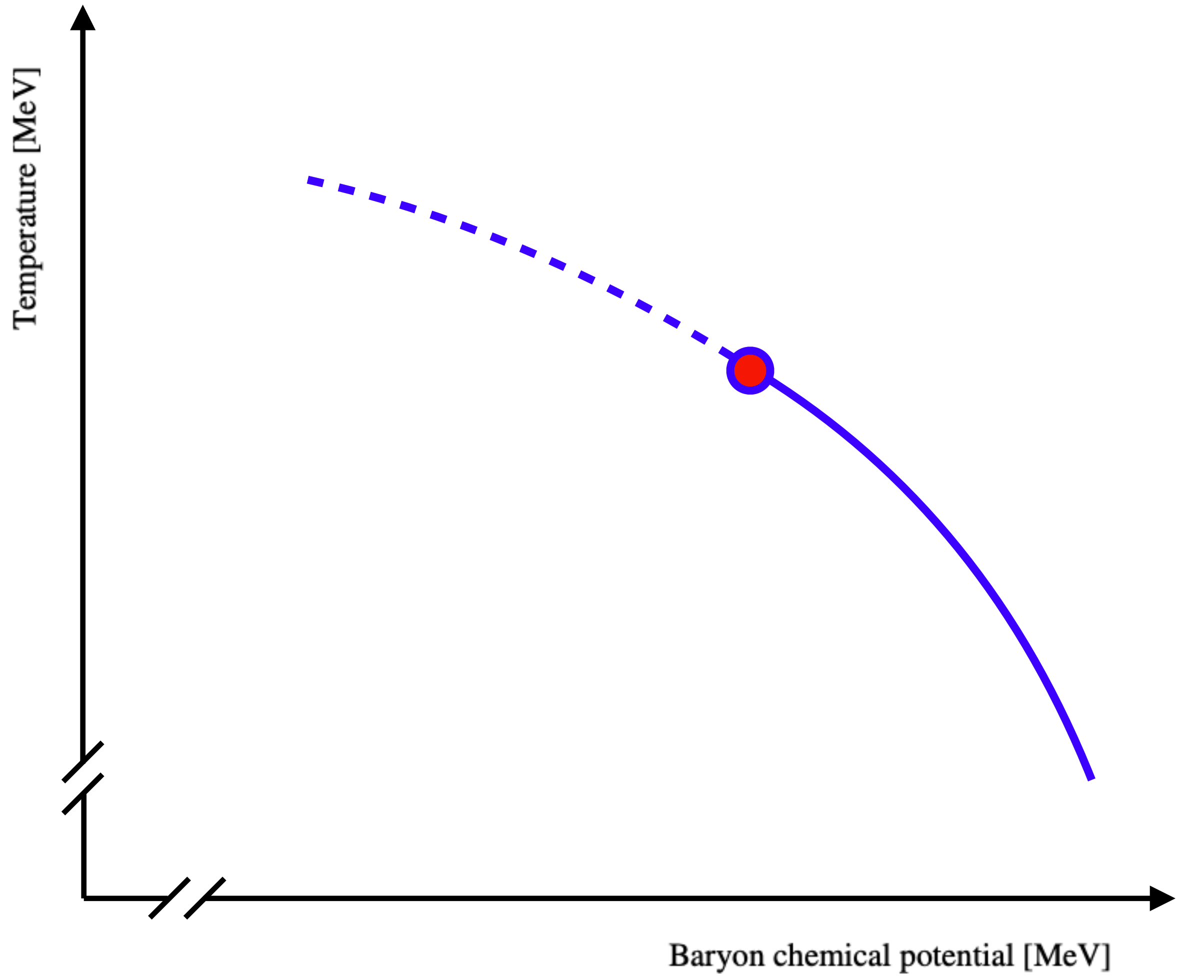
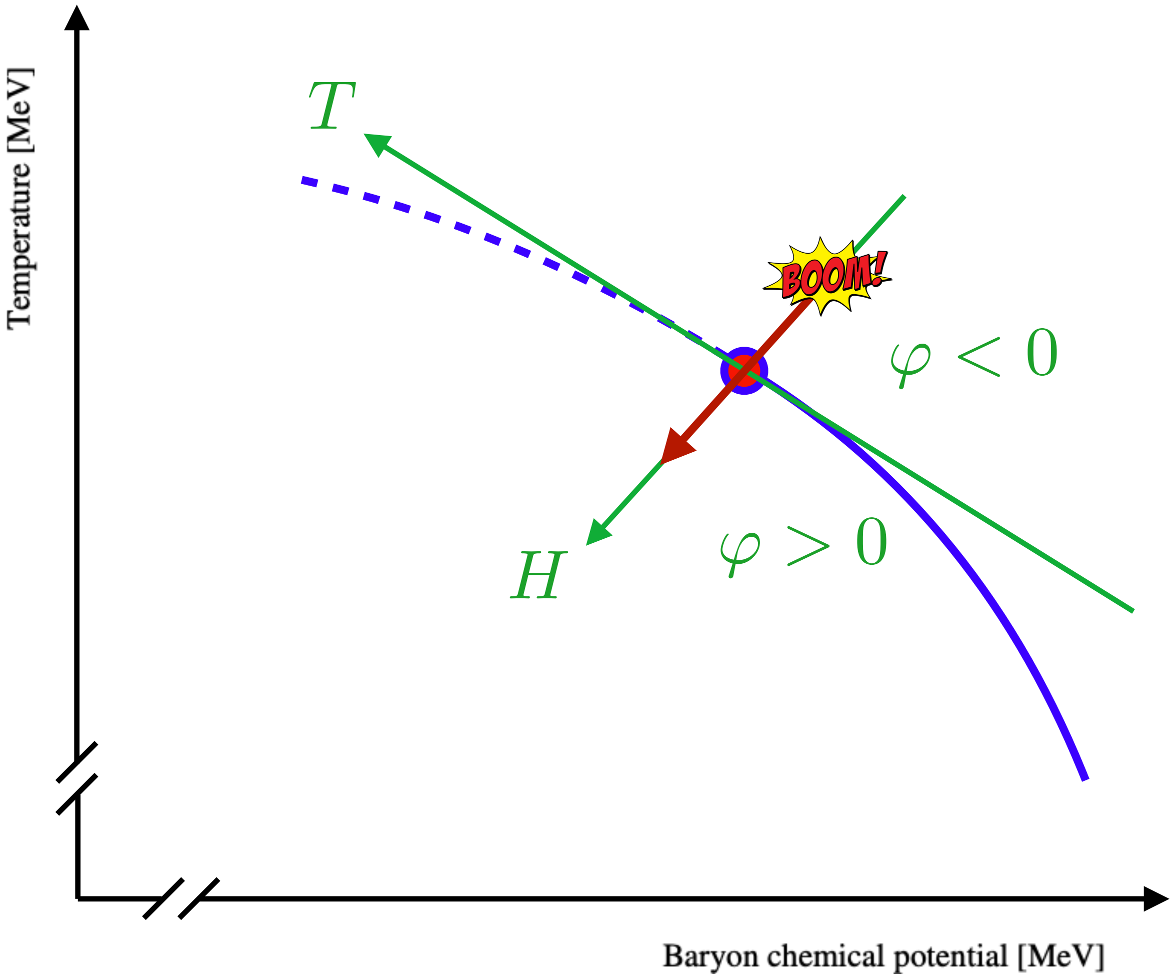
- Fireball is rapidly evolving ⇝ need to understand **time evolution** of critical fluctuations

Dynamics near critical point



Dynamics near critical point

map to **Ising model,**
study simplified trajectory:



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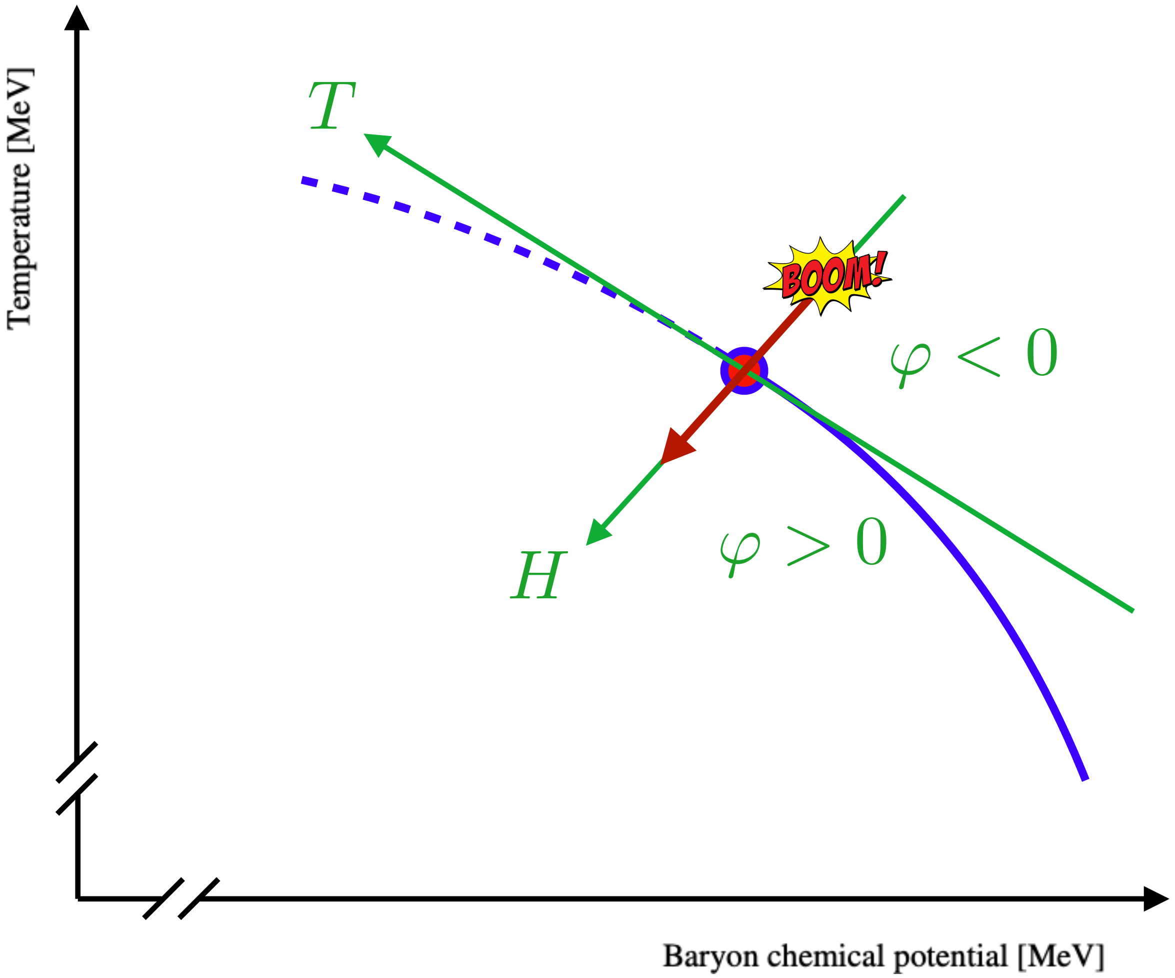

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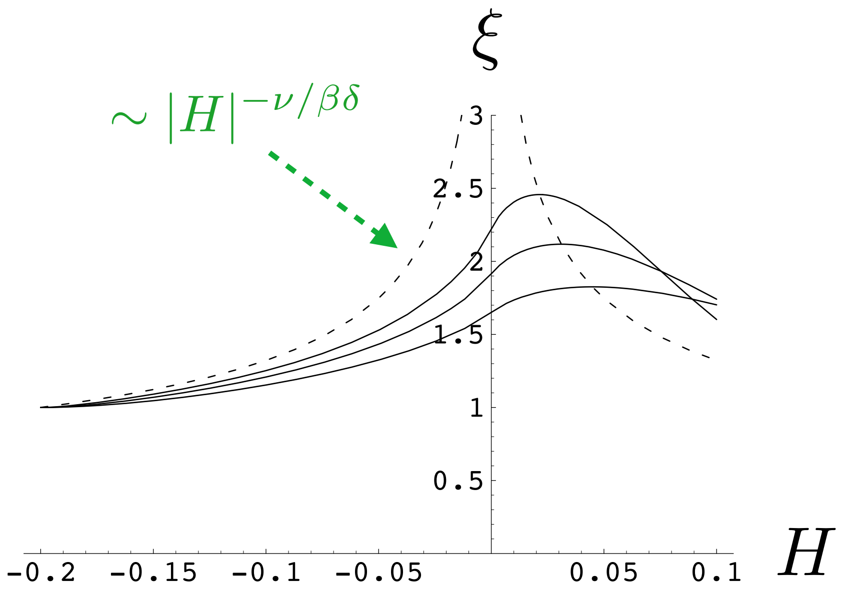

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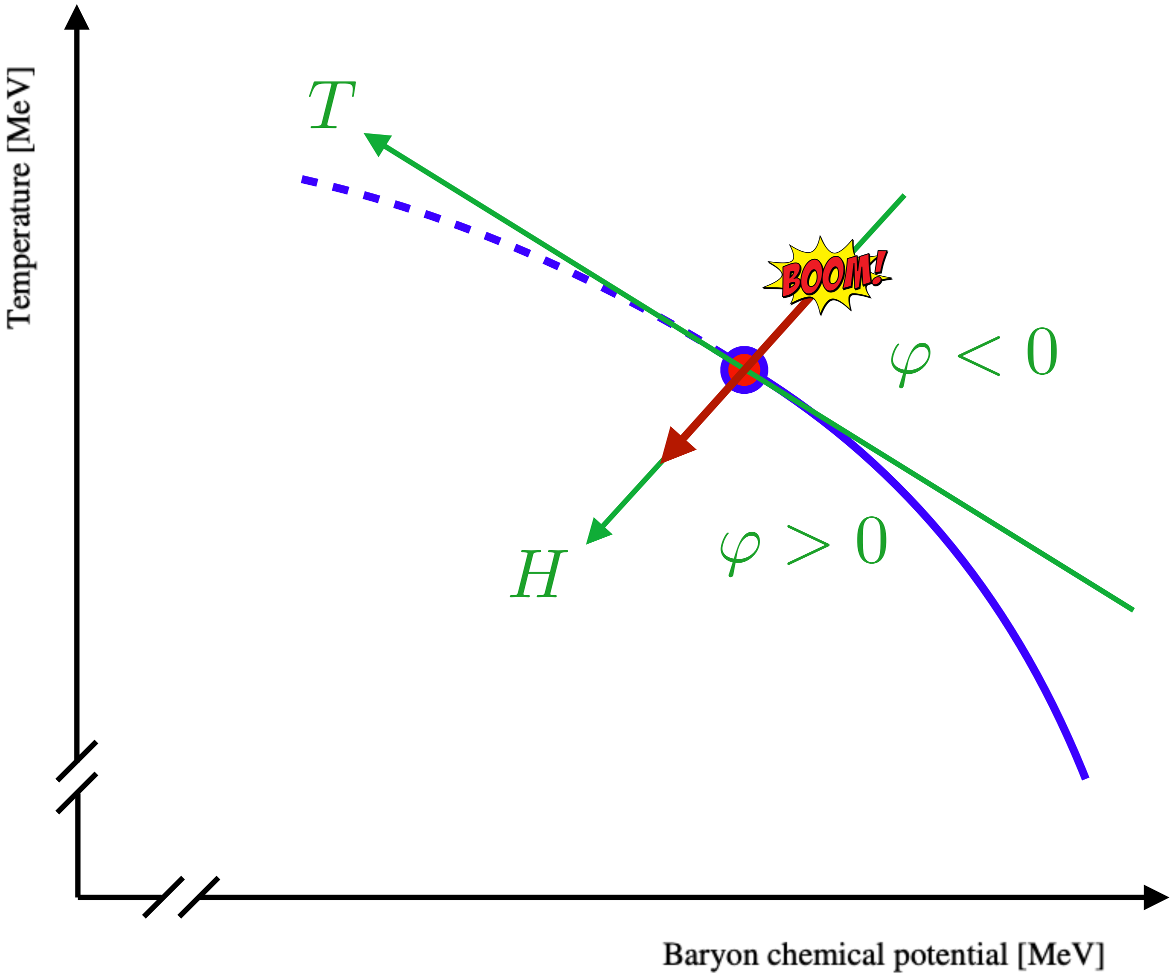

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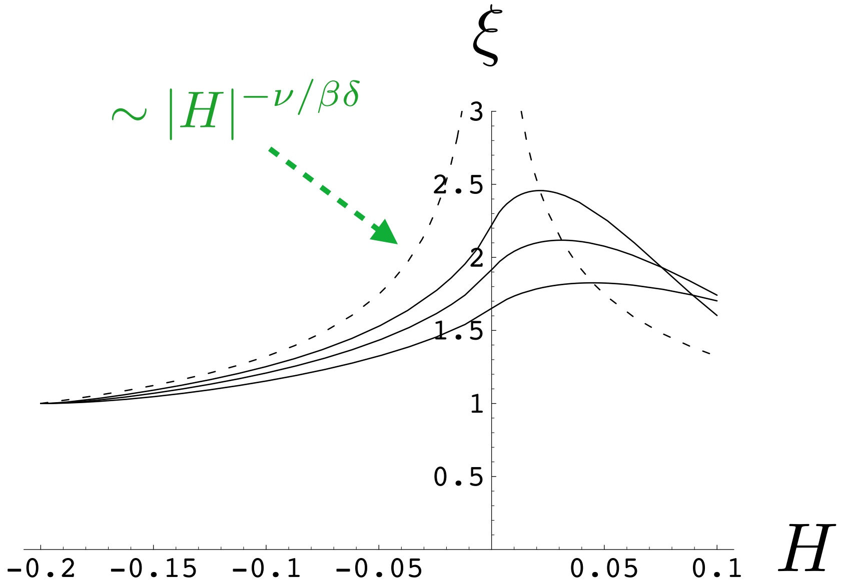

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⇝ **Critical mode falls out of (local) equilibrium!**

Critical dynamics

Reason:

critical slowing down

$$\xi_t \sim \xi^z$$

correction time

correlation length

Critical dynamics

Reason:

critical slowing down

dynamic critical exponent

$$\xi_t \sim \xi^{z}$$

correlation time

correlation length

Critical dynamics

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$$\xi_t \sim \xi^{z}$$

correlation time

correlation length

- z determined by **dynamic universality class**
- group theories by equations of motion for critical modes
- critical mode arbitrarily slow ⇝ **hydrodynamic theory**
  (form depends on: order parameter conserved or not, which other quantities are conserved, etc.)

Dynamic universality classes

Static universality classes split up into **dynamic** universality classes:

Hohenberg & Halperin, Rev. Mod. Phys. **49**, 435 (1977)

$\boldsymbol{Z_2}$

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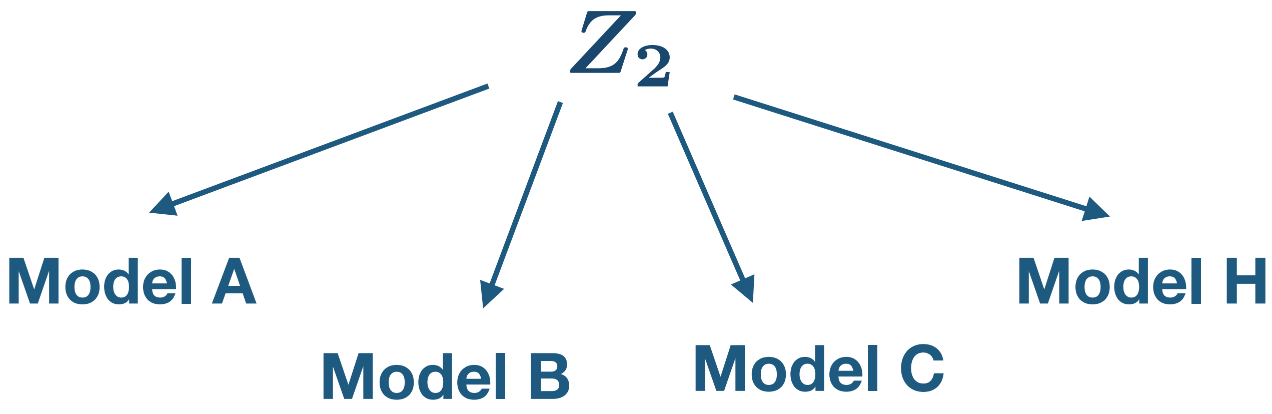

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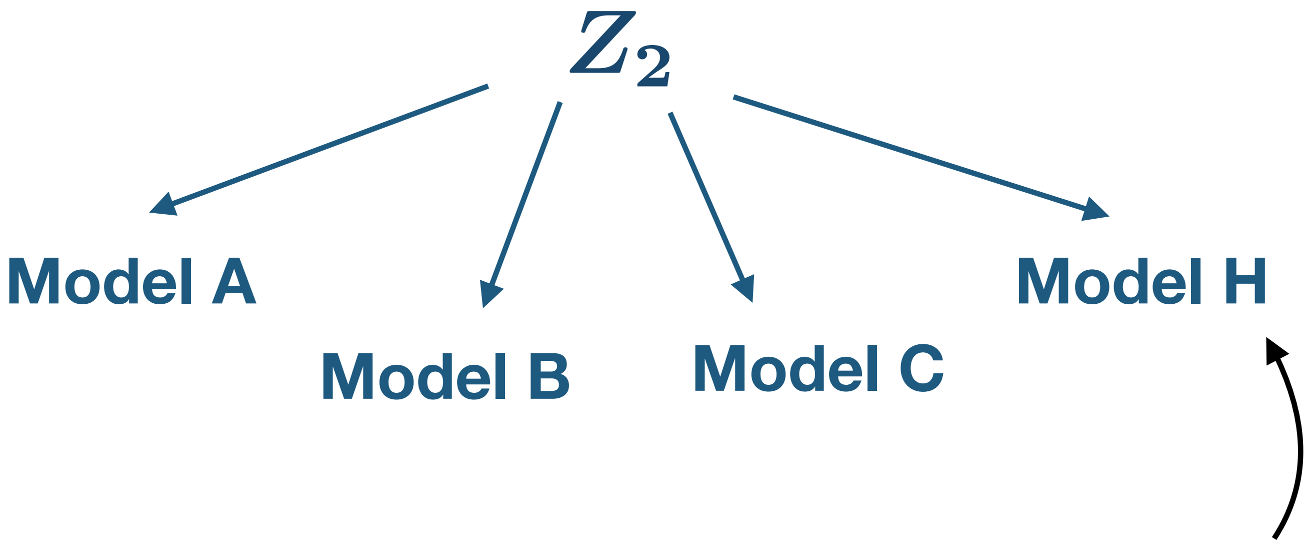

Dynamic universality class of QCD’s critical point

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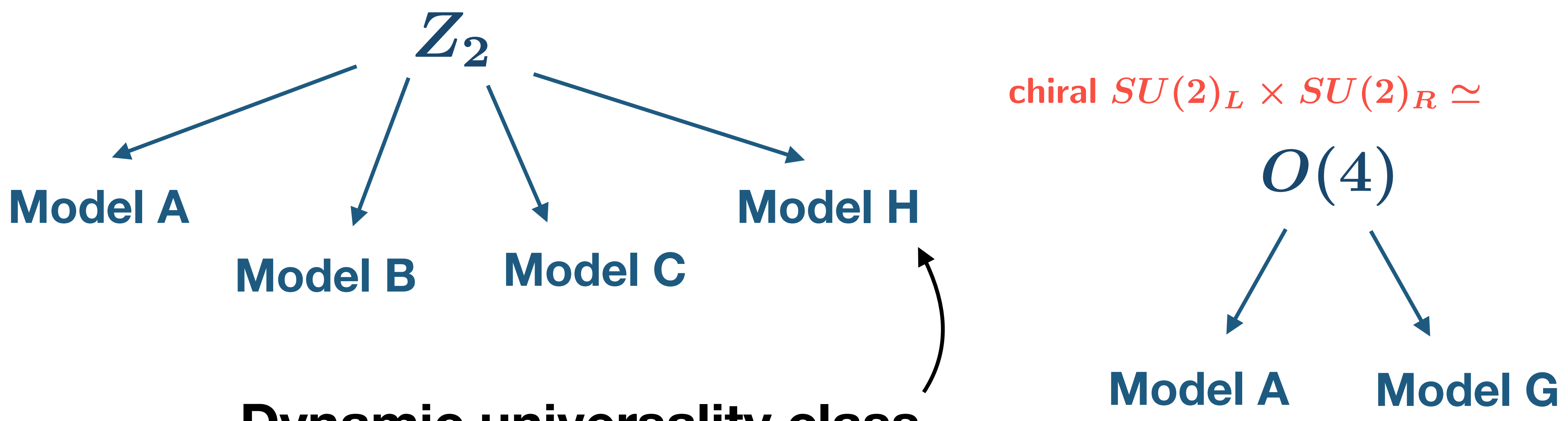

$\mathbf{Z_2}$

Model A

Model B

Model C

Model H

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chiral $SU(2)_L \times SU(2)_R \simeq$

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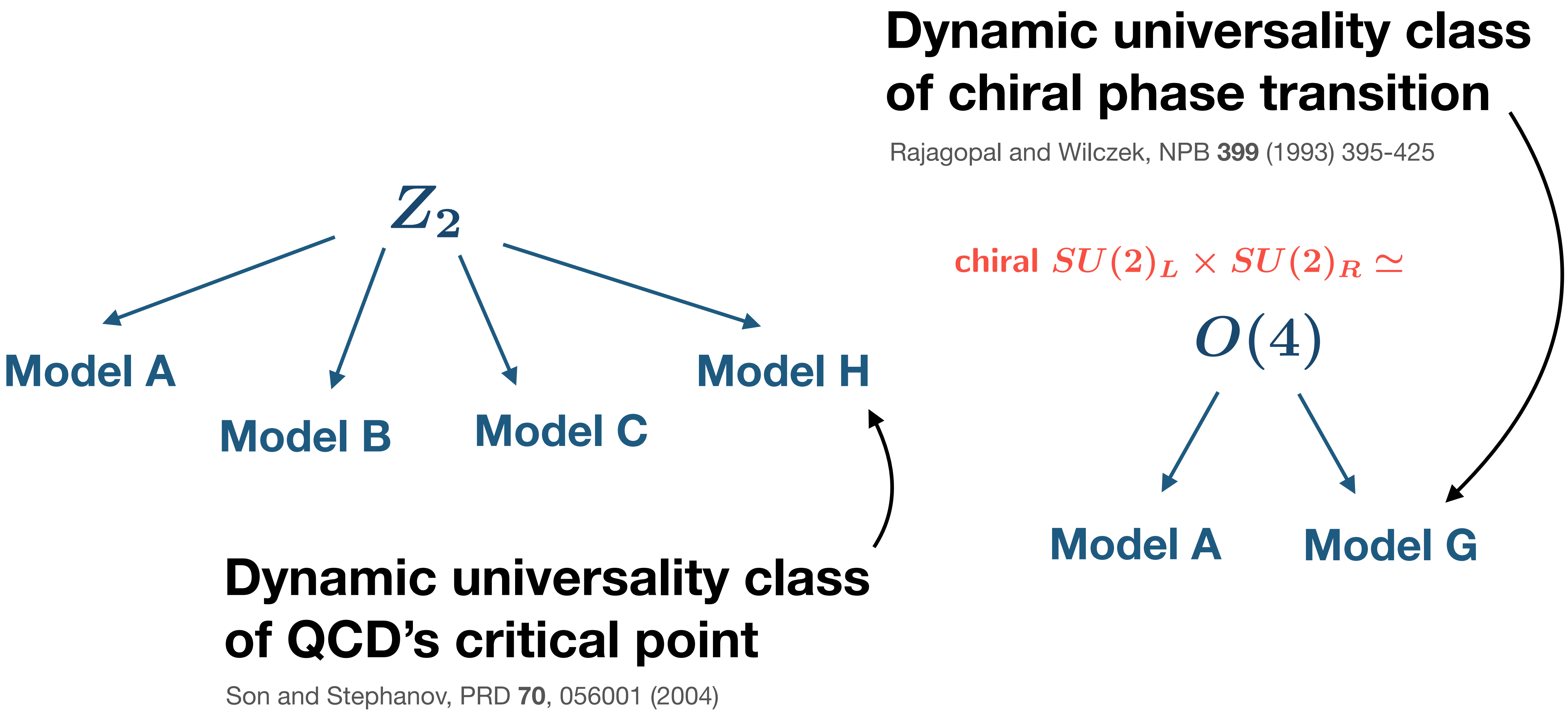

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Dynamic universality class of chiral phase transition

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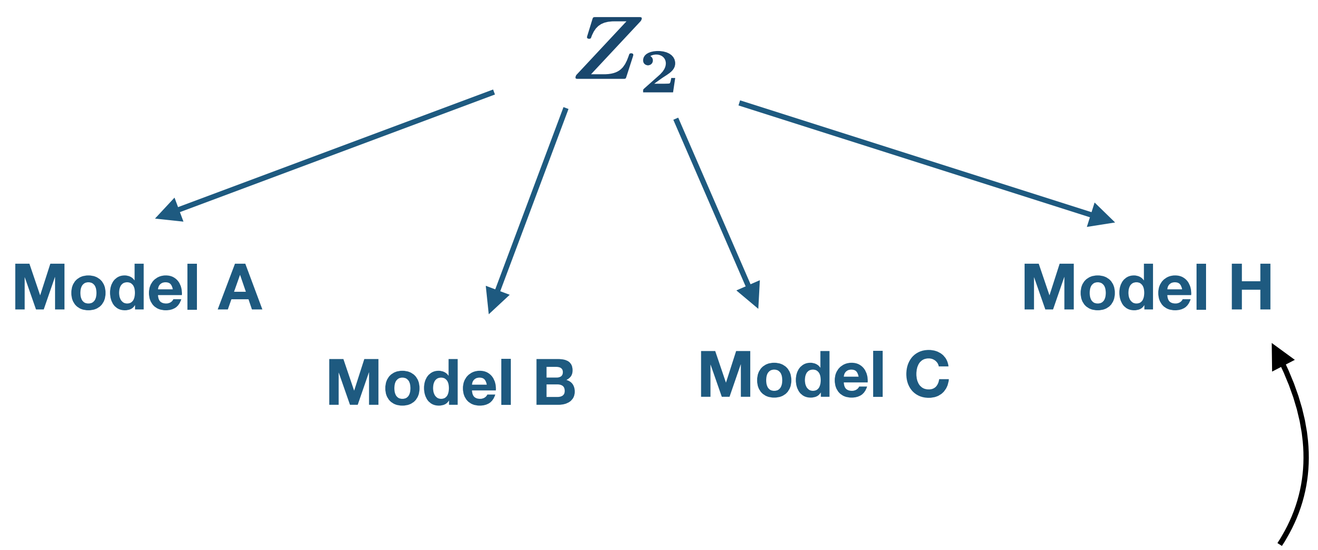

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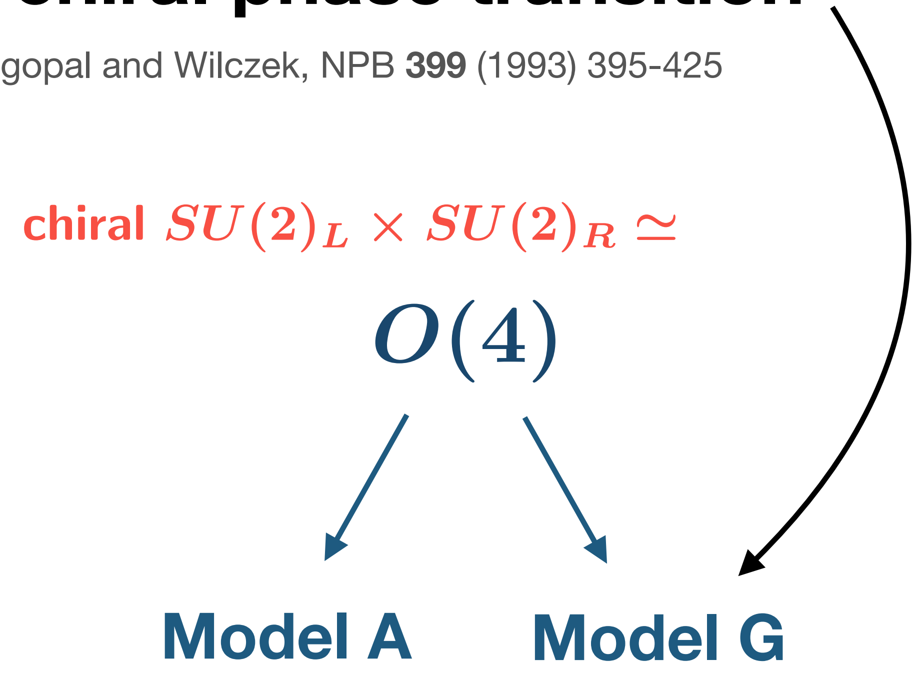

This work: predict **dynamic universal properties** of hot and dense QCD matter by studying simpler system from same dynamic universality class

Outline for remaining talk

1. Dynamic universality classes:
   QCD’s critical point, chiral phase transition
2. Functional renormalization group (FRG) flow
   for systems with reversible mode couplings
3. Results for fixed points & dynamic critical exponents

Dynamic universality class of critical point

Order parameter: $\phi \sim \delta(s/n)$ (entropy per baryon)

Son and Stephanov,
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FRG:
Chen, Tan, Fu, arXiv:2406.00679
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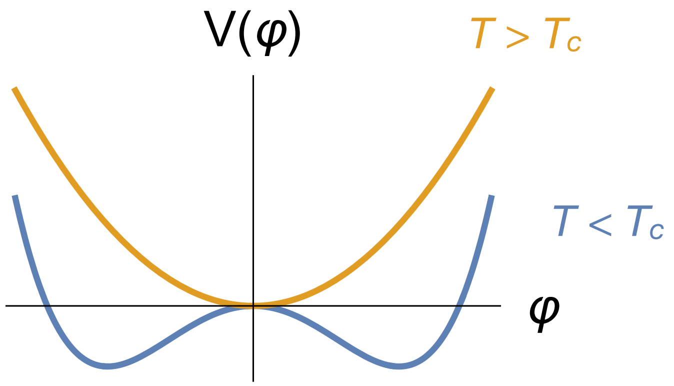
$$F = \int d^d x \left\{ \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{24}\phi^4 + \frac{\vec{j}^2}{2\rho} \right\}$$



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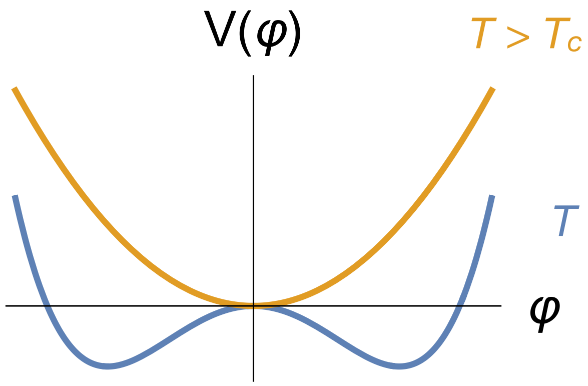

Equations of motion:

$$\frac{\partial\phi}{\partial t} = \sigma\vec{\nabla}^2\frac{\delta F}{\delta\phi} + \theta + g\{\phi,\vec{j}\}\cdot\frac{\delta F}{\delta\vec{j}}$$

order parameter

(transverse)
momentum density

$$\frac{\partial j_l}{\partial t} = \mathcal{T}\left[\eta\vec{\nabla}^2\frac{\delta F}{\delta j_l} + \xi_l + g\{j_l,\phi\}\frac{\delta F}{\delta\phi} + g\{j_l,j_m\}\frac{\delta F}{\delta j_m}\right]$$

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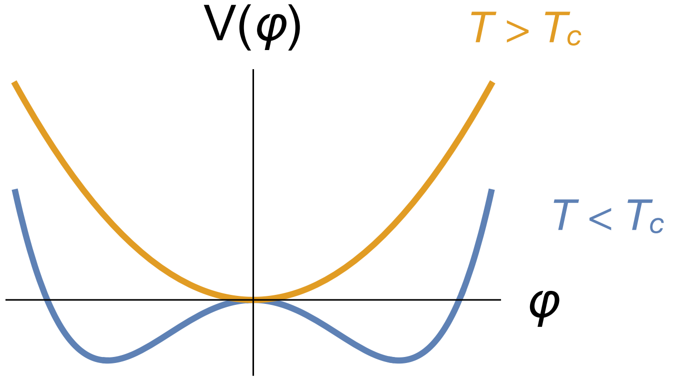

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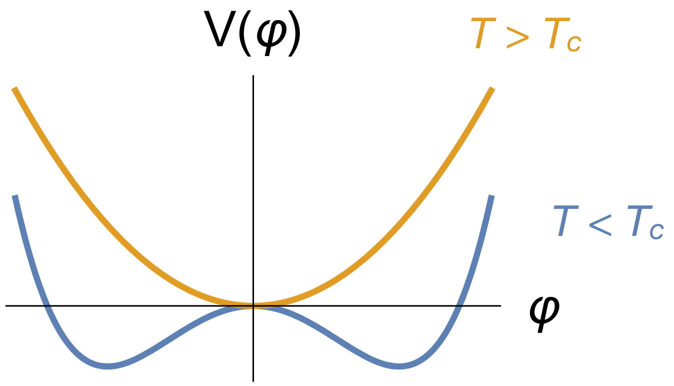

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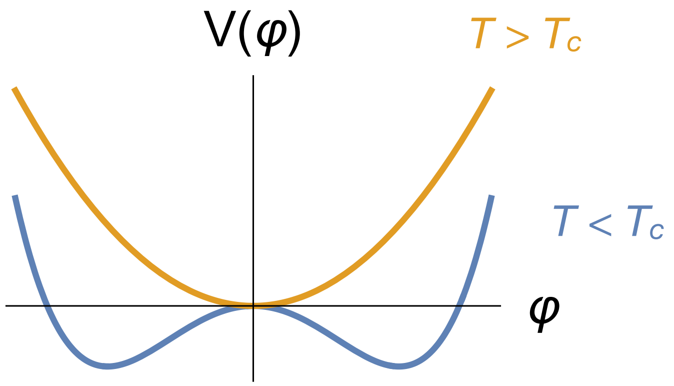

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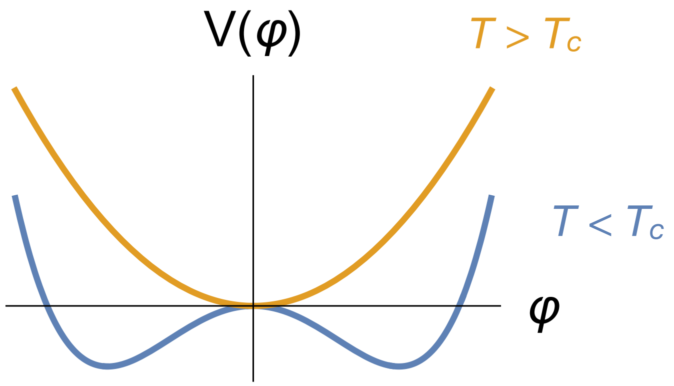

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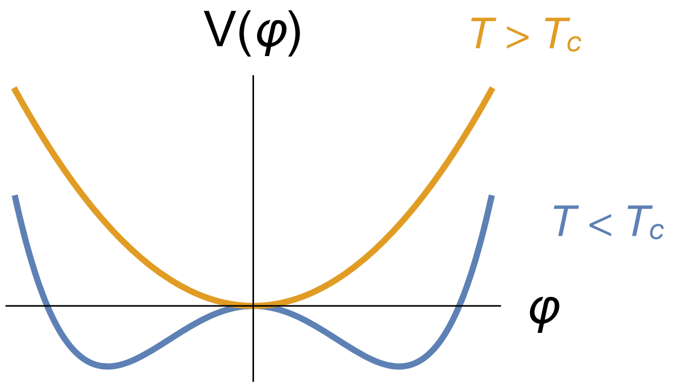

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diffusion noise reversibility

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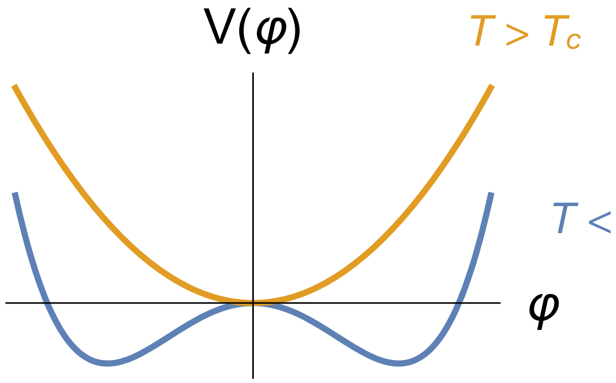

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diffusion noise reversibility convection

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CRC-TR 211

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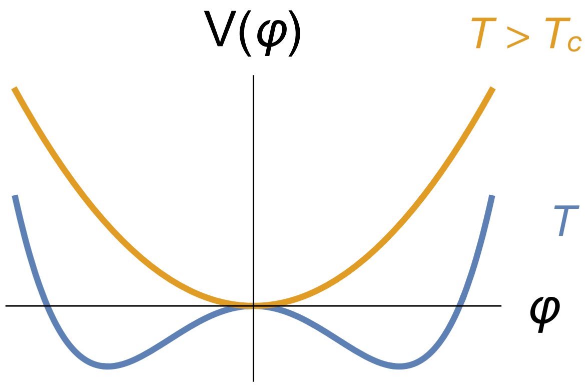

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Model H

$z = 4 - \eta_\perp - x_\sigma$

Liquid-gas critical point in pure fluid

Dynamic universality class of chiral phase transition

Chiral order parameter: $\phi = (\sigma, \vec{\pi})$

Rajagopal and Wilczek,
NPB **399** (1993) 395-425

FRG:
JR, Ye, Schlichting, von Smekal, JHEP **01**, 118 (2025)

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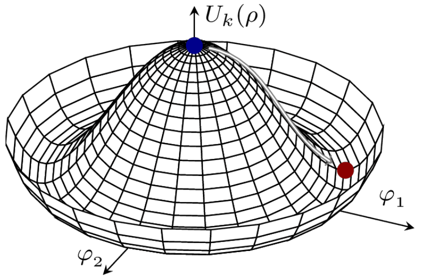

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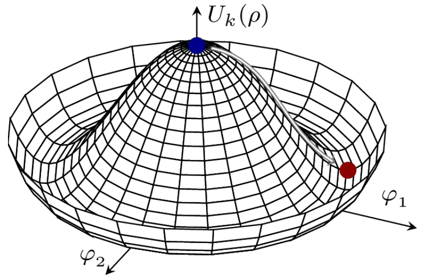

Equations of motion:

$$\frac{\partial \phi_a}{\partial t} = -\Gamma^\phi \frac{\delta F}{\delta \phi_a} + \theta_a + \frac{g}{2}\{\phi_a, n_{bc}\}\frac{\delta F}{\delta n_{bc}}$$

order parameter

iso-(axial-)vector charge densities

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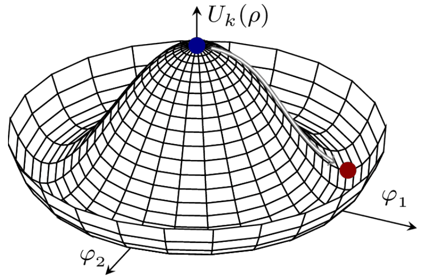

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damping noise

FDR

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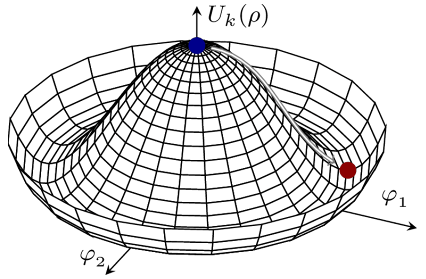

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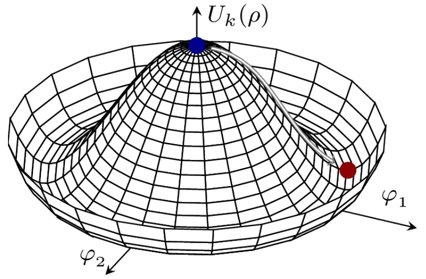

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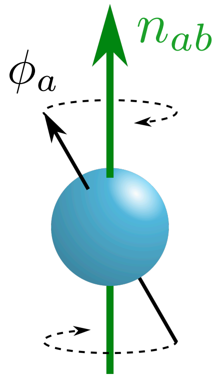
damping noise Larmor precession

FDR



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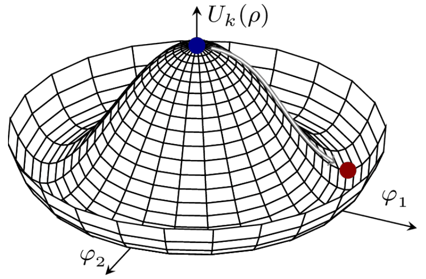

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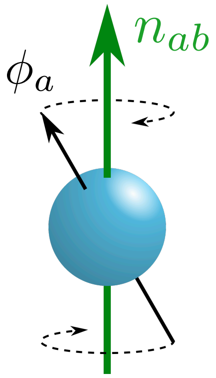

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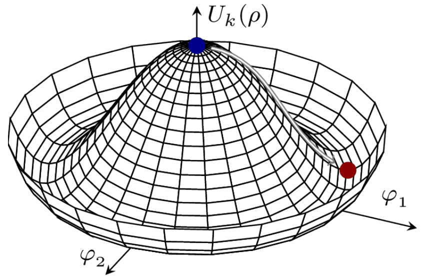

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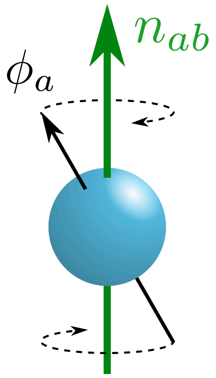

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Model G

$z = d/2$

$O(4)$ Heisenberg **antiferro**magnet

FRG:
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Temporal gauge symmetry (here for Model G)

- Equations of motion in Poisson-bracket formulation

$$\begin{aligned}
\partial_t \phi_a &= -\Gamma_0 \frac{\delta F}{\delta \phi_a} + \theta_a + \frac{g}{2} \{\phi_a, n_{bc}\} \frac{\delta F}{\delta n_{bc}} \\
\partial_t n_{ab} &= \gamma \vec{\nabla}^2 \frac{\delta F}{\delta n_{ab}} \qquad + \vec{\nabla} \cdot \vec{\zeta}_{ab} + g \{n_{ab}, \phi_c\} \frac{\delta F}{\delta \phi_c} \\
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- Apply external ‘magnetic’ field $\mathcal{H}_{ab}$: $F \to F - \frac{1}{2}\int d^d x\, \mathcal{H}_{ab} n_{ab}$

‘magnetization’

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CRC-TR 211

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Covariant time-derivatives,
in which $\mathcal{H}_{ab}$ = zero-component
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Covariant time-derivatives,
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Homogeneous term
drops out here

- Equations of motion **invariant** under **time-gauged** $O(4)$ transformations

$$\phi(t, \vec{x}) \to O(t)\phi(t, \vec{x})$$
$$n(t, \vec{x}) \to O(t) n(t, \vec{x}) O^T(t)$$

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- **Goal:** preserve during FRG flow (next)

2. Functional renormalization group (FRG) flow for systems with reversible mode couplings

Idea of the functional renormalization group (FRG)

- Wilson: introduce **infrared cutoff** to suppress fluctuations $p \lesssim k$

$$S \to S + \Delta S_k \qquad \Delta S_k = \int_{xx'} \phi(x) R_k(x, x') \phi(x')$$

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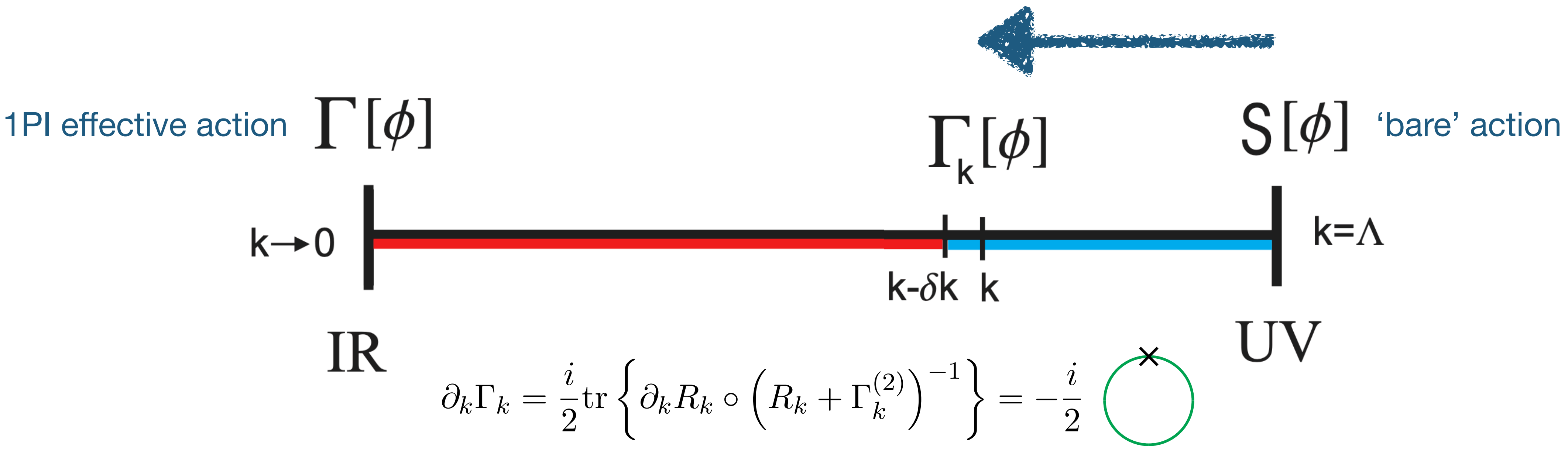
- Exact flow equation:

integrate out fluctuations
‘momentum shell by momentum shell’



$$\partial_k \Gamma_k = \frac{i}{2} \mathrm{tr} \left\{ \partial_k R_k \circ \left(R_k + \Gamma_k^{(2)} \right)^{-1} \right\} = -\frac{i}{2}$$

C. Wetterich, Phys. Lett. B **301** (1993) 90-94

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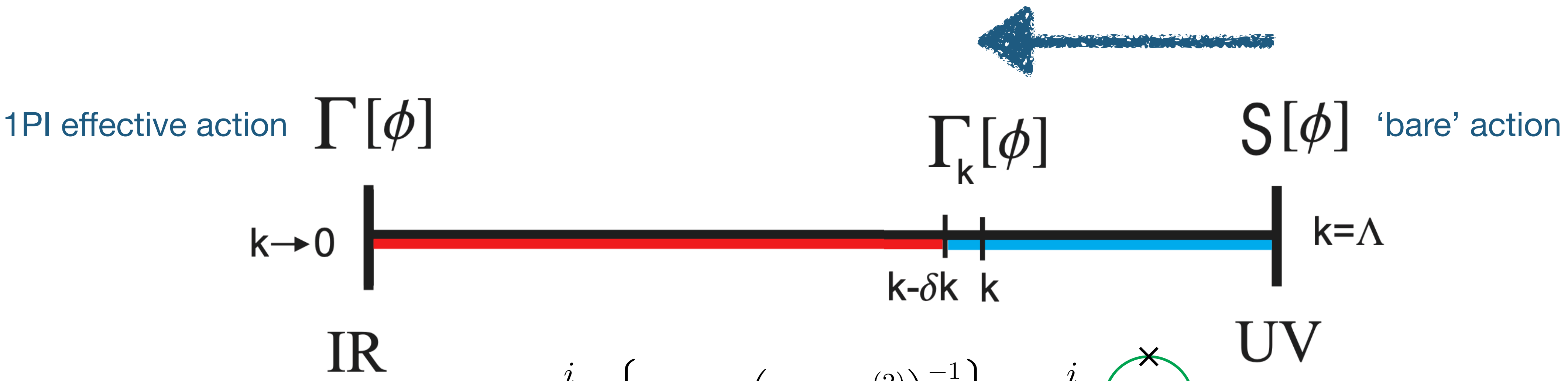

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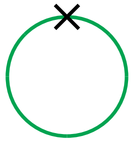

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- Quantitative results for (e.g.) critical exponents & amplitude ratios

Balog et al., PRL **123**, 240604 (2019), Dupuis et al., Phys. Rept. **910** (2021) 1-114

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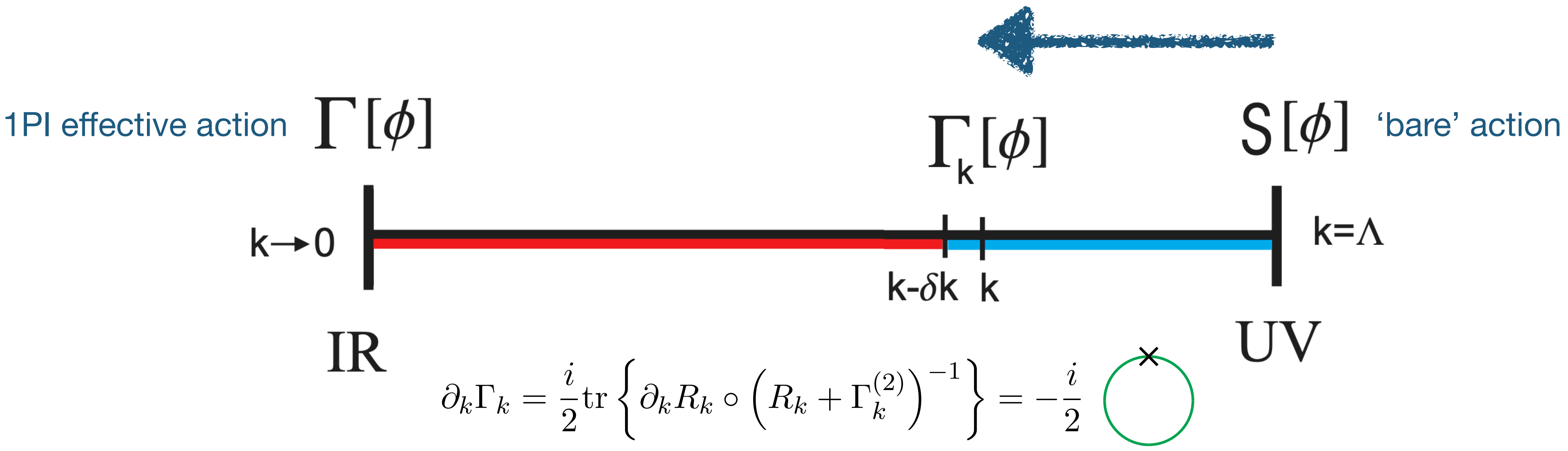

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- Scale invariance at critical point ⇝ fixed point of FRG flow

FRG flow for systems with reversible mode couplings

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FRG flow for systems with reversible mode couplings

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✓ **Problem solved:** temporal gauge symmetry becomes **extended** symmetry of effective MSR action Γ

see also Canet, Delamotte, Wschebor, PRE **93** (2016) 6, 063101
Floerchinger, Grossi, PRD **105** (2022) 8, 085015

FRG flow for systems with reversible mode couplings

- Similarly, add regulators also on level of LGW free energy:

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- Truncation:

$$m^2 \to m_k^2 \qquad \sigma \to \sigma_k \qquad \Gamma^\phi \to \Gamma^\phi_k$$

$$\lambda \to \lambda_k \qquad \eta \to \eta_k \qquad \gamma \to \gamma_k$$

(g, χ, ρ protected from renormalization)

3. Results for fixed points & dynamic critical exponents

Fixed points of the FRG flow (statics)

First look at flow of LGW free energy ⇝ flow of (static) couplings m_k^2, λ_k

Z_2

$O(4)$

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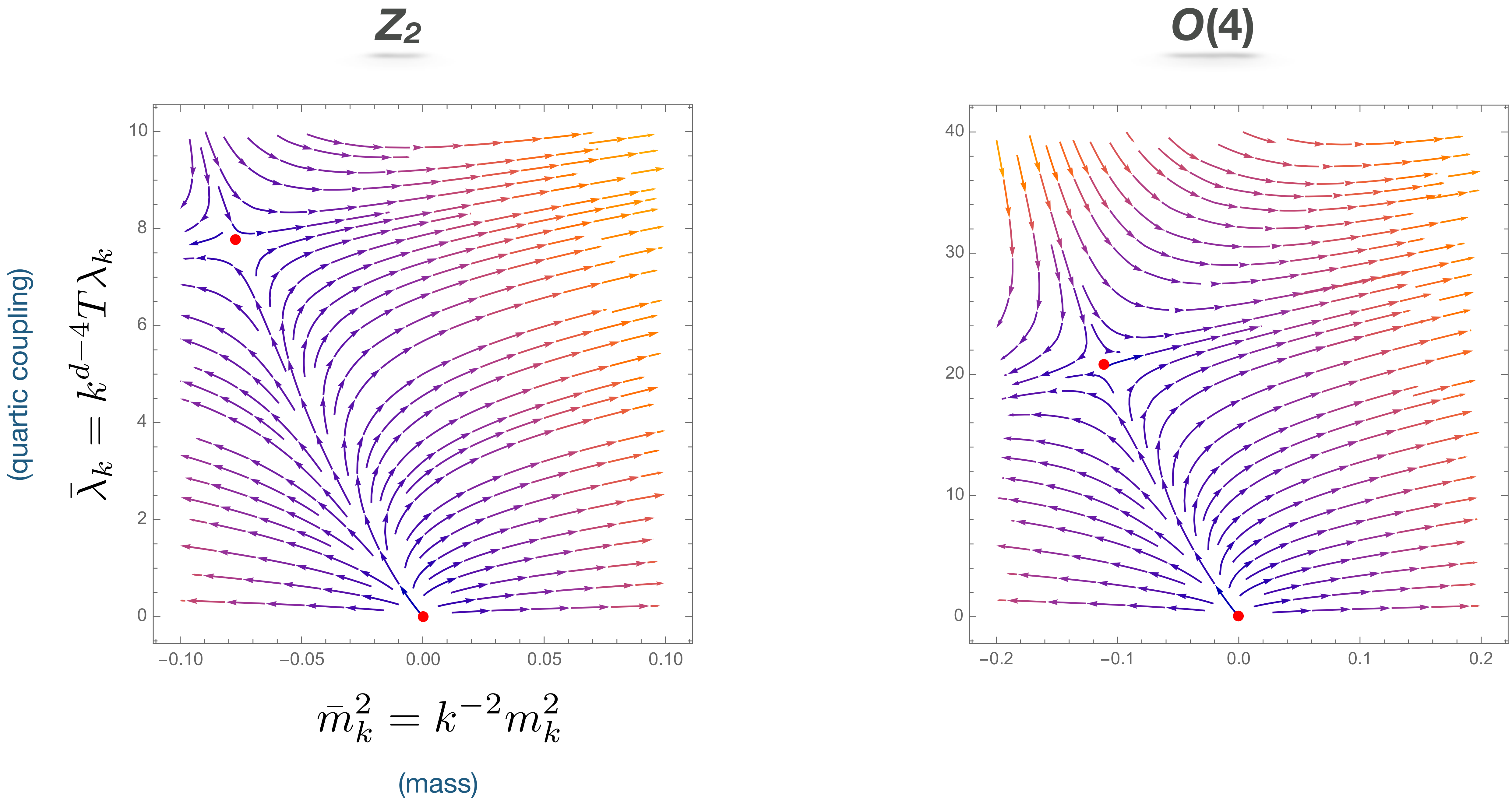

(quartic coupling)

(mass)

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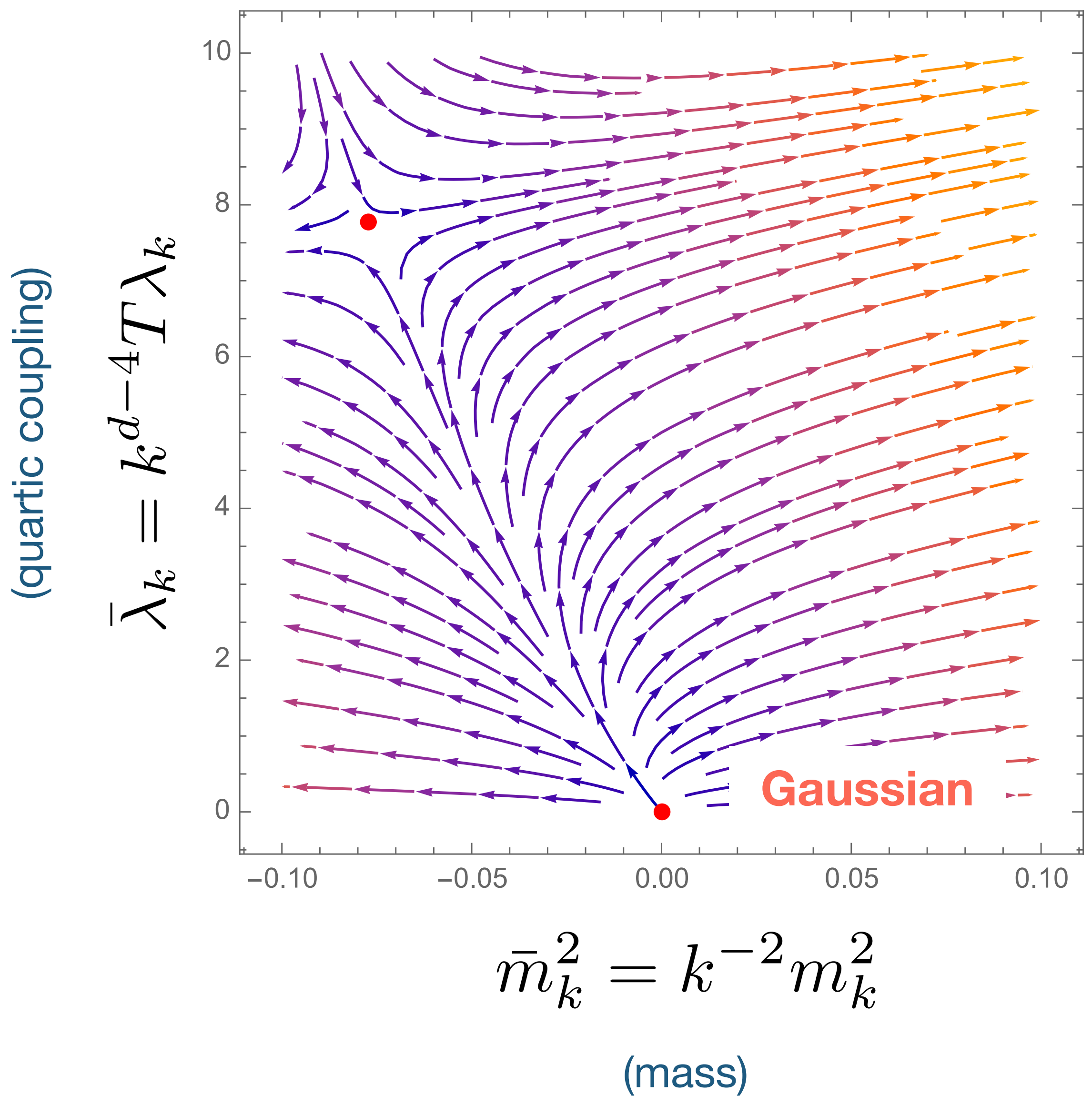

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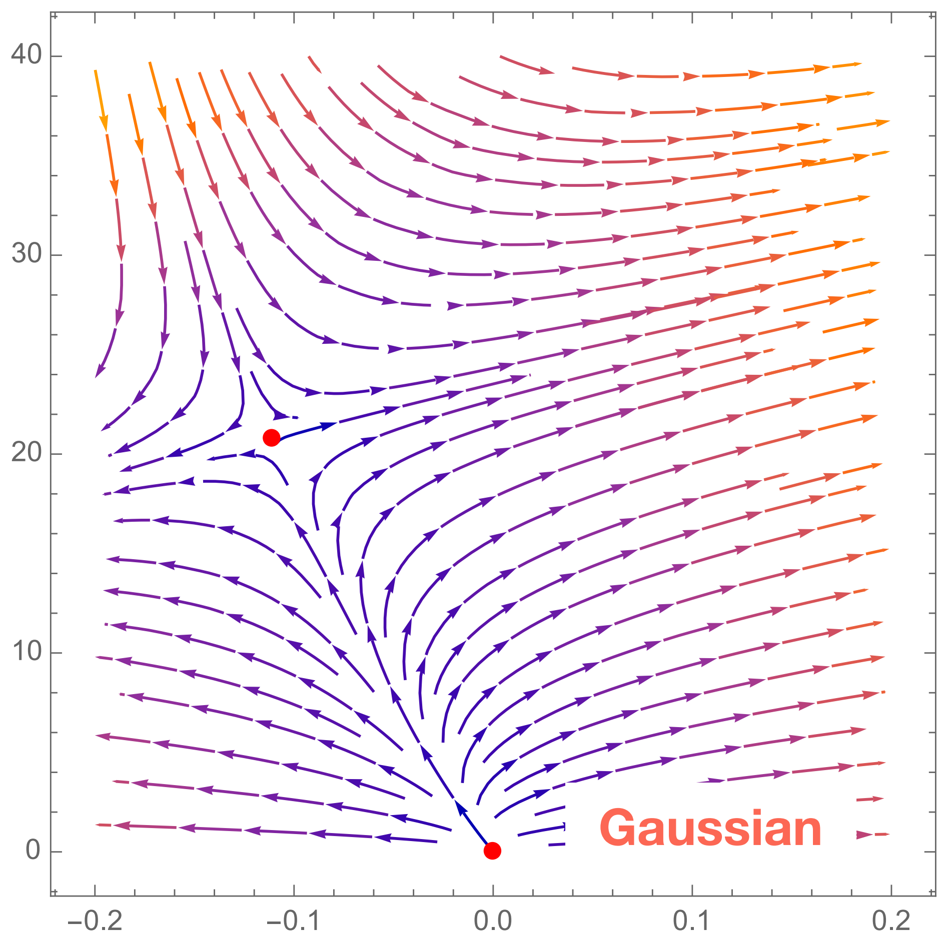

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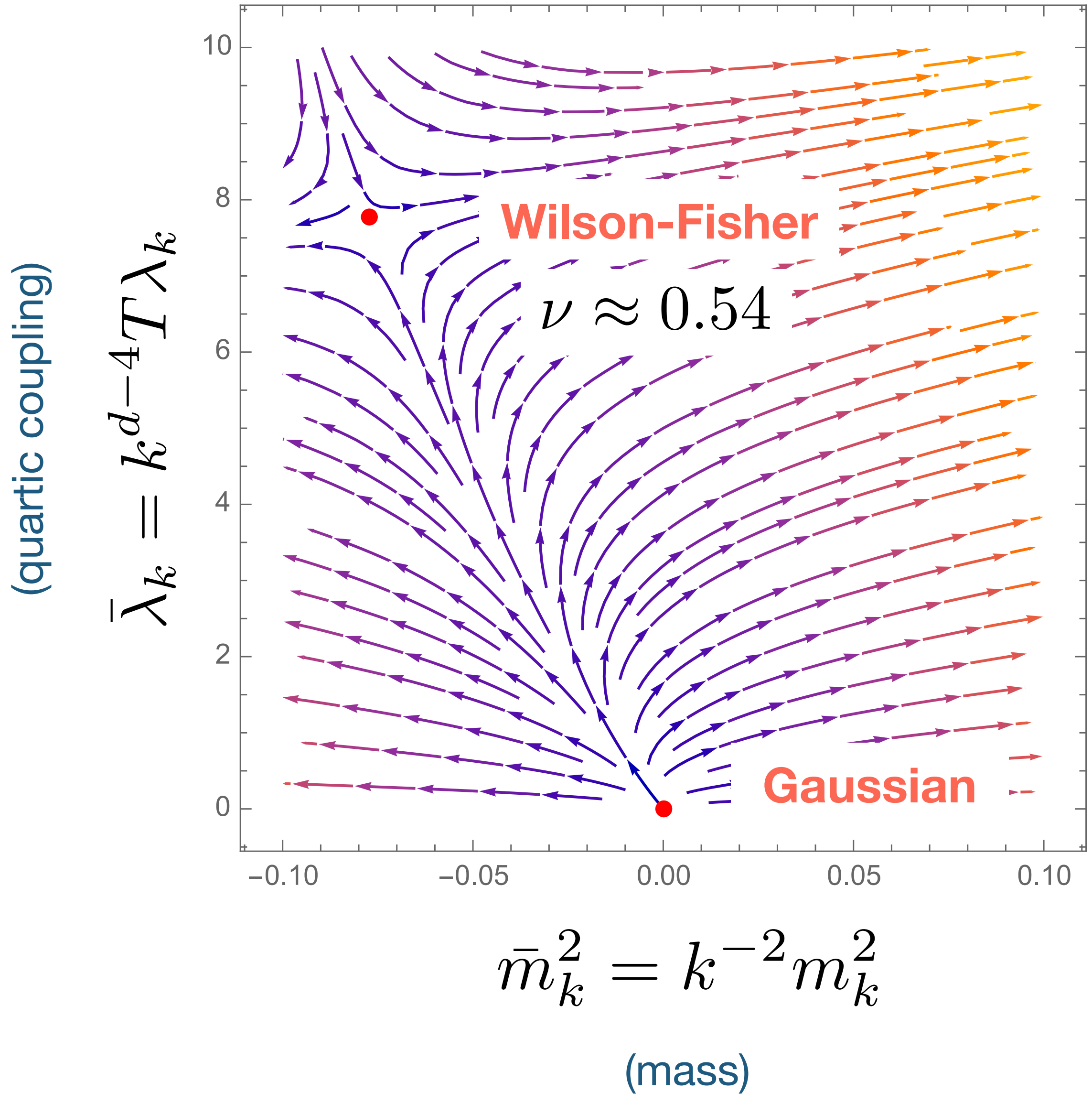

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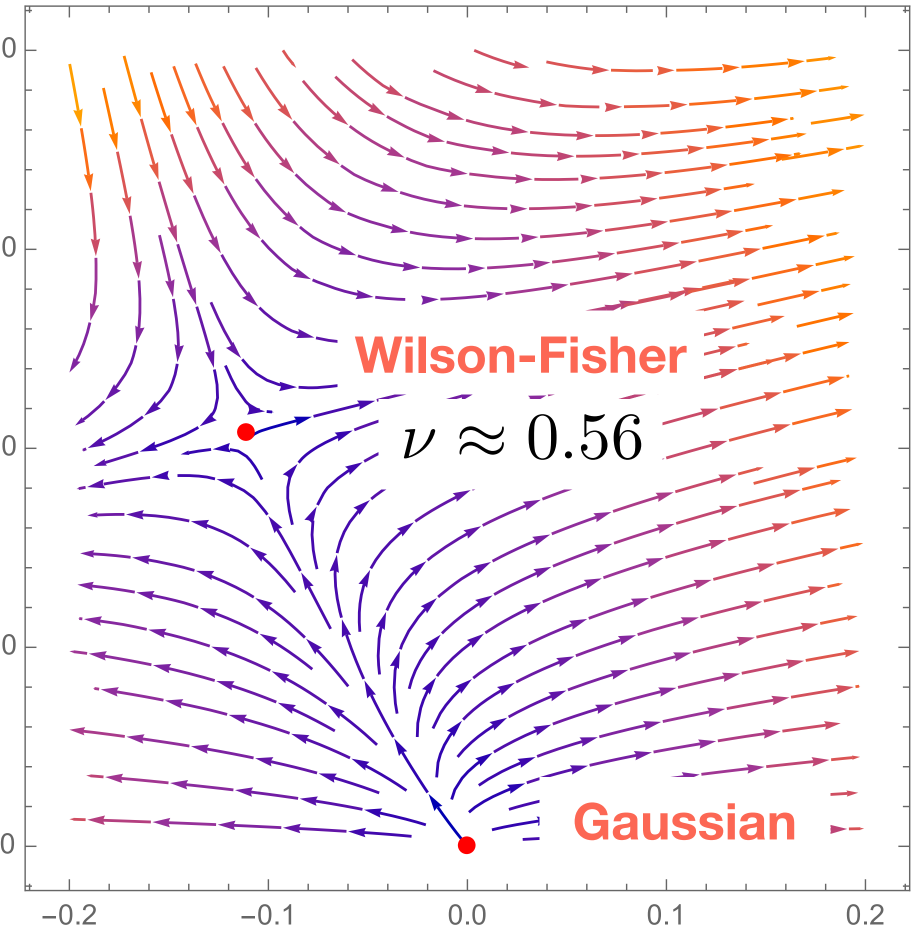

Fixed points of the FRG flow (dynamics)

Model G

$$w_G = \chi \frac{\Gamma_k^\phi}{\gamma_k} \qquad f_G \propto \frac{T\, k^{d-4}}{\Gamma_k^\phi \gamma_k}$$

Model H

$$w_H = \rho \frac{\sigma_k k^2}{\eta_k} \qquad f_H \propto \frac{T\, k^{d-4}}{\sigma_k \eta_k}$$

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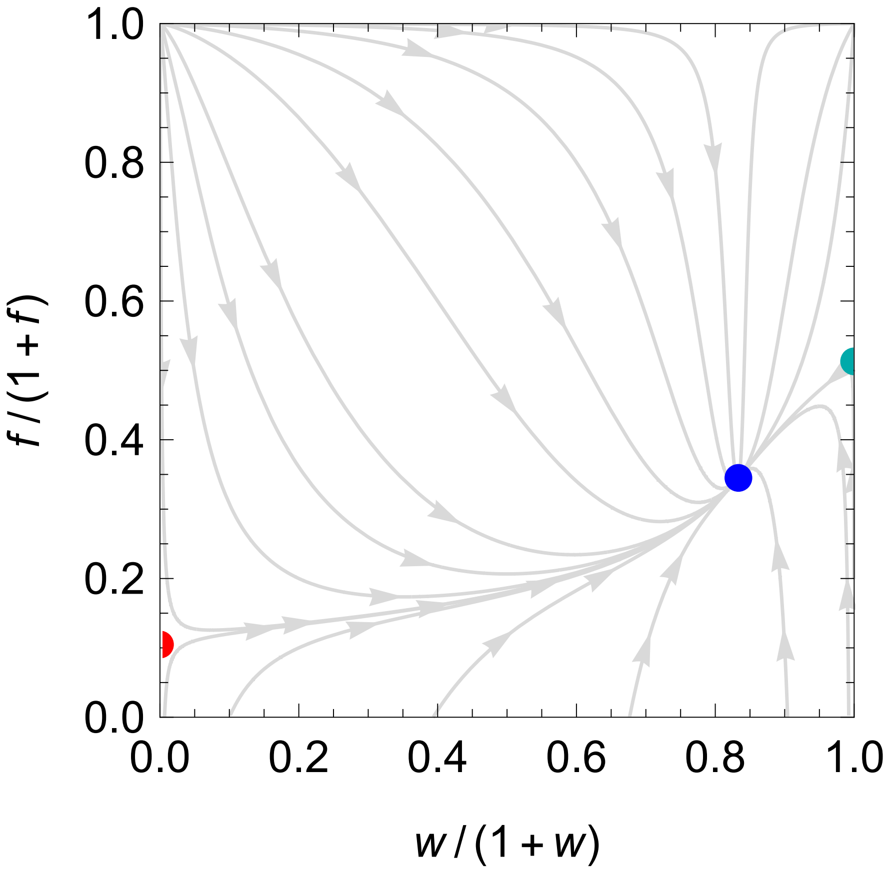

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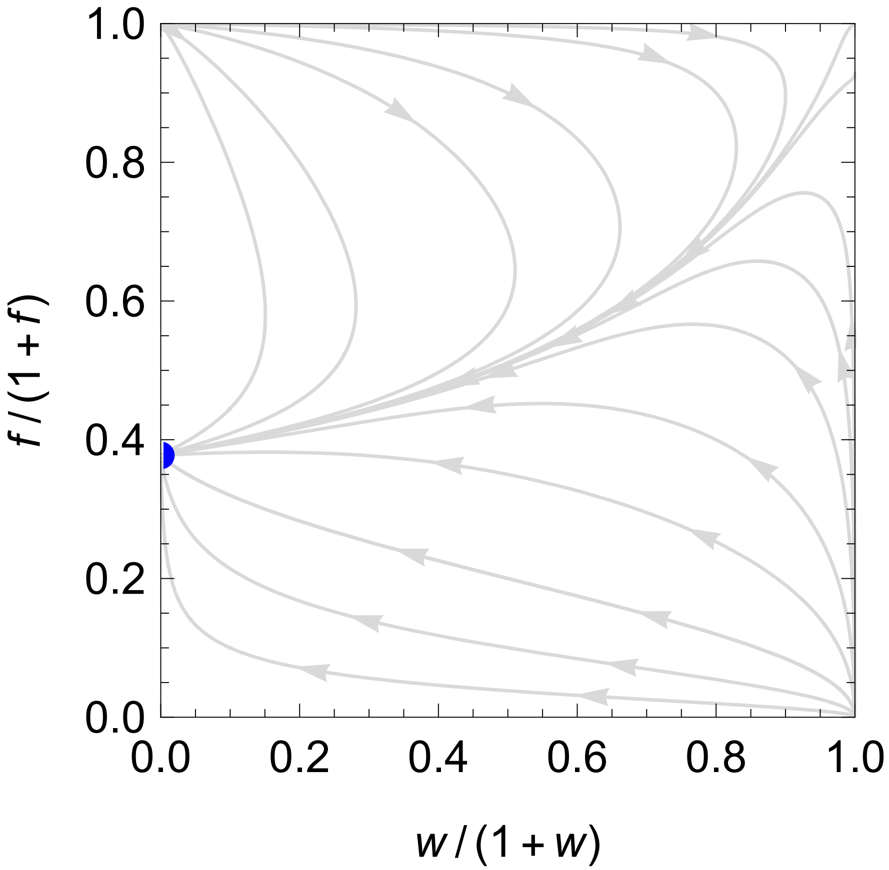
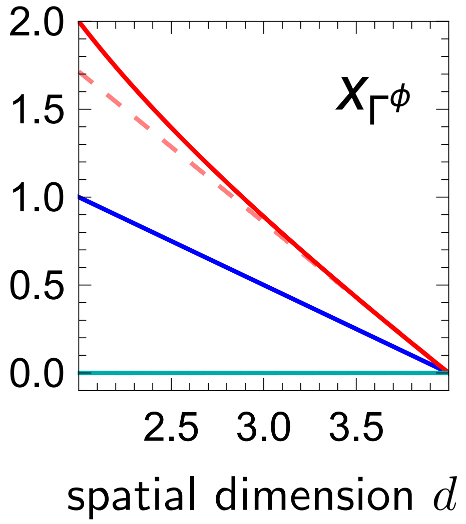


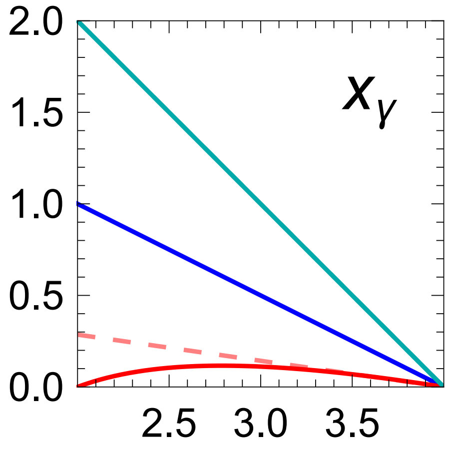
Dynamic critical exponents

Model G



$$\Gamma^\phi \sim k^{-x_{\Gamma^\phi}}$$

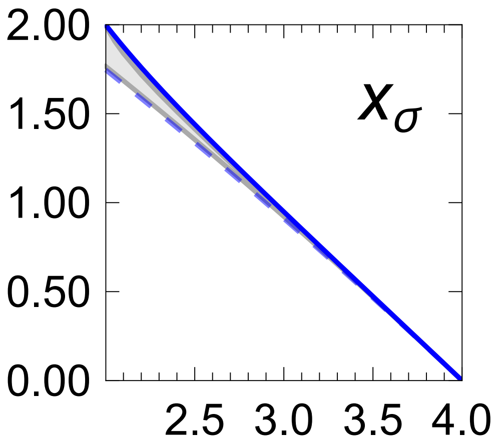
(order-parameter damping rate)



$$\gamma \sim k^{-x_\gamma}$$

(charge mobility)

Model H



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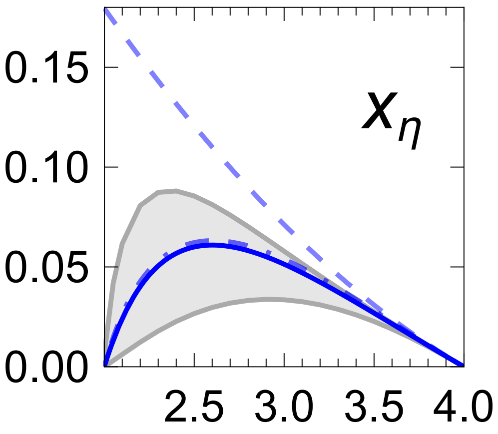
($\propto$ heat conductivity)



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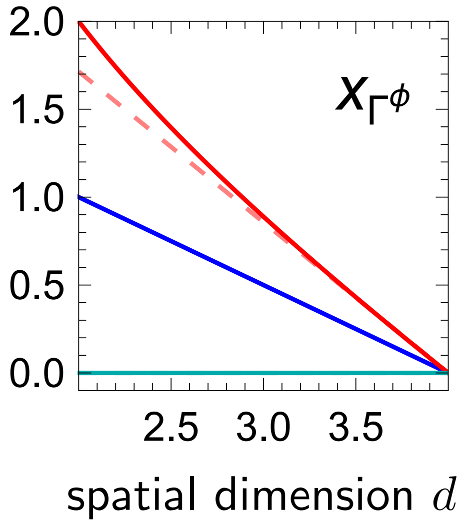

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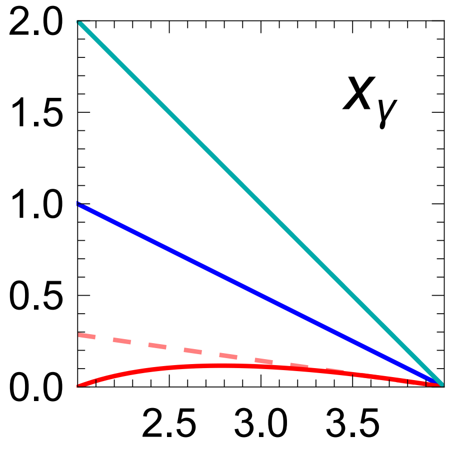

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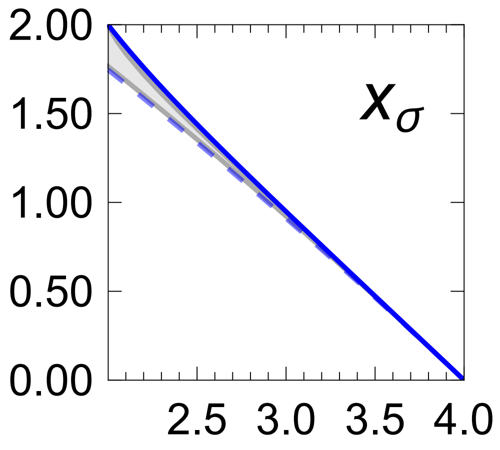

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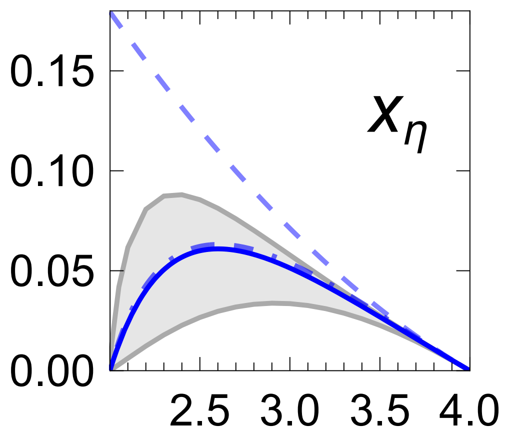

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- Model H $(d = 3)$:

	ϕ^4	LPA, $\phi = 0$	LPA', $\phi = \phi_{\text{min}}$
$x_\sigma \approx$	0.949	0.942	0.922
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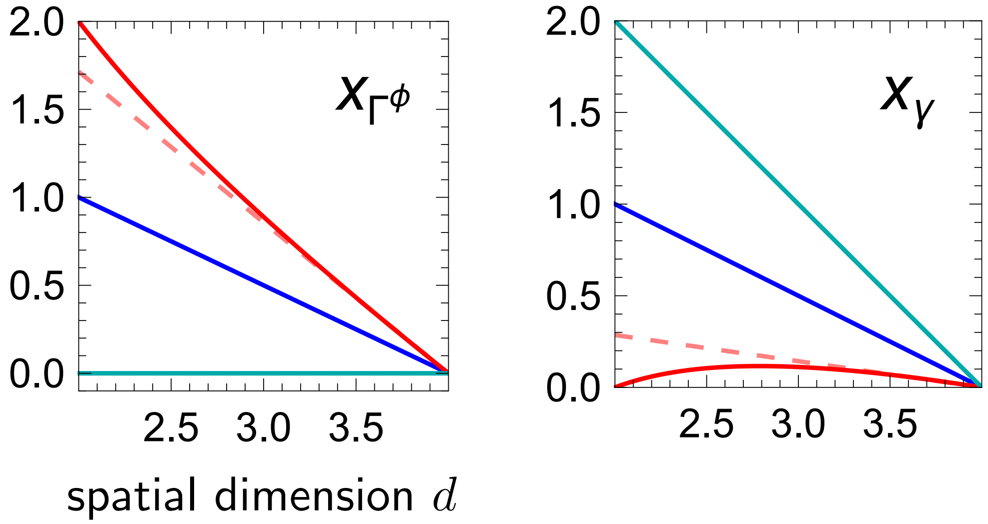

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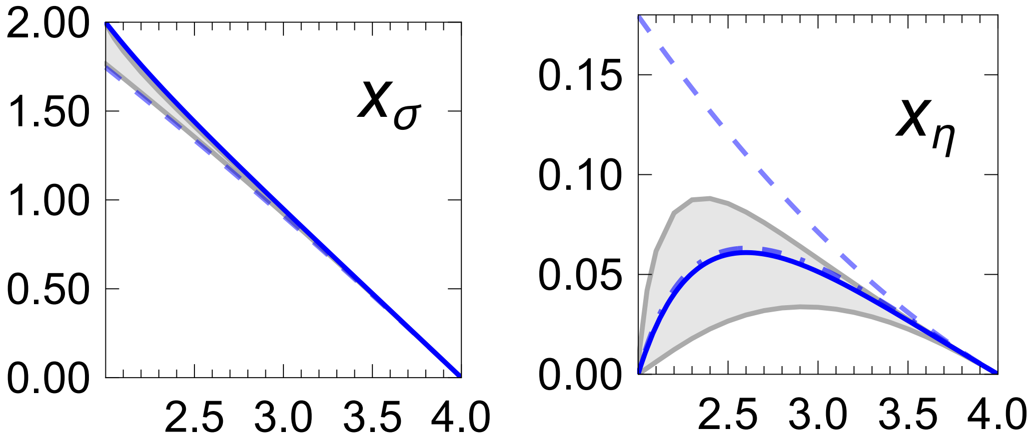

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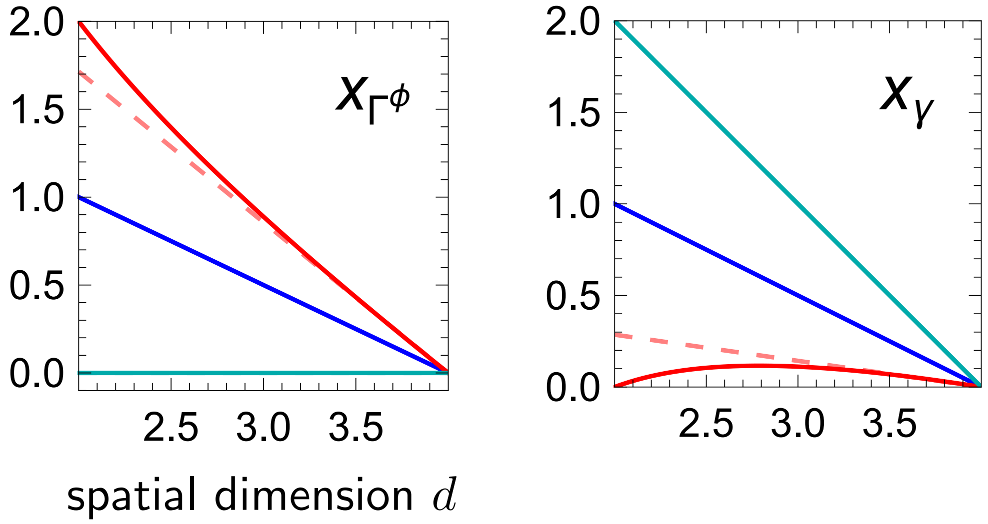

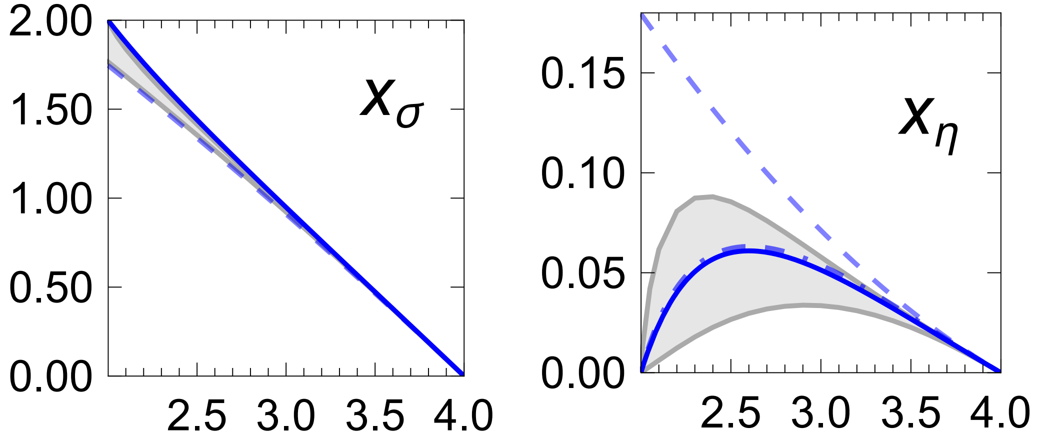

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- strong-scaling relation: $x_{\Gamma^\phi} + \eta_\perp = x_\gamma$ **(only Model G)** $\implies z_\phi = z_n = d/2$

Summary & Outlook

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- FRG flow for systems with reversible mode couplings

  Model G: JR, Ye, Schlichting, von Smekal, JHEP **01**, 118 (2025)

  Model H: JR, Ye, Schlichting, von Smekal, arXiv:2409.14470

Outlook:

- dynamics of **Model G** for non-vanishing external fields (quark masses)
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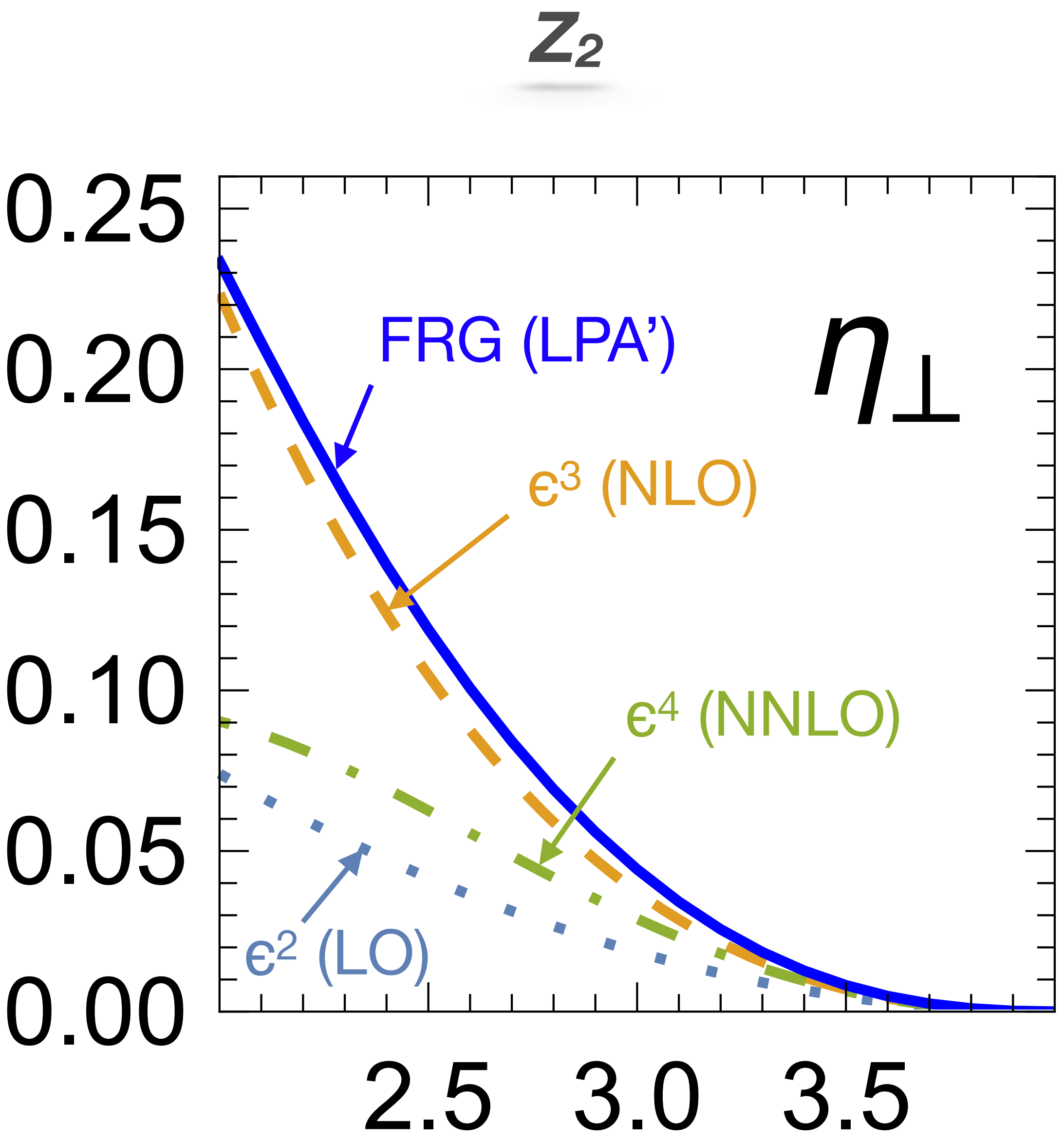
Thank you for your attention!

Backup

Anomalous dimension



Result from ϵ-expansion:

$$\eta = \frac{\varepsilon^2(N+2)}{2(N+8)^2}\left\{1 + \frac{(-N^2+56N+272)}{4(N+8)^2}\varepsilon + \frac{1}{16(N+8)^4}\left[-5N^4 - 230N^3 \right.\right.$$
$$\left.\left. +1124N^2 + 17920N + 46144 - 384(5N+22)(N+8)\zeta(3)\right]\varepsilon^2\right\} + O\left(\varepsilon^5\right)$$

Zinn-Justin, *Quantum Field Theory and Critical Phenomena,* Oxford University Press (2021)

Detailed formulas (Model H)

- Generating functional

$$Z[H,\tilde{H},\vec{A},\tilde{\vec{A}}] = \int \mathcal{D}\phi\,\mathcal{D}\tilde{\phi}\,\mathcal{D}n\,\mathcal{D}\tilde{n}\ \exp\Big\{ iS + i\int_x \big(\tilde{H}\phi + \tilde{A}_l j_l\big) +$$

$$i\int_x H\big(-\sigma_k\vec{\nabla}^2\tilde{\phi} + g\{\phi, j_m\}\mathcal{T}_{mo}\tilde{j}_o\big) + i\int_x A_l\big(-\eta_k\vec{\nabla}^2\mathcal{T}_{lo}\tilde{j}_o + g\mathcal{T}_{lm}\{j_m,\phi\}\tilde{\phi} + g\mathcal{T}_{lm}\{j_m,j_n\}\mathcal{T}_{no}\tilde{j}_o\big)\Big\}$$

- MSR action

$$S = \int_x \Big\{ -\tilde{\phi}\left(\frac{\partial\phi}{\partial t} - \sigma\vec{\nabla}^2\frac{\delta F}{\delta\phi} - g\{\phi,\vec{j}\}\cdot\frac{\delta F}{\delta\vec{j}}\right)$$

$$-\tilde{j}_l\left(\frac{\partial j_l}{\partial t} - \mathcal{T}_{lm}\left[\eta\vec{\nabla}^2\frac{\delta F}{\delta j_m} + g\{j_m,\phi\}\frac{\delta F}{\delta\phi} + g\{j_m,j_n\}\frac{\delta F}{\delta j_n}\right]\right)$$

$$-iT\tilde{\phi}\sigma\vec{\nabla}^2\tilde{\phi} - iT\tilde{j}_l\eta\mathcal{T}_{lm}\vec{\nabla}^2\tilde{j}_m\Big\}$$

- Composite operators

$$\tilde{\Phi} \equiv -\sigma_k\vec{\nabla}^2\tilde{\phi} + g\{\phi, j_m\}\mathcal{T}_{mo}\tilde{j}_o$$

$$\tilde{J}_l \equiv -\eta_k\vec{\nabla}^2\mathcal{T}_{lo}\tilde{j}_o + g\mathcal{T}_{lm}\{j_m,\phi\}\tilde{\phi} + g\mathcal{T}_{lm}\{j_m,j_n\}\mathcal{T}_{no}\tilde{j}_o$$

Detailed FRG setup (Model H)

- effective average MSR action

$$\Gamma_k[\phi, \tilde{\Phi}, \vec{j}, \tilde{\vec{J}}] \equiv \sup_{H, \tilde{H}, \vec{A}, \tilde{\vec{A}}} \big\{ -i \log Z_k[H, \tilde{H}, \vec{A}, \tilde{A}] - \int_x \big(\tilde{H}\phi + H\tilde{\Phi} + \tilde{A}_l j_l + A_l \tilde{J}_l \big) \big\}$$

- full (truncated) propagators

$$G^{R/A}_{\phi,k}(\omega, \vec{p}) = -\frac{\sigma_k \vec{p}^2}{\pm i\omega - \sigma_k \vec{p}^2 (m_k^2 + \vec{p}^2 + R_k^\phi(\vec{p}))}\,, \qquad G^{R/A}_{j,k}(\omega, \vec{p}) = -\frac{\eta_k \vec{p}^2}{\pm i\omega - \eta_k \vec{p}^2 (1/\rho + R_k^j(\vec{p}))}$$

$$iF_{\phi/j,k}(\omega, \vec{p}) = \frac{T}{\omega} \Big(G^R_{\phi/j,k}(\omega, \vec{p}) - G^A_{\phi/j,k}(\omega, \vec{p}) \Big)$$

- (truncated) 1PI vertex functions

$$\Gamma_k^{\tilde{\Phi}\phi j_l}(p, q, r) = -g\, \frac{r^0\, (\mathcal{T}_{\vec{r}}\vec{p})_l}{\eta_k \vec{r}^2\, \sigma_k \vec{p}^2}$$

$$\Gamma_k^{\tilde{J}_l \phi\phi}(p, q, r) = g\, \frac{q^0\, (\mathcal{T}_{\vec{p}}\vec{q})_l}{\eta_k \vec{p}^2\, \sigma_k \vec{q}^2} + g\, \frac{r^0\, (\mathcal{T}_{\vec{p}}\vec{r})_l}{\eta_k \vec{p}^2\, \sigma_k \vec{r}^2}$$

$$\Gamma_k^{\tilde{\Phi}\tilde{\Phi}\phi\phi}(p, q, r, s) = \frac{2ig^2 T}{\sigma_k \vec{p}^2\, \sigma_k \vec{q}^2} \left(\frac{\mathcal{T}_{lm}(\vec{p}+\vec{r})}{\eta_k (\vec{p}+\vec{r})^2} + \frac{\mathcal{T}_{lm}(\vec{q}+\vec{r})}{\eta_k (\vec{q}+\vec{r})^2} \right) r_l s_m$$

$$\Gamma_k^{\tilde{J}_l \tilde{J}_m \phi\phi}(p, q, r, s) = 2ig^2 T\, \frac{(\mathcal{T}_{\vec{p}}(\vec{p}+\vec{r}))_l\, (\mathcal{T}_{\vec{q}}(\vec{q}+\vec{s}))_m}{\eta_k \vec{p}^2\, \eta_k \vec{q}^2\, \sigma_k (\vec{p}+\vec{r})^2} + 2ig^2 T\, \frac{(\mathcal{T}_{\vec{p}}(\vec{p}+\vec{s}))_l\, (\mathcal{T}_{\vec{q}}(\vec{q}+\vec{r}))_m}{\eta_k \vec{p}^2\, \eta_k \vec{q}^2\, \sigma_k (\vec{q}+\vec{r})^2}$$

Detailed FRG flow equations

- projection onto kinetic coefficients (Model H)

$$
\begin{aligned}
\partial_k\sigma_k &= -\frac{\sigma_k^2}{2iT}\lim_{\vec{p}\to 0}\vec{p}^2\lim_{\omega\to 0}\frac{\delta^2\partial_k\Gamma_k}{\delta\tilde{\Phi}(-p)\delta\tilde{\Phi}(p)}\bigg|_0\\
\partial_k\eta_k &= -\frac{\eta_k^2}{2iT}\lim_{\vec{p}\to 0}\vec{p}^2\lim_{\omega\to 0}\frac{\mathcal{T}_{lm}(\vec{p})}{d-1}\frac{\delta^2\partial_k\Gamma_k}{\delta\tilde{J}_l(-p)\delta\tilde{J}_m(p)}\bigg|_0
\end{aligned}
$$

- analytical result:

$$
\begin{aligned}
\partial_k\sigma_k &= \frac{2g^2\Omega_d k^{d-1}T}{(2\pi)^d}\,\frac{d-1}{d-2}\,\frac{1}{\eta_k}\left(\frac{\sigma_k^2}{(\eta_k/\rho+\sigma_k(k^2+m_k^2))^2}-\frac{1}{(k^2+m_k^2)^2}\right)\\
\partial_k\eta_k &= -\frac{g^2\Omega_d k^{d+1}T}{(2\pi)^d\,(2+d)\sigma_k(k^2+m_k^2)^3}
\end{aligned}
$$

- analytical result (Model G):

$$
\begin{aligned}
\partial_k\Gamma_k^\phi &= \frac{g^2\,(N-1)\,d\,\Omega_d k^{d-1}T}{(2\pi)^d\,\left(k^2+m_k^2\right)\,\gamma_k}\left\{\frac{\Gamma_k^\phi}{k^2\gamma_k/\chi+\Gamma_k^\phi(k^2+m_k^2)}-\frac{2+(d-4)\;{}_2F_1\left(1,\frac{d-2}{2};\frac{d}{2};-\frac{k^2\gamma_k/\chi}{\Gamma_k^\phi(k^2+m_k^2)}\right)}{(d-2)\,\left(k^2+m_k^2\right)}\right\}\\
\partial_k\gamma_k &= -\frac{2g^2\Omega_d k^{d+1}T}{(2\pi)^d\,\Gamma_k^\phi\,\left(k^2+m_k^2\right)^3}
\end{aligned}
$$

Beta functions

- dynamic couplings (Model G):

$$w_G \equiv \chi \frac{\Gamma_k^\phi}{\gamma_k}\,, \qquad f_G \equiv \frac{d\,\Omega_d\, g^2 T}{(2\pi)^d}\,\frac{k^{d-4}}{\Gamma_k^\phi \gamma_k}$$

$$\begin{aligned} k\partial_k f_G &= f_G(d-4) + \\ &\qquad f_G^2\left(\frac{2}{d(1+\bar{m}_k^2)^3} - (N-1)I_d(\bar{m}^2, w_G)\right) \\ k\partial_k w_G &= w_G f_G\left[\frac{2}{d(1+\bar{m}_k^2)^3} + (N-1)I_d(\bar{m}^2, w_G)\right] \end{aligned}$$

with $I_d(\bar{m}^2, w_G) \equiv -\frac{1}{(1+\bar{m}^2)^2}\left\{\frac{1}{1+(1+\bar{m}^2)w_G} + \frac{4-d}{d-2}\left[1 - {}_2F_1\left(1, \frac{d-2}{2}; \frac{d}{2}; -\frac{1}{(1+\bar{m}^2)w_G}\right)\right]\right\}$

- dynamic couplings (Model H):

$$w_H \equiv \rho\,\frac{\sigma_k k^2}{\eta_k}\,, \qquad f_H \equiv \frac{d\,\Omega_d\, g^2 T}{(2\pi)^d}\,\frac{k^{d-4}}{\sigma_k \eta_k}$$

$$\begin{aligned} k\partial_k f_H &= f_H(d-4) \\ &\quad - f_H^2\frac{2}{d-2}\left(\frac{(d-1)}{d(1/w_H + (1+\bar{m}^2))^2} - \frac{d-1}{d(1+\bar{m}^2)^2}\right) \\ &\quad + f_H^2\frac{1}{d(d+2)}\frac{1}{(1+\bar{m}^2)^3} \\ k\partial_k w_H &= 2w_H + w_H f_H\left[\frac{1}{d(d+2)}\frac{1}{(1+\bar{m}^2)^3}\right. \\ &\quad \left. + \frac{2}{d-2}\left(\frac{(d-1)}{d(1/w_H + (1+\bar{m}^2))^2} - \frac{d-1}{d(1+\bar{m}^2)^2}\right)\right]. \end{aligned}$$

Fixed points & critical exponents (dynamic sector)

critical exponents:

- Model G (strong-scaling FP):

$$f_G^* = \frac{(4-d)d(1+\bar{m}^2)^3}{4}$$

$$I_d(\bar{m}^{*2}, w_G^*) = -\frac{2}{(N-1)d(1+\bar{m}^{*2})^3}$$

numerical inversion: $\rightsquigarrow \; w_G^*$

$$x_{\Gamma^\phi} = x_\gamma = 2 - \frac{d}{2}$$

- Model G (weak-scaling FP 1):

$$f_G^* = \frac{(4-d)(d-2)d(1+\bar{m}^{2*})^3}{2d(N(1+\bar{m}^{2*})-\bar{m}^{2*})-4}$$

$$w_G^* = 0$$

$$x_{\Gamma^\phi} = \frac{(N-1)(4-d)d(1+\bar{m}^{2*})}{d(N(1+\bar{m}^{2*})-\bar{m}^{2*})-2}$$

$$x_\gamma = \frac{(4-d)(d-2)}{d(N(1+\bar{m}^{2*})-\bar{m}^{2*})-2}$$

- Model G (weak-scaling FP 2):

$$f_G^* = \frac{1}{2}(4-d)d(1+\bar{m}^{2*})^3$$

$$w_G^* = \infty$$

$$x_{\Gamma^\phi} = 0 \qquad x_\gamma = d$$

- Model H:

$$f_H^* = \frac{4-d}{\frac{2}{d-2}\frac{d-1}{d(1+\bar{m}^2)^2} + \frac{1}{d(d+2)}\frac{1}{(1+\bar{m}^2)^3}}$$

$$w_H^* = 0$$

$$x_\sigma = \frac{4-d}{\frac{2}{d-2}\frac{d-1}{d(1+\bar{m}^2)^2} + \frac{1}{d(d+2)}\frac{1}{(1+\bar{m}^2)^3}} \frac{2}{d-2}\frac{d-1}{d(1+\bar{m}^2)^2} \tag{26}$$

$$x_\eta = \frac{4-d}{\frac{2}{d-2}\frac{d-1}{d(1+\bar{m}^2)^2} + \frac{1}{d(d+2)}\frac{1}{(1+\bar{m}^2)^3}} \frac{1}{d(d+2)}\frac{1}{(1+\bar{m}^2)^3} \tag{27}$$

Include momentum dependence in Model G

- **Strong-scaling** of charge diffusion coefficient in Model G

$$D_n(\boldsymbol{p},\tau) = s^{2-z} D_n(s\boldsymbol{p}, s^{1/\nu}\tau)$$

$$\rightsquigarrow \quad D_n(p,\tau) \sim \tau^{-\nu(2-z)} \underline{\mathcal{L}(\tau^{-\nu}\bar{p})} \qquad \bar{p} = f^+ p$$

JR, Ye, Schlichting, von Smekal, arXiv:2403.04573

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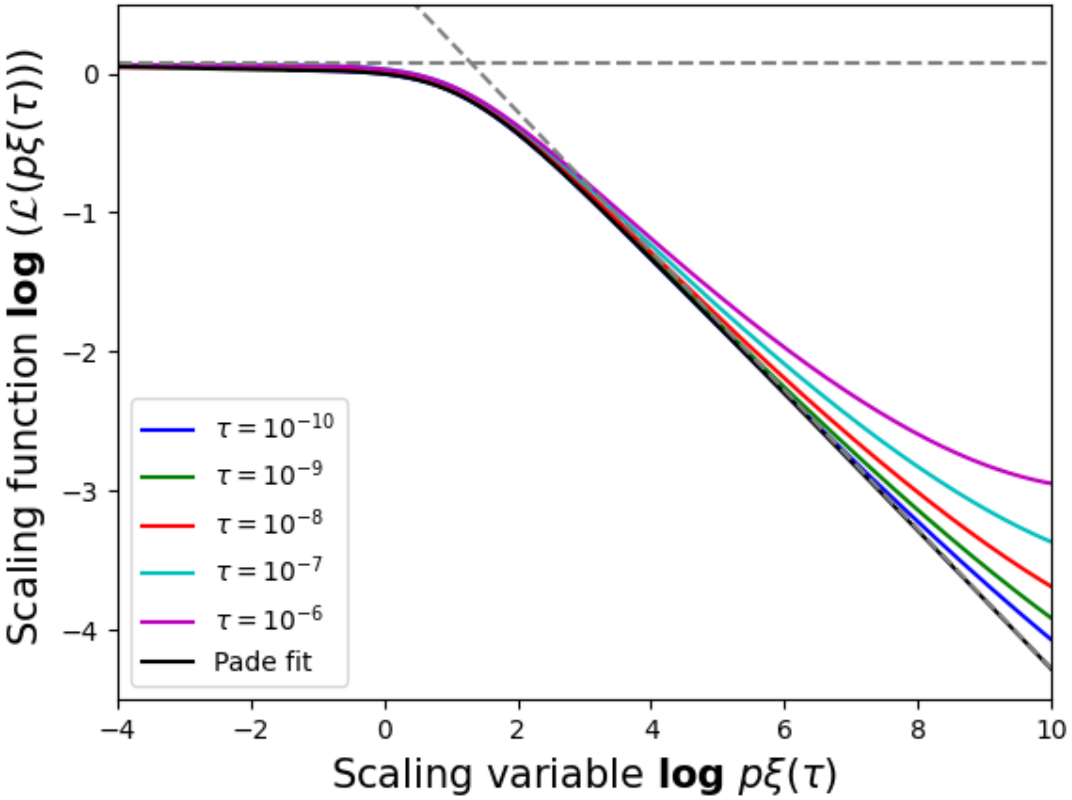

dynamic universal scaling function

JR, Ye, Schlichting, von Smekal, arXiv:2403.04573

Dynamic (hyper-)scaling relations

- What can we say about the scaling exponents? Investigate fixed-point equation of f:

$$\Gamma^{\phi} \sim k^{-x_{\Gamma^{\phi}}} \qquad \sigma \sim k^{-x_{\sigma}}$$

$$\gamma \sim k^{-x_{\gamma}} \qquad \eta \sim k^{-x_{\eta}}$$

$$f_G \equiv \frac{d\,\Omega_d\,g^2 T}{(2\pi)^d}\,\frac{k^{d-4}}{\Gamma^{\phi}_k \gamma_k}$$

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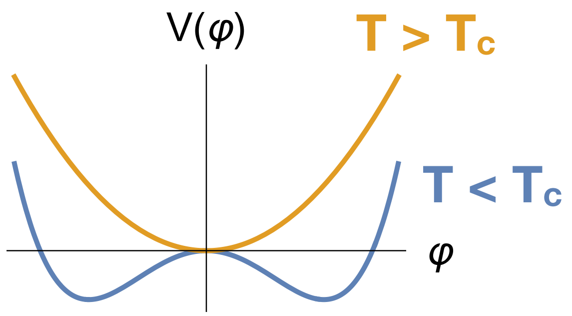
only at **‘strong-scaling’ fixed point** of Model G: also **‘strong-scaling’** relation

Dynamic universality class of chiral transition

Statics: $O(4)$ Landau-Ginzburg-Wilson (LGW) functional



$$F = \int d^d x \left\{ \frac{1}{2}(\partial^i \phi_a)(\partial^i \phi_a) + \frac{m^2}{2}\phi_a\phi_a + \frac{\lambda}{4!N}(\phi_a\phi_a)^2 \right.$$
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six conserved iso-vector and
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(higher-order terms irrelevant)

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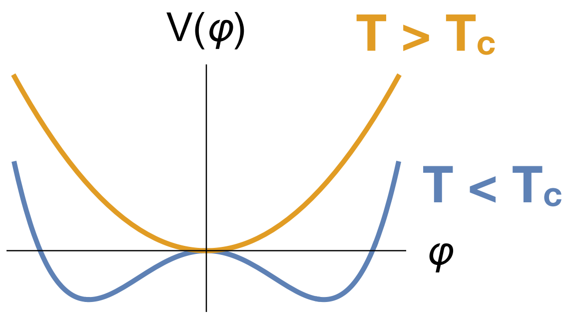

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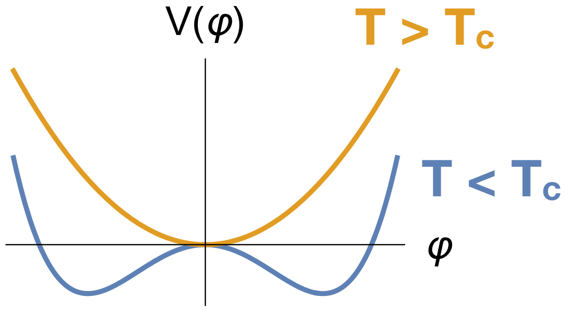

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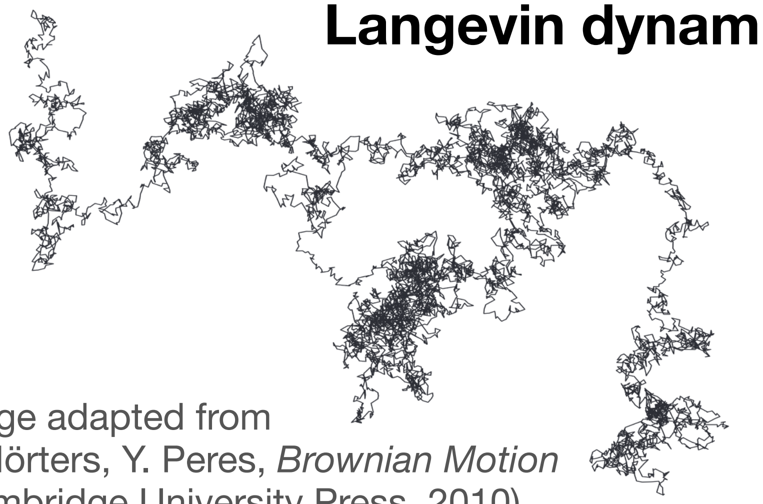
Langevin dynamics



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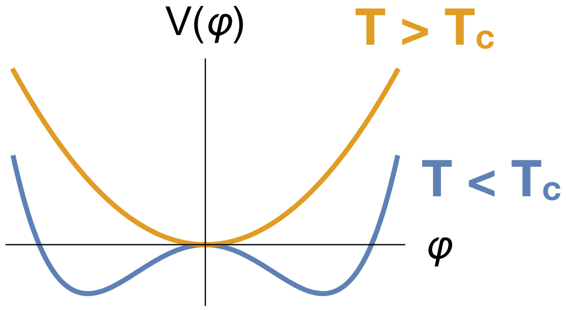

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damping/diffusion

$$\frac{\partial \phi_a}{\partial t} = \boxed{-\Gamma_0 \frac{\delta F}{\delta \phi_a}}$$

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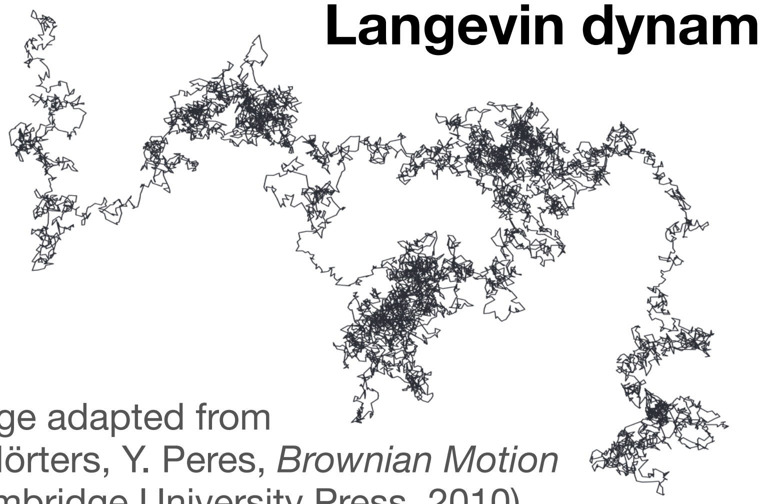

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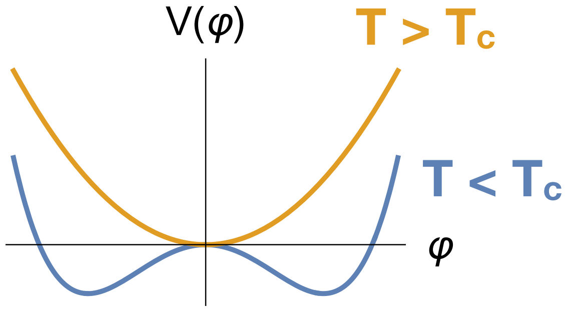

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[see Landau & Lifshitz, *Statistical Physics, Part 1* (Butterworth-Heinemann, Oxford, 1980)]

damping/diffusion stochastic forces

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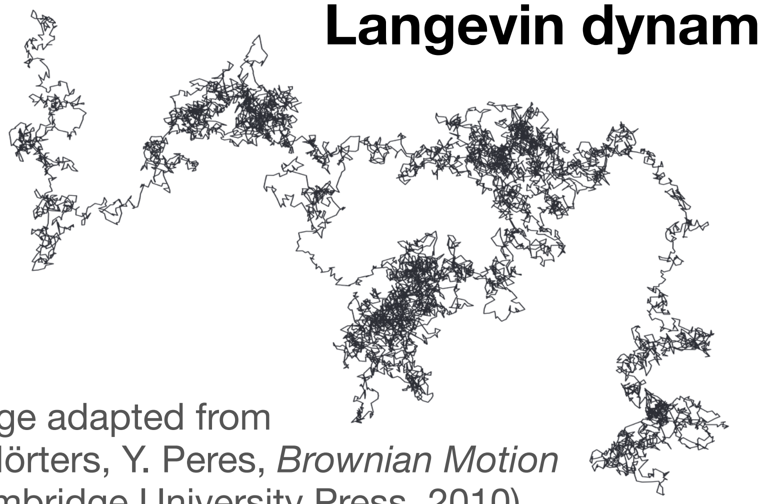

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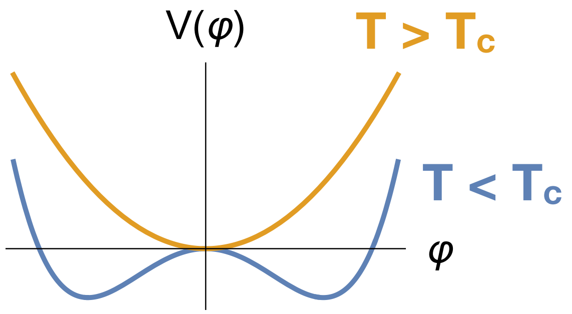

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fixed by fluctuation-dissipation (Einstein) relations, e.g.:

damping/diffusion

stochastic forces

$$\langle \theta_a(x) \theta_b(x') \rangle = 2\Gamma_0 T \delta_{ab} \delta(x - x')$$

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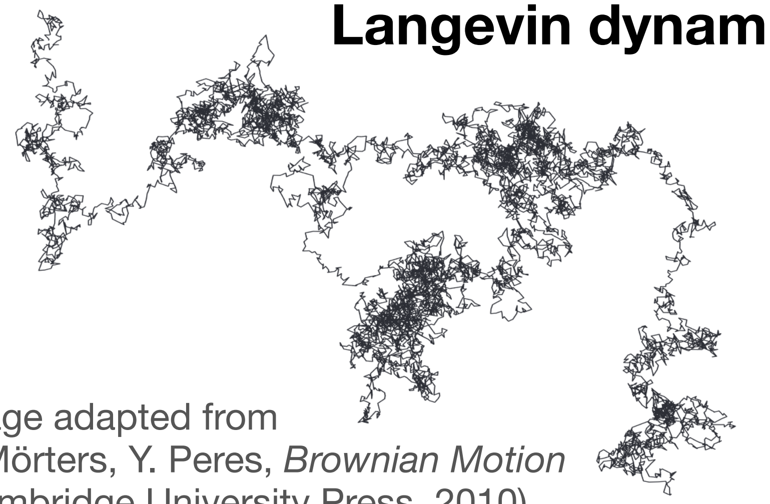

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$$
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(n's generate $O(4)$ transformations)

- **Reversible (ideal) part:** Poisson bracket technique

$$\frac{\partial \phi_a}{\partial t} = \{\phi_a, F\}$$

$$\frac{\partial n_{ab}}{\partial t} = \{n_{ab}, F\}$$

$$\{\cdot,\cdot\} \to -i[\cdot,\cdot]$$

conserve F exactly

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(n's generate $O(4)$ transformations)

- **Reversible (ideal) part:** Poisson bracket technique

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conserve F exactly

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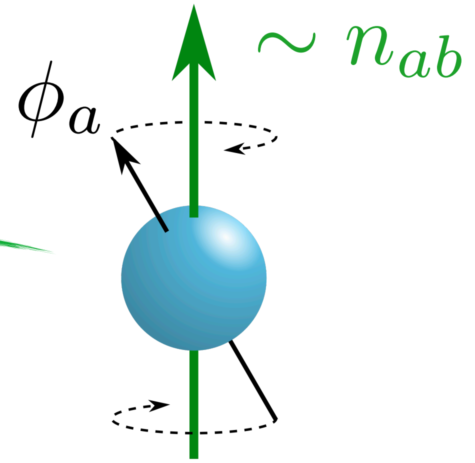

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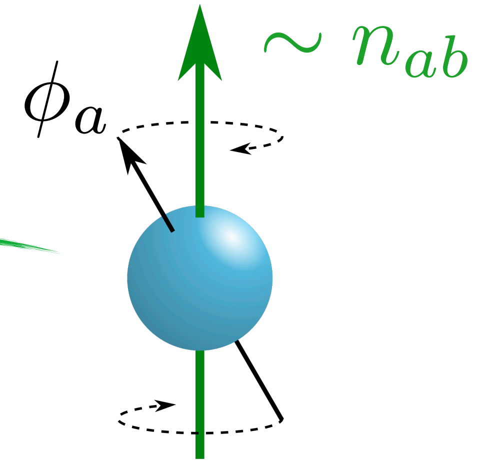

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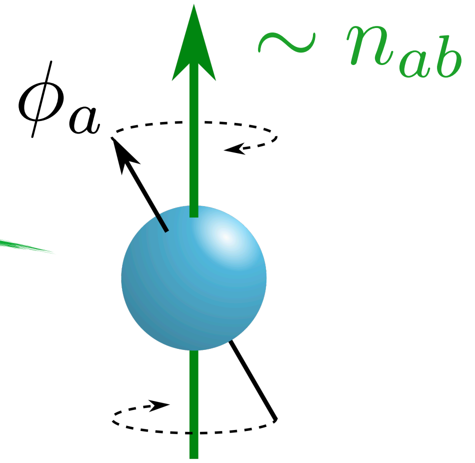

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Summary of Schwinger-Keldysh formalism

Goal: compute **non-equilibrium** correlation functions

⤳ Path integral requires **doubling number of fields:**

L.V. Keldysh, Sov. Phys. JETP **20** (1965) 1018

$$\langle O(t)\rangle = \mathrm{tr}\left(O(t)\rho_0\right)$$

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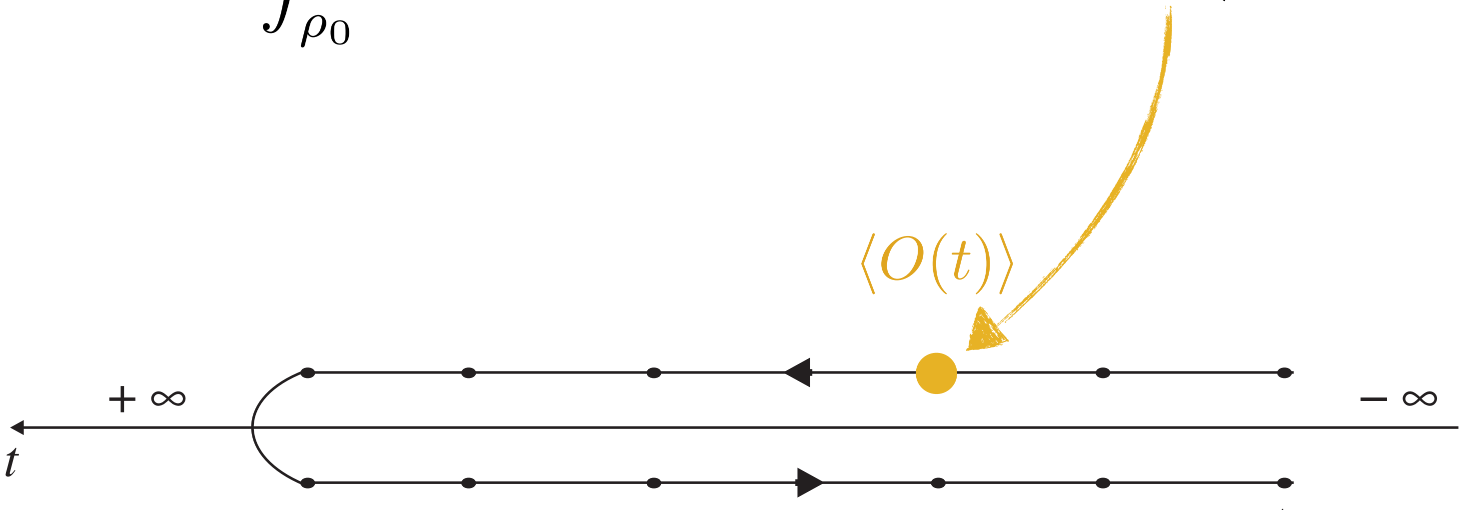

Figure adapted from
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closed-time path

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⤳ in particular: **direct access** to **real-time** Green functions

$$G^K(t,t') = i\langle\{\phi(t),\phi(t')\}\rangle$$

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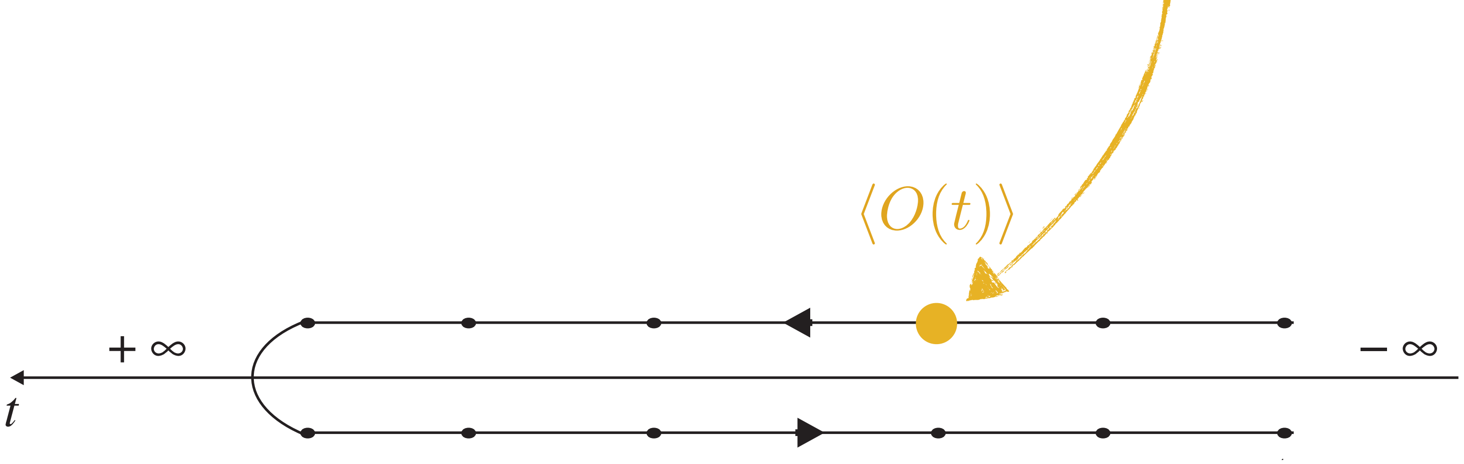

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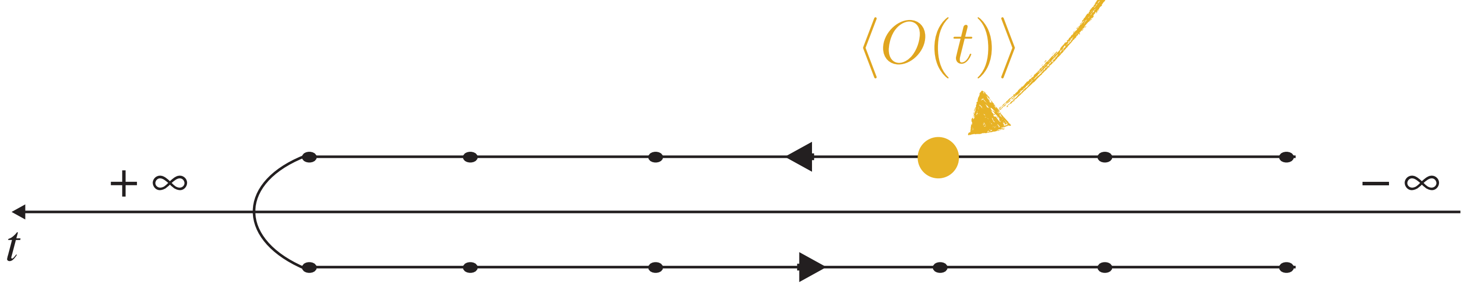

⇝ **Causal structure** built into the formalism!

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closed-time path

Summary of Martin-Siggia-Rose (MSR) formalism

in classical simulations:

solve Langevin equation

$$\partial_t^2 \varphi + \gamma \partial_t \varphi = -\frac{\delta F}{\delta \varphi} + \xi(x)$$

$$\langle \xi(x) \rangle = 0$$

$$\langle \xi(x) \xi(x') \rangle = 2\gamma T \delta(x - x')$$

However, we need:

Path-integral formulation

for (real-time) FRG

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Introduce (Hubbard) response field $\tilde{\varphi}$

However, we need:

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deterministic part of eom’s

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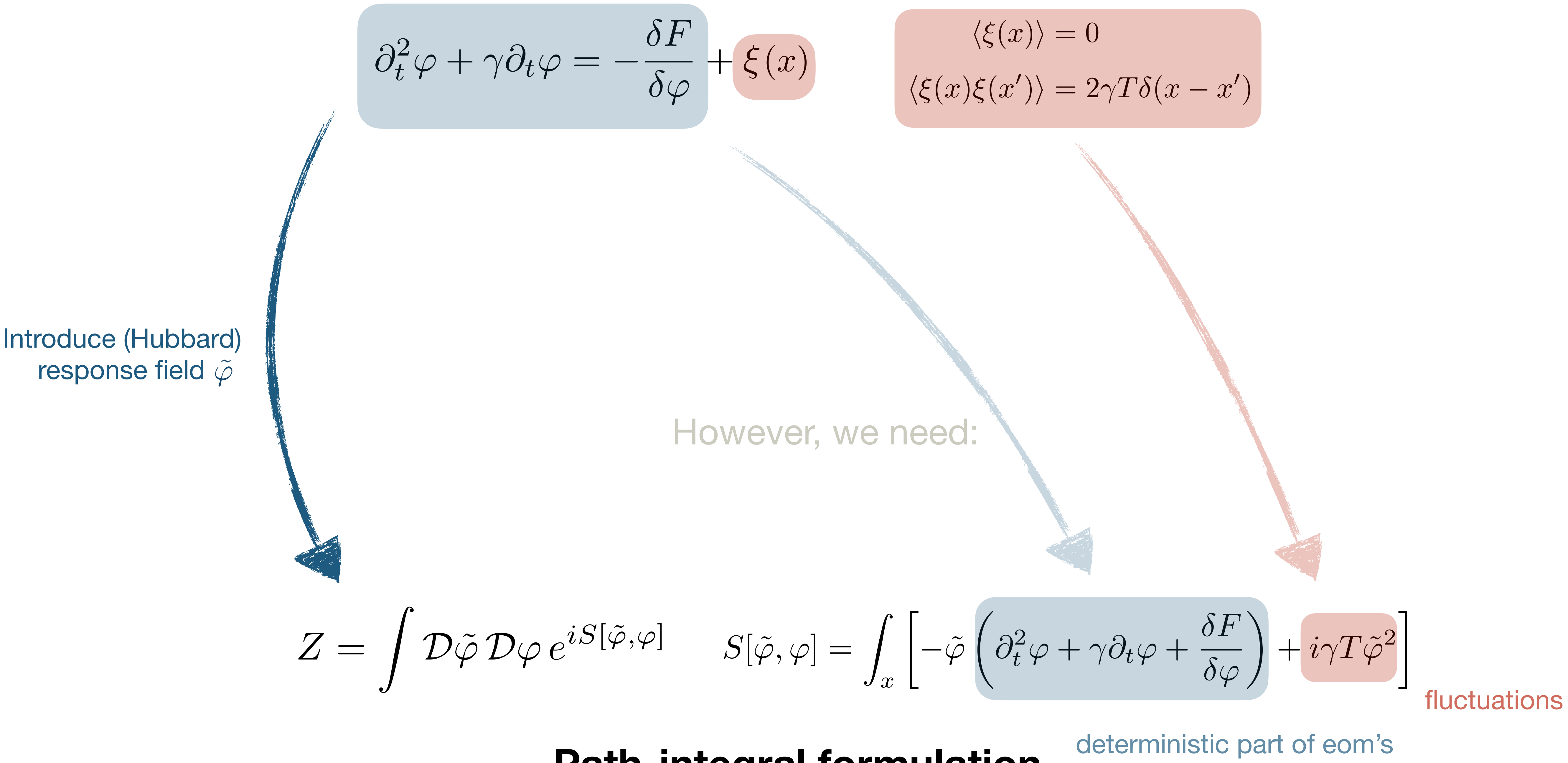

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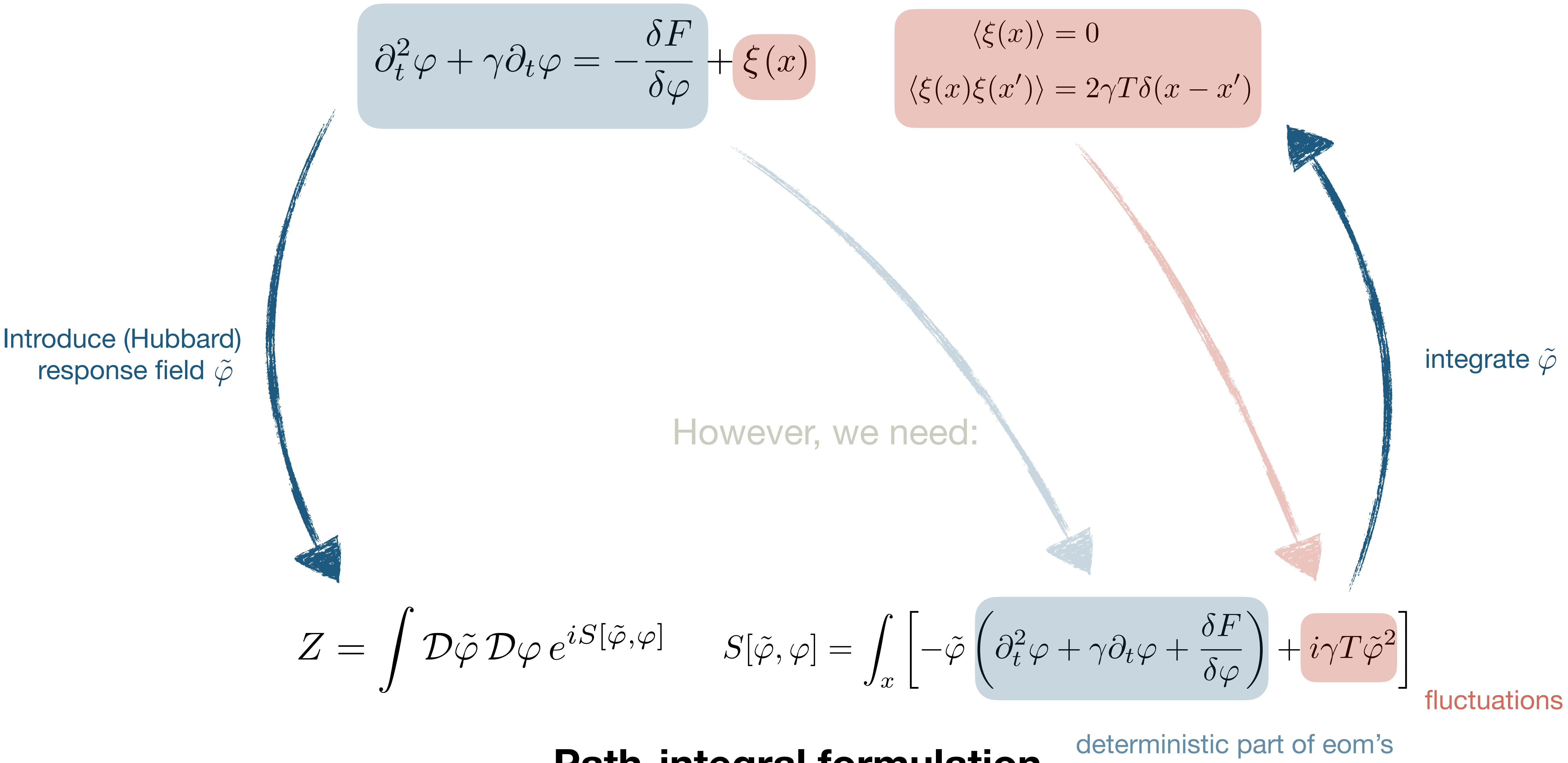

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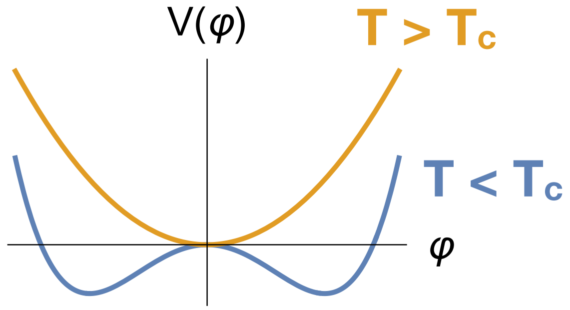
Dynamic universality classes in more detail

Statics: Landau-Ginzburg-Wilson functional

$$F = \int d^d x \, \left\{ \frac{1}{2} (\vec{\nabla} \varphi)^2 + V(\varphi) \right\}$$



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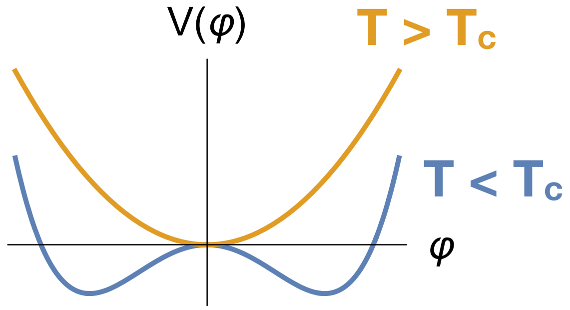

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Gaussian white noise

describes particle submerged in heat bath:

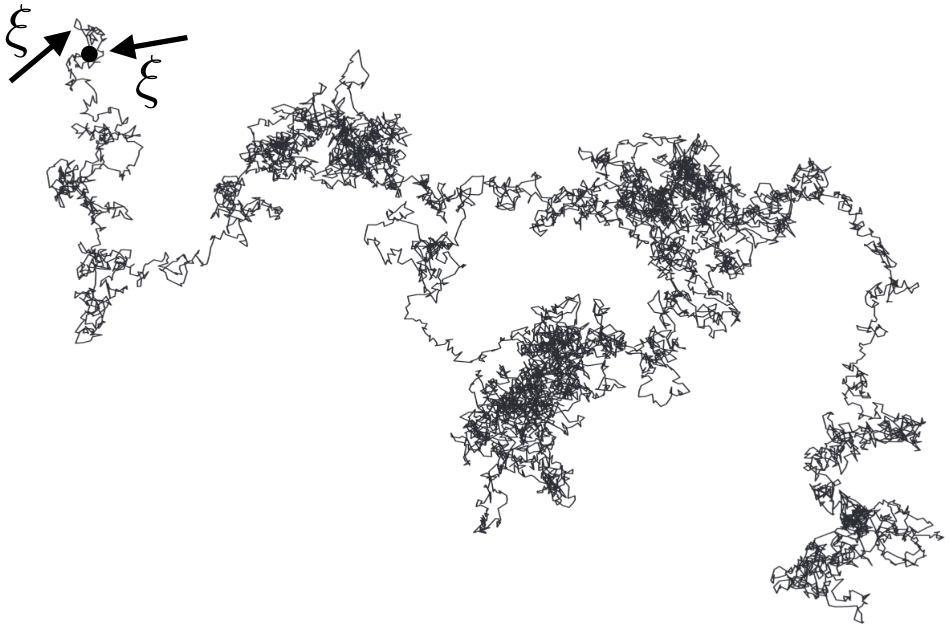

Image adapted from P. Mörters, Y. Peres, *Brownian Motion* (Cambridge University Press, 2010)

Dynamic universality classes in more detail

Model A
$z = 2 + c\eta$

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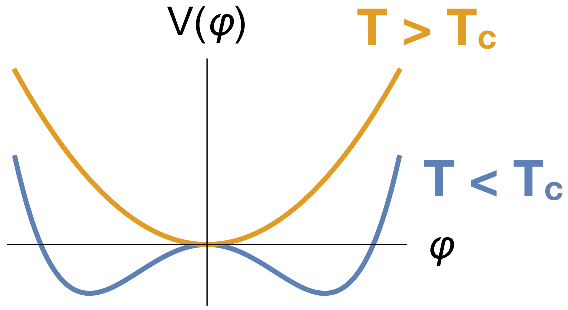

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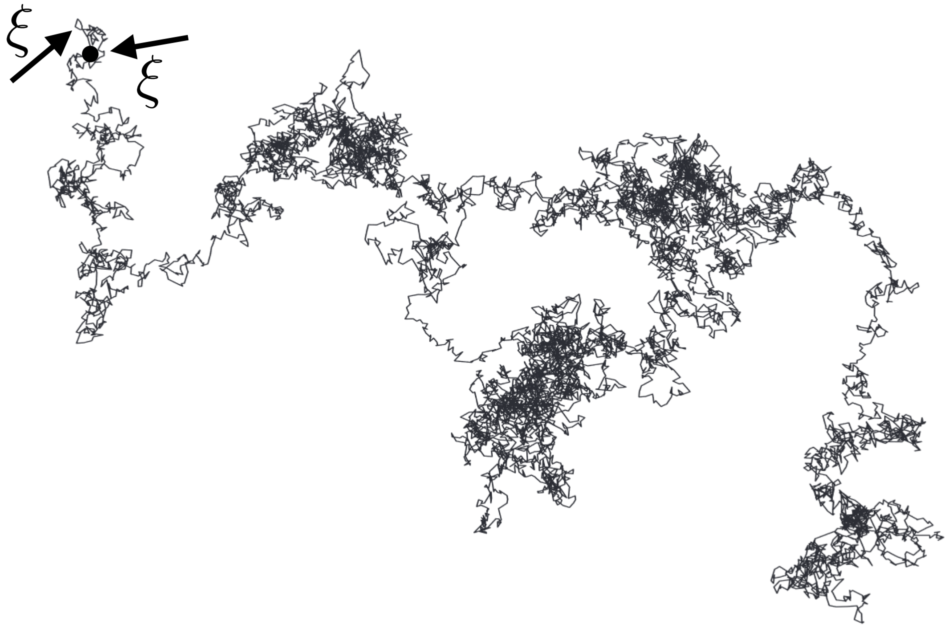

- No conservation laws here! ⇝ **Model A**
- **Slow modes** determine critical dynamics

(e.g. densities of conserved quantities) (generally true!)

Image adapted from P. Mörters, Y. Peres, *Brownian Motion* (Cambridge University Press, 2010)

Dynamic universality classes in more detail

Model B
$z = 4 - \eta$

Statics: Landau-Ginzburg-Wilson functional

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$$\partial_t n = \bar{\lambda} \vec{\nabla}^2 \frac{\delta F}{\delta n} + \vec{\nabla} \cdot \vec{\zeta}$$

diffusive!

- Critical dynamics dominated by diffusion ⤳ **Model B**
- Include hydrodynamic shear modes of energy-momentum tensor ⤳ **Model H**

Dynamic universality classes in more detail

Model C

$z = 2 + \alpha/\nu$

Statics: Landau-Ginzburg-Wilson functional

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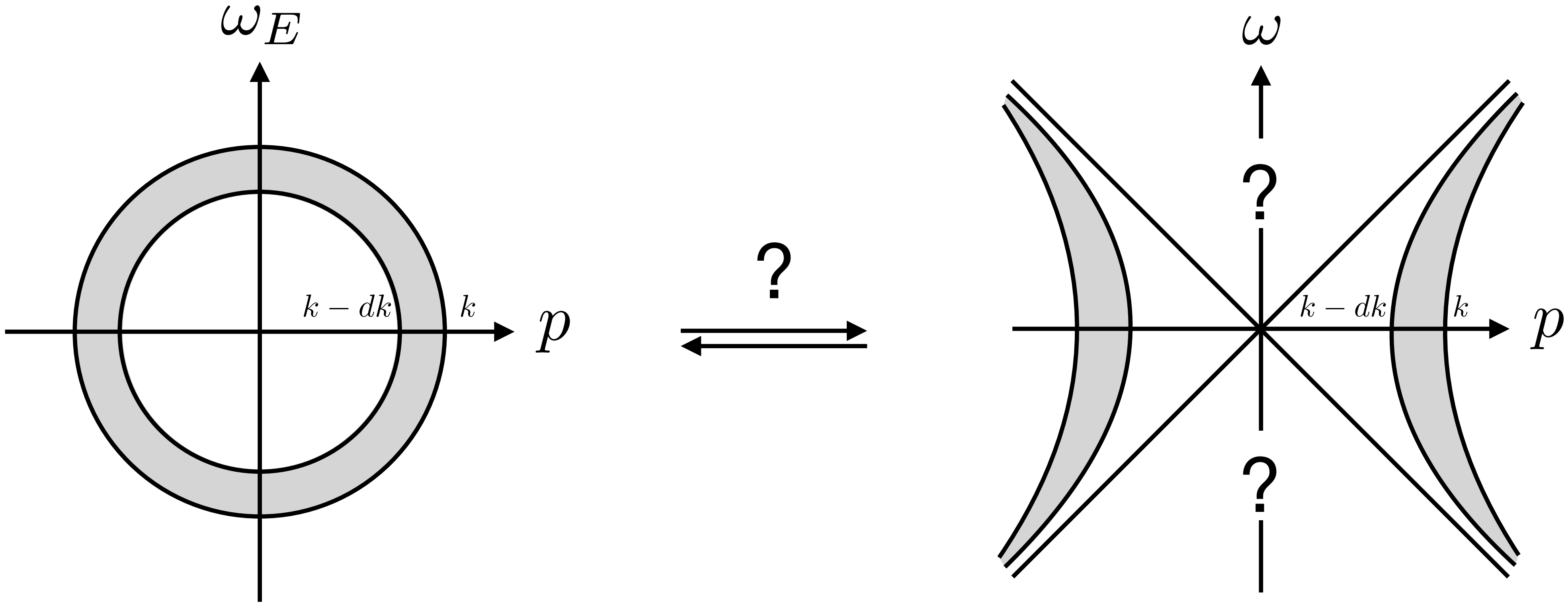
diffusive!

- Order parameter not conserved but interacts non-linearly with conserved (energy) density ⤳ **Model C**

Wilson’s RG in Minkowski spacetime?



Wilsonian renormalization in Euclidean spacetime

vs.

Wilsonian renormalization in Minkowski spacetime

Conceptually straightforward:
integrate out (hyper-)spheres
no need to worry about causality (at least naively)

Conceptually intricate:
integrate hyperboloids?
timelike momenta?
causal structure of propagators?
…

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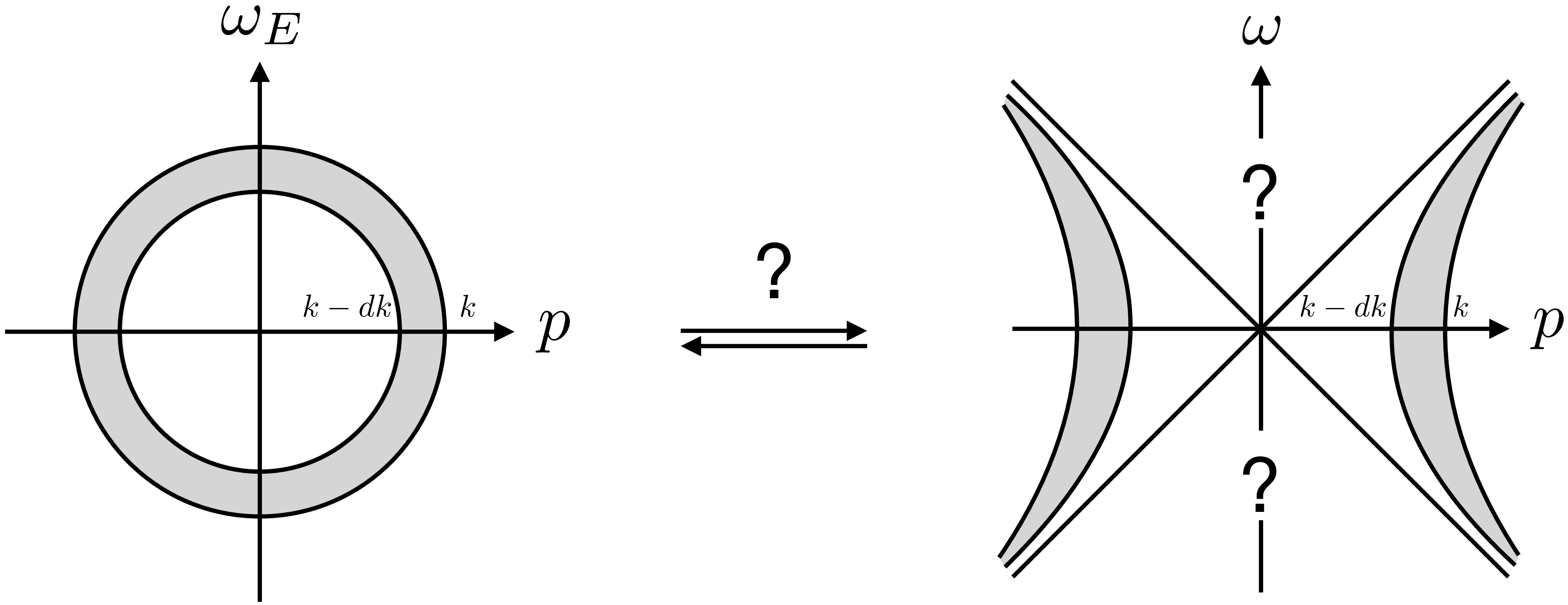

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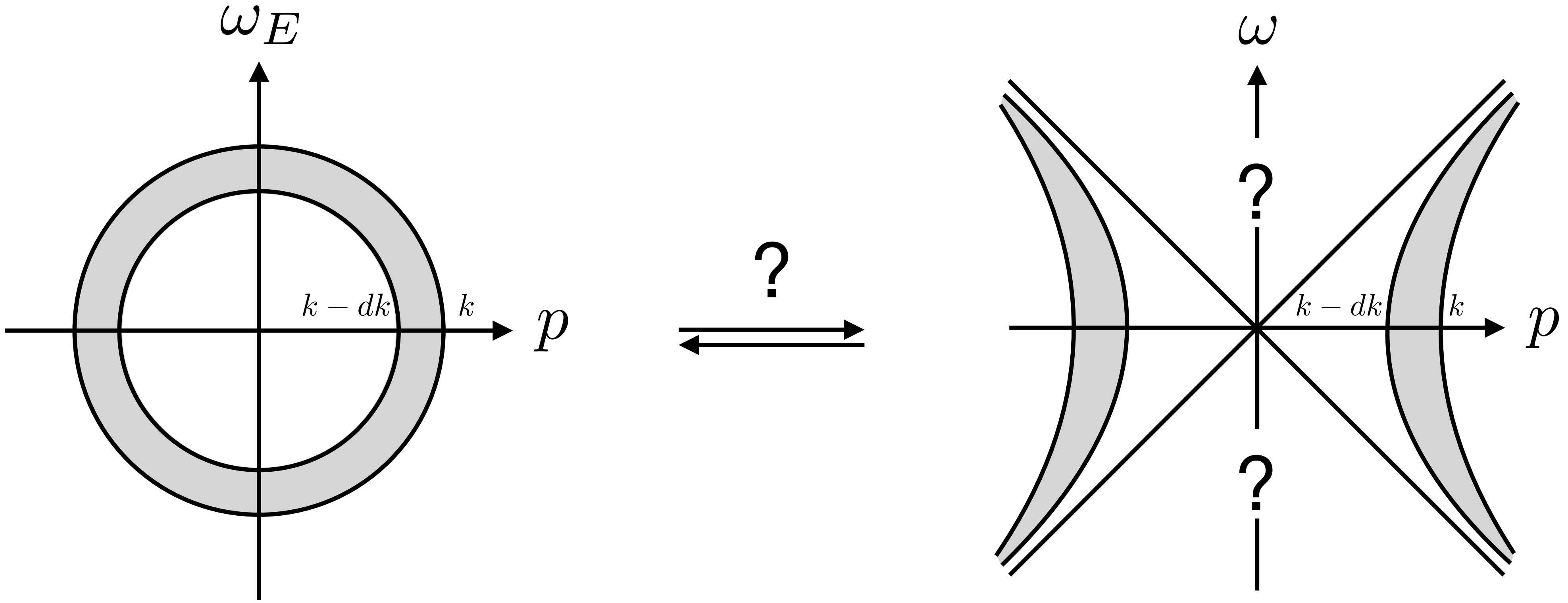

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Problem: Frequency-dependent regulators usually violate **causal structure**

Solution: Interpret regulator as fictitious scale-dependent heat bath

⇒ **Spectral representation**

JR, L. von Smekal, JHEP **10**, 065 (2023)

Causal regulators

Solution: Observe that regulator is a self-energy

- Self-energies generally inherit **causal structure**

⤳ **Spectral representation** from (subtracted) Kramers-Kronig relations

$$R_k^{R/A}(\omega,\boldsymbol{p}) = R_k^{R/A}(0,\boldsymbol{p}) - \int_0^\infty \frac{d\omega'}{2\pi}\,\frac{2\omega^2 J_k(\omega',\boldsymbol{p})}{\omega'((\omega \pm i\varepsilon)^2 - \omega'^2)}$$

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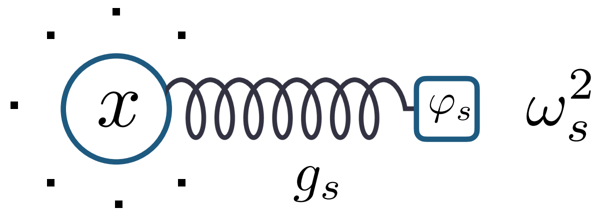
⤡

- Interpret as coupling to **fictitious heat bath:**

(Hubbard-Stratonovich transformation)



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⤳ Spectral density encodes **spectrum of bath oscillators**

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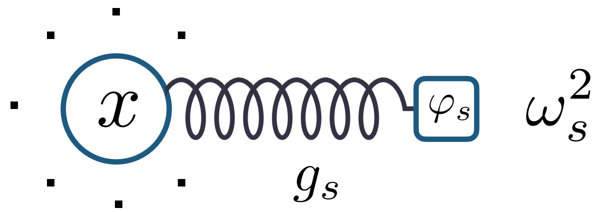


here: $J_k(\omega) = \pi \sum_s \frac{g_s^2(k)}{\omega_s(k)} \left(\delta(\omega - \omega_s(k)) - \delta(\omega + \omega_s(k))\right)$

⤳ **Physical** only for **positive-semidefinite** spectral densities $J_k(\omega, \boldsymbol{p}) \geq 0 \quad (\omega > 0)$

⤳ Spectral density encodes **spectrum of bath oscillators**

QM example for causal regulator

$$R_k^{R/A}(\omega) = R_k^{R/A}(0) - \int_0^\infty \frac{d\omega'}{2\pi} \frac{2\omega^2 J_k(\omega')}{\omega'((\omega \pm i\varepsilon)^2 - \omega'^2)} \quad \text{in} \quad \Gamma_k^{(2)\,R}(\omega) = (\omega + i\varepsilon)^2 - m^2 + R_k^R(\omega)$$

- spectral density: ⇝ **Regulator** (retarded part):

$$J_k(\omega) = 2k\omega\, e^{-\omega^2/k^2} = 2\,\mathrm{Im}\, R_k^R(\omega)$$

- assume UV finiteness:

$$\Delta M_{UV}^2(k) = -R_k^{R/A}(0) + \underbrace{\int_0^\infty \frac{d\omega'}{\pi} \frac{J_k(\omega')}{\omega'}}_{\geq 0} \overset{!}{=} 0$$

(positivity)

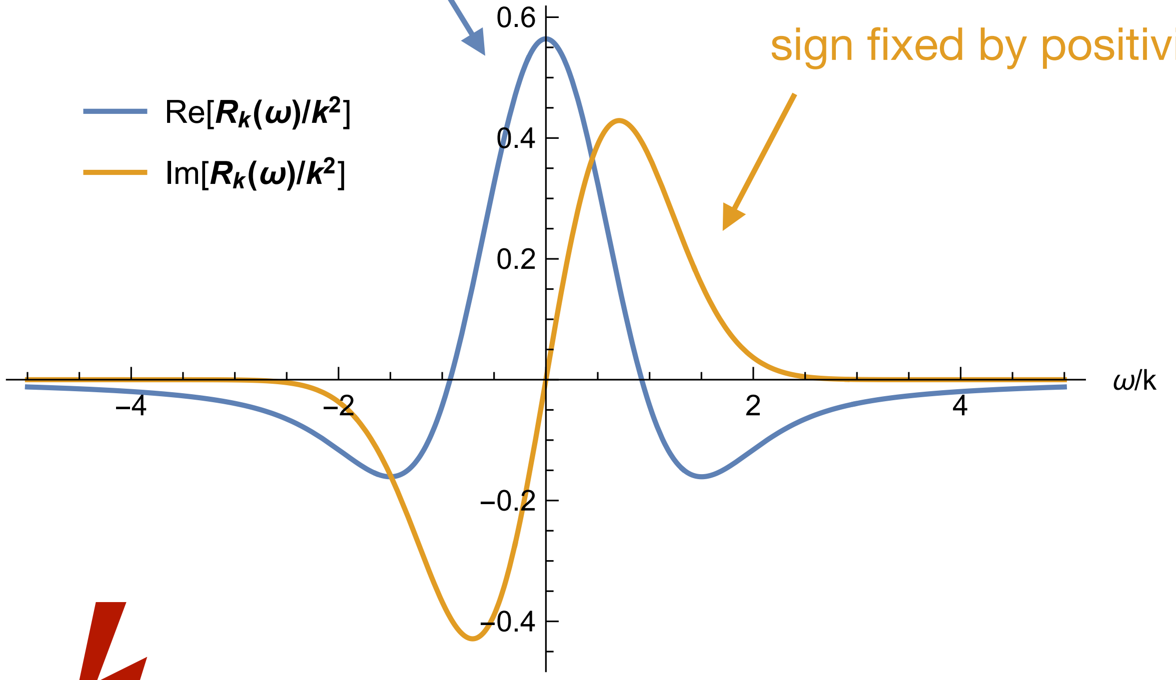
⇒ IR mass shift:

$$\Delta M_{IR}^2(k) = -R_k^{R/A}(0) < 0$$

is negative!

negative mass shift!

sign fixed by positivity



Solution: choose IR mass shift $\Delta M_{IR}^2(k) > 0$ positive (at cost of **UV finiteness**)