Deconfinement as percolation of electric center fluxes in QCD

Benasque, 12 February 2025

Lorenz von Smekal

Milad Ghanbarpour,
***Quark Numbers and Percolation in QCD*, PhD thesis, JLU, Dec. 2024**

PRD 106 (2022) 054513

HFHF Helmholtz Forschungsakademie Hessen für FAIR

CRC–TR 211 Strong-interaction matter under extreme conditions

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Hilbert Space

Kogut & Susskind, PRD 11 (1975) 395

- **Form of states:**

$$|\psi\rangle = \sum \left(f(U) \otimes |\psi_F\rangle \right)$$

↑ set of spatial link variables

- **Implement Gauss law (physical States):**

$$\hat{\rho}(\Omega)\,|\psi\rangle = |\psi\rangle$$

↑ generates spatial gauge transformations

- **transform at single site:**

$$\hat{\rho}(\Omega) \to \hat{\Pi}_i(\Omega) \prod_{j\sim i} \hat{\Pi}_{\langle i,j\rangle}(\Omega)$$

$$\big(\hat{\Pi}_{\langle i,j\rangle}(\Omega)\, f\big)(U) = \begin{cases} f(\{\dots,\Omega^\dagger U_{\langle i,j\rangle},\dots\}) \\ f(\{\dots,U_{\langle j,i\rangle}\Omega,\dots\}) \end{cases}$$

- **generated by:**

$\hat{Q}_i^a$ color charges

$\hat{E}^a_{\langle i,j\rangle}$ color-electric fluxes

Hilbert Space

- **local Gauss law:**

$$\hat{Q}_i^a = -\sum_{j\sim i} \hat{E}^e_{\langle i,j\rangle}$$

in physical states

but charges/fluxes not gauge invariant,
and don't commute

- **restrict to Z_3 center:**

$$\hat{Q}_i^z = \hat{\Pi}_i(z) \qquad \hat{E}^z_{\langle i,j\rangle} = \hat{\Pi}_{\langle i,j\rangle}(z) \qquad z \in \{1, e^{\frac{2\pi}{3}i}, e^{\frac{4\pi}{3}i}\}$$

gauge invariant and commute

- **local center charge and flux:**

$$q, e \in \{0, 1, 2\}$$

$$\hat{Q}_i^z \,|q,e\rangle = z^{q_i}\,|q,e\rangle \qquad \hat{E}^z_{\langle i,j\rangle}\,|q,e\rangle = z^{e_{\langle i,j\rangle}}\,|q,e\rangle$$

- **decompose:**

$$\mathcal{H} = \oplus_{\{q,e\}} \mathcal{H}_{\{q,e\}}$$

Physical States

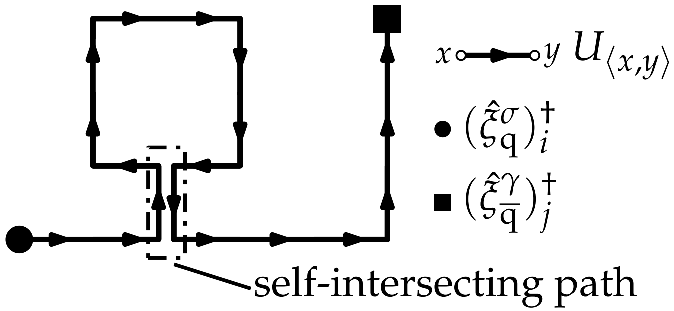
- **local Z_3 Gauss law:**

$$q_i + \sum_{j\sim i} e_{\langle i,j\rangle} = 0 \mod 3$$

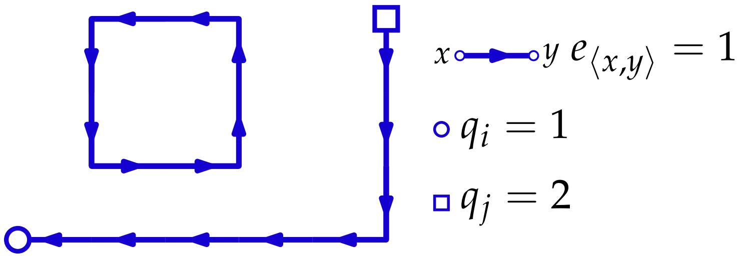
- **physical center charge / flux states:**

$$\mathcal{H}_{\text{phys}} = \oplus_{\{q,e\}_{\text{phys}}} \mathcal{H}_{\{q,e\}}$$

- **mesonic state:**



creation operator



center charge / flux configuration

Physical States

- project onto these sectors:

$$\hat{P}_i(q) = \frac{1}{3} \sum_{z \in Z_3} z^{-q} \, \hat{Q}_i^z \qquad \hat{P}_{\langle i,j \rangle}(e) = \frac{1}{3} \sum_{z \in Z_3} z^{-e} \, \hat{E}^z_{\langle i,j \rangle}$$

- use Z_3 Gauss law:

$$\underbrace{\prod_{i \in V} \hat{Q}_i^z}_{= \hat{Q}_V^z} |\psi\rangle = \underbrace{\prod_{\langle i,j \rangle \in \mathcal{S}^*} \hat{E}^z_{\langle j,i \rangle}}_{= -\hat{\Phi}^z_{\mathcal{S} = \partial V}} |\psi\rangle$$

to implement charges via fluxes

- define projection operator

flux *e* through $\mathcal{S} = \partial V$

$$\hat{P}_{\mathcal{S}}(e) = \frac{1}{3} \sum_{z \in Z_3} z^{-e} \hat{\Phi}^z_{\mathcal{S}}$$

Mack, PLB 78 (1978) 263

Kijowski & Rudolph, J. Math. Phys. 43 (2002) 1796; *ibid*. 46 (2005) 032303

Transfer Operator

- **partition function:**

Lüscher, Com. Math. Phys. 54 (1977) 283

Borgs & Seiler, Com. Math. Phys. 91 (1983) 329

Palumbo, NPB 645 (2002) 309

Mitrjushkin, NPB (PS) 119 (2003) 326

$$Z(\beta,\mu) = \mathrm{Tr}\left(e^{\beta\mu\hat{N}_q}\hat{T}^{N_4}\hat{P}_0\right)$$

project on gauge-invariant states (with Gauss' law)

$$\hat{P}_0\,|\psi\rangle = \int \mathcal{D}h\,\hat{\varrho}(h)\,|\psi\rangle$$

- **transfer operator:**

$$\hat{T}\big(f(U)\otimes|\psi_F\rangle\big) = \int \mathcal{D}U'\,K(U,U')\,\big(f(U')\otimes|\psi_F\rangle\big)$$

- **with kernel:**

$$K(U,U') = T_F^\dagger(U)T_G^\dagger(U)S(U,U')T_G(U')T_F(U')$$ symmetric, Lüscher

$$K(U,U') = S(U,U')T_G^\dagger(U')T_G(U')\tilde{T}_F(U')$$ asymmetric, Milad

same PI representation of partition function

hermitian (Wilson fermion) Hamiltonian in time-continuum limit

Transfer Operator

- **apply center-flux operator:**

$$\big(\hat{E}^z_{\langle i,j\rangle} K\big)(U,U') = S(U^z,U')T_G^\dagger(U')T_G(U')\tilde{T}_F(U')$$

only acts on spatial link variables here, $U^z = \begin{cases} \{\dots, z^* U_{\langle i,j\rangle}, \dots\} \\ \{\dots, U_{\langle j,i\rangle} z, \dots\} \end{cases}$

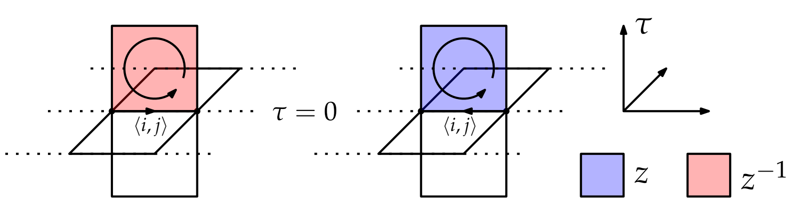
- **EVs of center-flux configurations:**

$$\Big\langle \prod_{\langle i,j\rangle\in\mathcal{S}^*} \hat{E}^z_{\langle i,j\rangle} \Big\rangle = \frac{1}{Z}\mathrm{Tr}\Big(\Big[\prod_{\langle i,j\rangle\in\mathcal{S}^*} \hat{E}^z_{\langle i,j\rangle}\Big] e^{\beta\mu\hat{N}_q}\hat{T}^{N_4}\hat{P}_0\Big)$$

single plaquette flip for $\langle \hat{E}^z_{\langle i,j\rangle}\rangle$

$$= \int \mathcal{D}[\dots]\, e^{-S_G^z(U,\{z\})}\, e^{-S_F(\overline{\psi},\psi,U,\mu)}$$

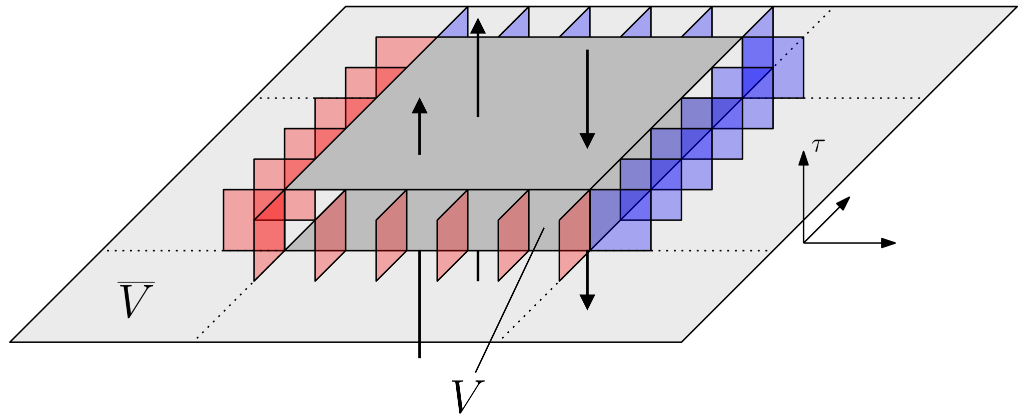
flip all temporal plaquettes $U_p \to z^* U_p$ with spatial link in $\mathcal{S}^*$



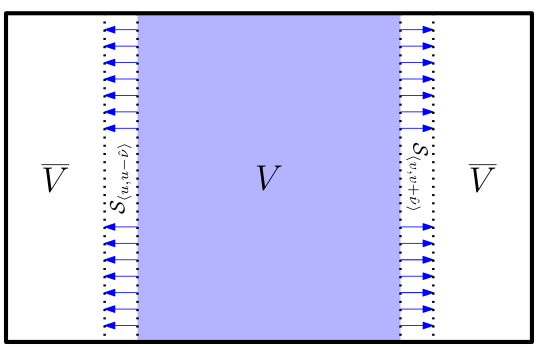
$\langle i,j\rangle$: forward backward link

Closed Center Vortex Sheets

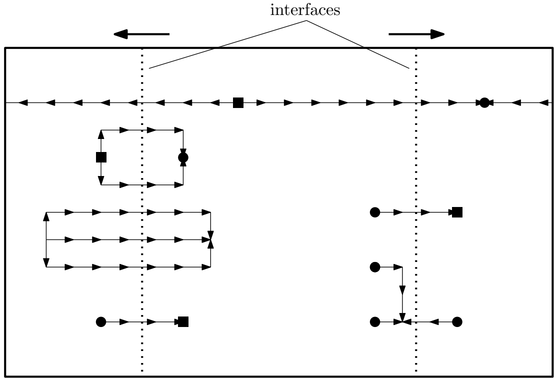
- **pure gauge theory**
  remove with variable transform
- **heavy-dense limit of QCD**
  static fermion determinant
- ***Z*$_3$-Fourier transform over closed center vortex sheets**



fix electric flux through $S = \partial V$

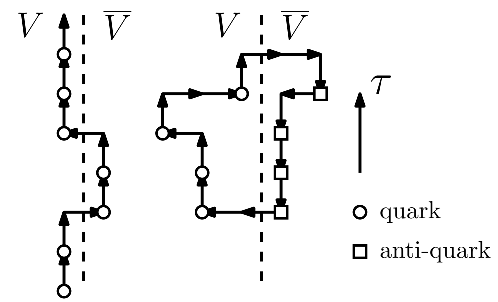


or net quark number mod. 3 inside



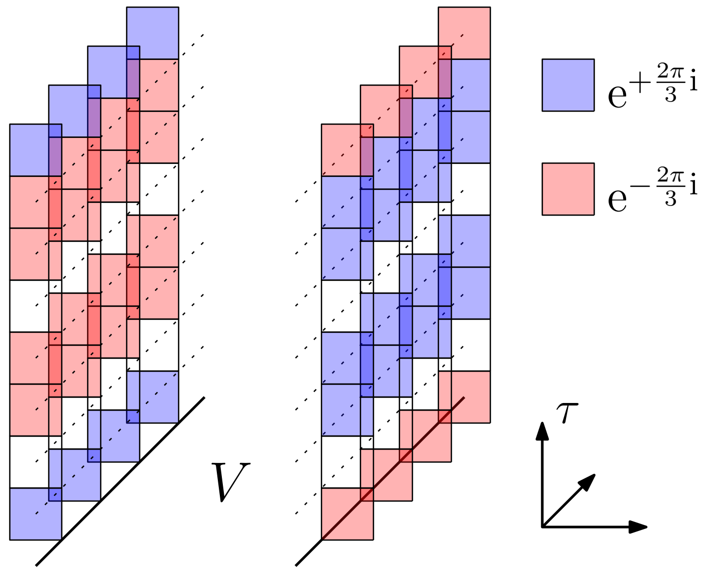
Quarks in a Finite Volume

- **with arbitrary spatial hops**
  (anti-)quarks/diquarks can hop in and out of V



- **introduce between *all* time slices**
  N_τ closed center-vortex sheets

- **Z_3-Fourier transforms**
  over N_τ closed center-vortex sheets
  → selective static membrane at $S = \partial V$
  (only hadrons can pass)



Quarks in a Finite Volume

- **fix charge in *V***

Ghanbarpour, LvS, PRD 106 (2022) 054513

$$Z(q_V \mod 3 = e) = \frac{1}{3^{N_\tau}} \sum_{\{z_\tau \in Z_3\}} \left[\prod_{\tau=1}^{N_\tau} z_\tau^{-e} \right] Z(\{z_\tau\})$$

↑

total charge (net quark number) modulo 3 in sub-volume *V*, write $q_V =_3 e$

with

$$Z(\{z_\tau\}) = \int \mathcal{D}[\ldots]\, \mathrm{e}^{-S_G(\{z_\tau\},U) - S_F(U,\overline{\psi},\psi)}$$

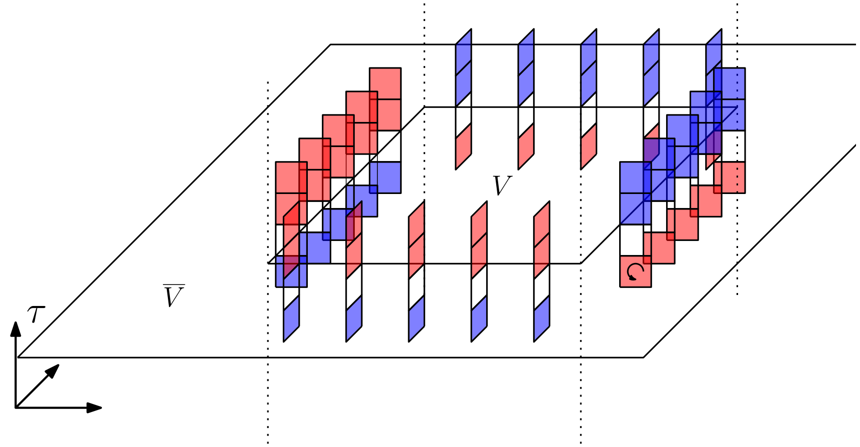
↑

only in gauge action

- **twisted plaquette action**

$$S_G(\{z_\tau\}, U) = -\frac{2}{g^2} \sum_p \mathrm{ReTr}\big(z(p) U_p\big)$$

$$z(p_{(i,\tau),\mu\nu}) = \begin{cases} z_\tau, & \nu = 4, \mu = k, \langle i, i+\hat{k} \rangle \in S^* \\ z_\tau^{-1}, & \nu = 4, \mu = k, \langle i+\hat{k}, i \rangle \in S^* \\ 1, & \text{otherwise} \end{cases}$$



- **effective Polyakov-loop theory**

(1 flavor Wilson)

$$Z_{\text{eff}} = \int \Big(\prod_i \mathrm{d}L_i \, J(L_i) \, Q(L_i) \Big) \prod_{\langle i,j \rangle} \big(1 + 2\lambda \, \mathrm{Re} L_i L_j^*\big)$$

Fromm, Langelage, Lottini, Philipsen, JHEP 01 (2012) 042
Langelage, Neuman, Philipsen, JHEP 09 (2014) 131

leading order hopping expansion

static fermion determinat → site factors

$$Q(L) = \big(1 + hL + h^2 L^* + h^3\big)^2 \big(1 + \bar{h} L^* + \bar{h}^2 L + \bar{h}^3\big)^2$$

where

Pietri, Feo, Seiler, Stamatescu, PRD 76 (2007) 114501

$$h(\mu) = e^{(\mu - m)/T}$$

$$\bar{h}(\mu) = h(-\mu)$$

Effective Potts Model

- **for QCD at strong coupling**

  with static fermion determinant

$$Z_{\text{eff}} = \mathcal{N} \sum_{\{z_i \in Z_3\}} \exp\left\{ \sum_{\langle i,j \rangle} 2\gamma \, \text{Re}\, z_i z_j^* \right\} \times$$

$$\left(\prod_i \left(1 + h z_i + h^2 z_i^* + h^3\right)^2 \left(1 + \bar{h} z_i^* + \bar{h}^2 z_i + \bar{h}^3\right)^2 \right)$$

with $\gamma = \frac{1}{3} \ln \left(\frac{1+2\lambda}{1-\lambda} \right)$

- **Roberge-Weiss symmetric**

  from global Z_3 symmetry

$$Z_{\text{eff}}(T, \mu = i\theta T) \equiv Z^I_{\text{eff}}(\theta) = Z^I_{\text{eff}}(\theta + 2\pi/3)$$

Equivalent Flux-Tube Model

- **flux-tube model representation (dual)**

Ghanbarpour, LvS, PRD 106 (2022) 054513

$$Z_{\text{eff}}(T,\mu) = \sum_{\{n,l\}_{\text{phys}}} \exp\left\{-\beta\Big(H(n,l) - \mu \sum_i q_i\Big)\right\}$$

analogous to:

Patel, NPB 243 (1984) 411

Bernard, DeGrand, DeTar, Gottlieb, Krasnitz, Sugar, Toussain, PRD 49 (1994) 6051

Condella & DeTar, PRD 61(2000) 074023

here with:

string tension

$$H(n,l) = \sum_{\langle i,j\rangle} \sigma |l_{\langle i,j\rangle}| + \sum_{i,s} m(n_{i,s} + \bar{n}_{i,s})$$

fluxes represented by link variables: $l_{\langle i,j\rangle} \in \{-1,0,1\}$

(anti-)quark occupation numbers: $n_{i,s} \in \{0,\ldots,3\}$ and $\bar{n}_{i,s} \in \{0,\ldots,3\}$ spin $s = \{\uparrow,\downarrow\}$

- ***Z*₃-Gauss' law:**

(Poisson equation)

$$\sum_{j\sim i} l_{\langle i,j\rangle} - \sum_s (n_{i,s} - \bar{n}_{i,s}) = 0 \mod 3$$

flux from volume around site *i*

$$\phi_i = q_i \mod 3$$

net-quark number modulo 3

Ensembles with Fixed Flux

- **Z_3-Fourier transform**

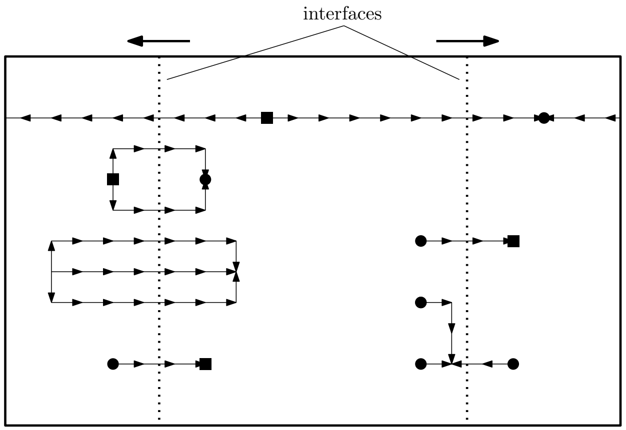
$$Z(q_V =_3 e) = \frac{1}{3} \sum_{z \in Z_3} z^{-e} \, Z_S(z)$$

Z_3-flux ensembles **Z_3-interface ensembles**

- **interface ensembles**

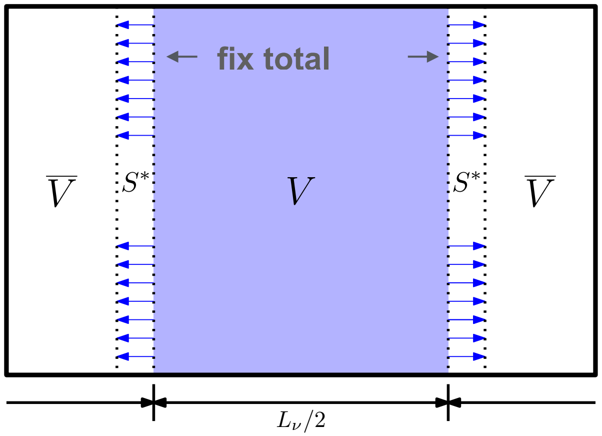
$$Z_S(z) = \sum_{\{z_i \in Z_3\}} \exp\Big\{ \sum_{\langle i,j\rangle} 2\gamma \, \mathrm{Re}\left(z^{-s_{\langle i,j\rangle}} \, z_i z_j^*\right) \Big\} \prod_i Q(z_i)$$

$$s_{\langle i,j\rangle} = \begin{cases} 1, & \langle i,j\rangle \in \mathcal{S}^* \\ -1, & \langle j,i\rangle \in \mathcal{S}^* \\ 0, & \text{otherwise} \end{cases}$$



total flux $e = 2 = q_V$

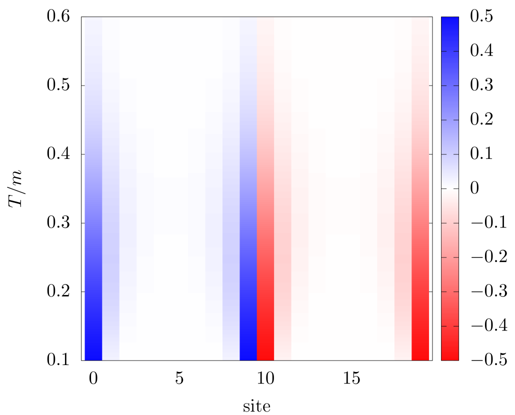
$\Leftrightarrow$



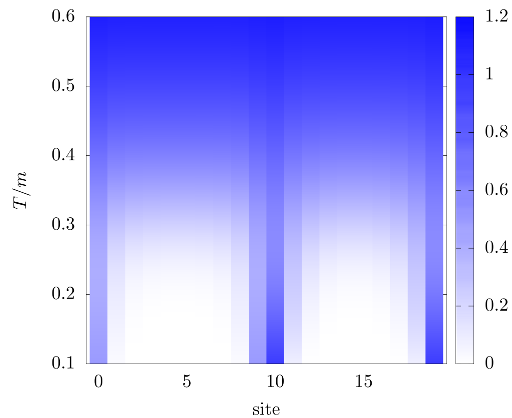
Test in (1+1)d

- **net quark number density**

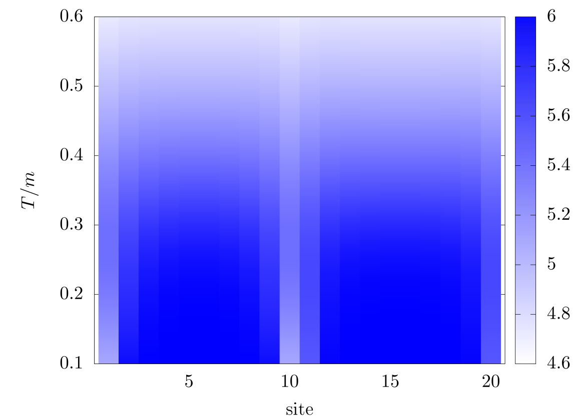
$L = 20\,, \quad \sigma a/m = 0.3$



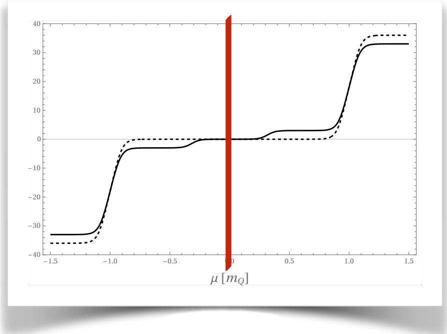
(a) $\mu/m = 0$ (mesonic)

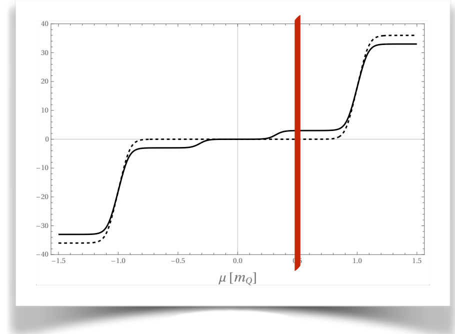


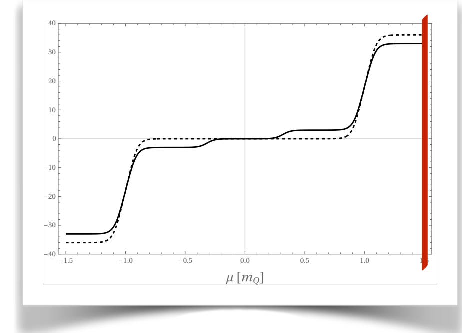
(b) $\mu/m = 0.5$ (baryonic)



(c) $\mu/m = 1.5$ (saturation)

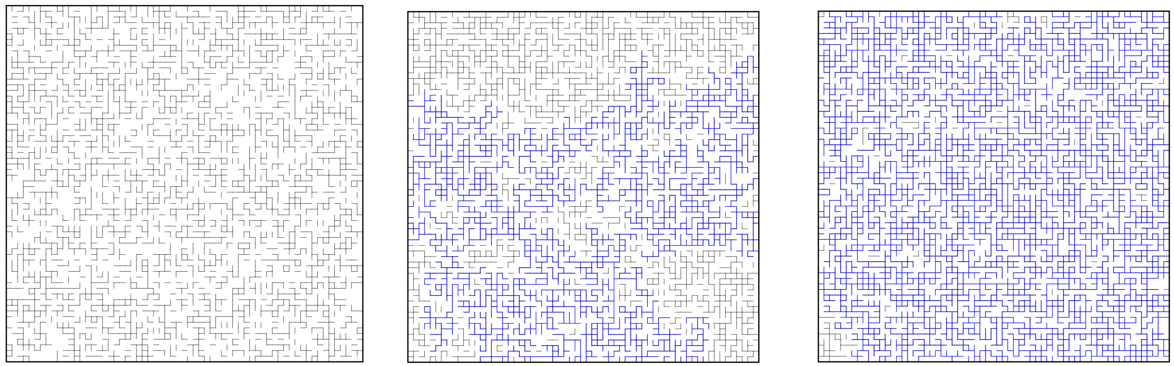






Bond Percolation

- **place bonds randomly:**



(a) $p = 0.4$ (b) $p = p_c = 0.5$ (c) $p = 0.6$

- **find spanning cluster:**

with probability

$$R_1(p, N) = \phi\big(A\,(p - p_c)N^{1/\nu}\big)$$

in two dimensions

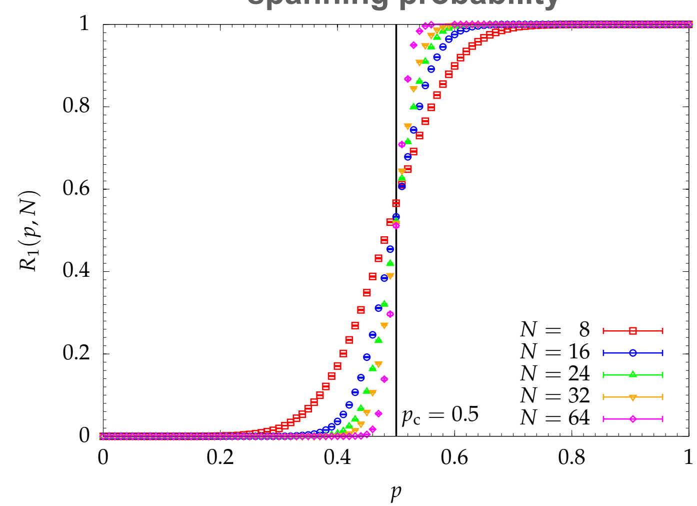
$$p_c = 1/2 \qquad \nu = 4/3$$

in three dimensions

$$p_c = 0.24881182(10) \qquad \nu^{-1} = 1.1410(15)$$

Wang, Zhou, Zang et al., PRE 87 (2013) 052107

spanning probability



Percolation in QCD

- **expectation value:**

  electric center-flux through link $\langle i,j \rangle$

$$\left\langle \hat{E}^z_{\langle i,j\rangle} \right\rangle = \frac{1}{Z} \mathrm{Tr}\Big(\hat{E}^z_{\langle i,j\rangle} \, e^{\beta\mu \hat{N}_q} \hat{T}^{N_4} \hat{P}_0 \Big)$$

- **probability:**

  of obtaining value $e \in \{0,1,2\}$

$$p(e_{\langle i,j\rangle}) = \left\langle \hat{P}_{\langle i,j\rangle}(e) \right\rangle = \frac{1}{3} \sum_{z\in Z_3} z^{-e} \left\langle \hat{E}^z_{\langle i,j\rangle} \right\rangle$$

$$= \frac{1}{3} \sum_{z\in Z_3} z^{-e_{\langle i,j\rangle}} \left\langle e^{\frac{2}{g^2} \mathrm{ReTr}\big([z^* - 1] U_p\big)} \right\rangle$$

- **bond probability:**

$$p_b = 1 - p(e_{\langle i,j\rangle} = 0)$$

$$= \frac{2}{3} \left\langle 1 - \cosh\left(\frac{\sqrt{3}}{g^2} \mathrm{ImTr}\, U_p \right) e^{-\frac{3}{g^2} \mathrm{ReTr}\, U_p} \right\rangle$$

Bond Probability in QCD

- **strong-coupling limit:** $p_b \to 0$ $< p_c$, never have percolation, confinement

- **at weak coupling, high *T*:**

$$p_b \to \frac{N_c - 1}{N_c} = \begin{cases} 1/2\,, & N_c = 2 \\ 2/3\,, & N_c = 3 \\ 1\,, & N_c \to \infty \end{cases}$$

asymptotically larger than p_c
in all cases, percolating electric fluxes
deconfinement

- **spanning probability:**

$$R_1(T,\mu,L) = \sum_{\{q,e\}\in\mathcal{R}_1} \frac{1}{Z} \mathrm{Tr}\Big(\hat{P}_{\{q,e\}}\, e^{\beta\mu\hat{N}_q} \hat{T}^{N_4} \hat{P}_0\Big)$$

Percolation in Potts Model

- ***q*-state Potts, Boltzmann factor:**

$$\omega(\{s,b\}) = \prod_{\langle i,j\rangle} \left(e^{-K}\delta_{b_{\langle i,j\rangle},0} + (1-e^{-K})\delta_{b_{\langle i,j\rangle},1}\delta_{s_i,s_j}\right) \prod_i e^{h\delta_{s_i,0}}$$

site-bond representation

Edwards & Sokal, PRD 38 (1988) 2009

- **place bond:** $b_{\langle i,j\rangle} \in \{0,1\}$ with probability $1-e^{-K}$

  between like nearest-neighbor spins $s_i \in \{0,1,\dots q-1\}$

- **infinite external field:** $h \to \infty$ $\rightsquigarrow$ bond percolation

  with bond probability $p = 1-e^{-K}$, $K = J/T$ controlled by temperature

- **vanishing external field:** $h \to 0$,

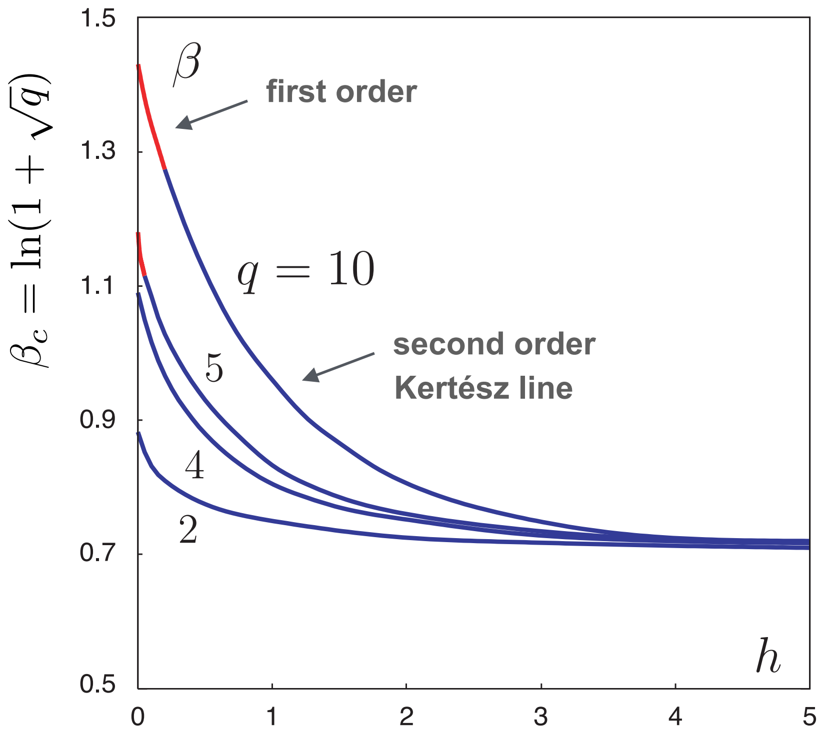
  if $p = p_c$ at $T = T_b > T_c$ $\rightsquigarrow$ bond percolation in ordered phase below T_c

  lose at Curie temperature T_c

Percolation in Potts Model

- ***q*-state Potts, 2 dimensions:**

Blanchard, Gandolfo, Laanait, Ruiz, Satz, J. Phys. A 41 (2008) 085001



$$\beta_b = 1/T_b = \ln 2$$

$$(p_c = 1/2)$$

Percolation in Flux-Tube Model

- spanning probability:

$$R(T,\mu,L) = \frac{1}{Z_{\text{flux}}} \sum_{\{n,l\}\in\mathcal{R}} \exp\big\{-\beta\big(H(n,l)-\mu q\big)\big\}$$

set of percolating configs $\mathcal{R}$: contain at least one cluster of bond configurations spanning the entire volume in at least one direction

- simulate with worm algorithm

Prokof'ev & Svistunov, PRL 87 (2001) 160601
Korzec & Vierhaus, 2011, CPC 182 (2011) 1477
Delgado, Evertz, Gattringer, CPC 183 (2012) 1920
Rindlisbacher, Akerlund, de Forcrand, NPB (2016) 542

- measure with fully-dynamic connectivity algorithm

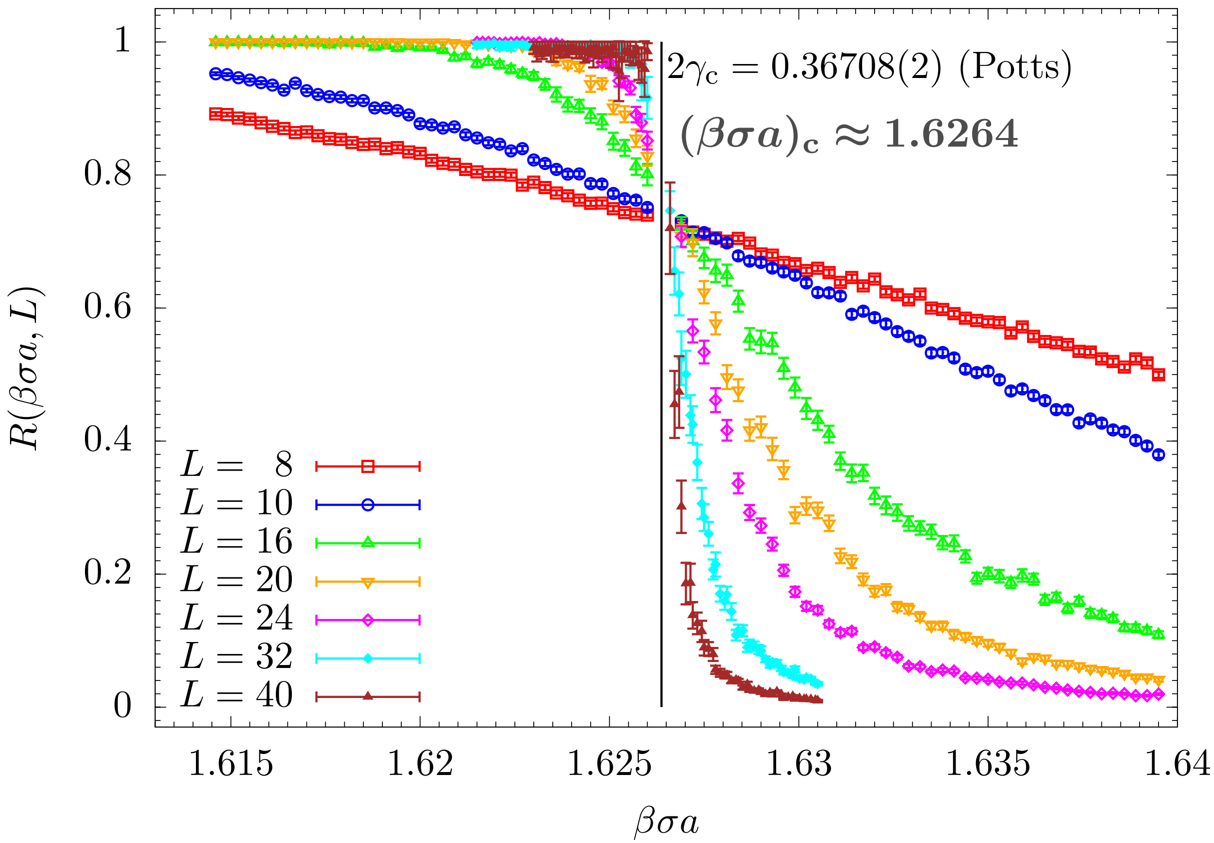
Holm, Lichtenberg, Thorup, J. ACM 48 (2001) 723
Alexandru, Bergner, Schaich, Wenger, PRD 97 (2018) 114503

Percolation in Flux-Tube Model

- **infinitely heavy quarks**

  Z_3-Potts (1st order transition)

$m \to \infty, \ \mu = 0$



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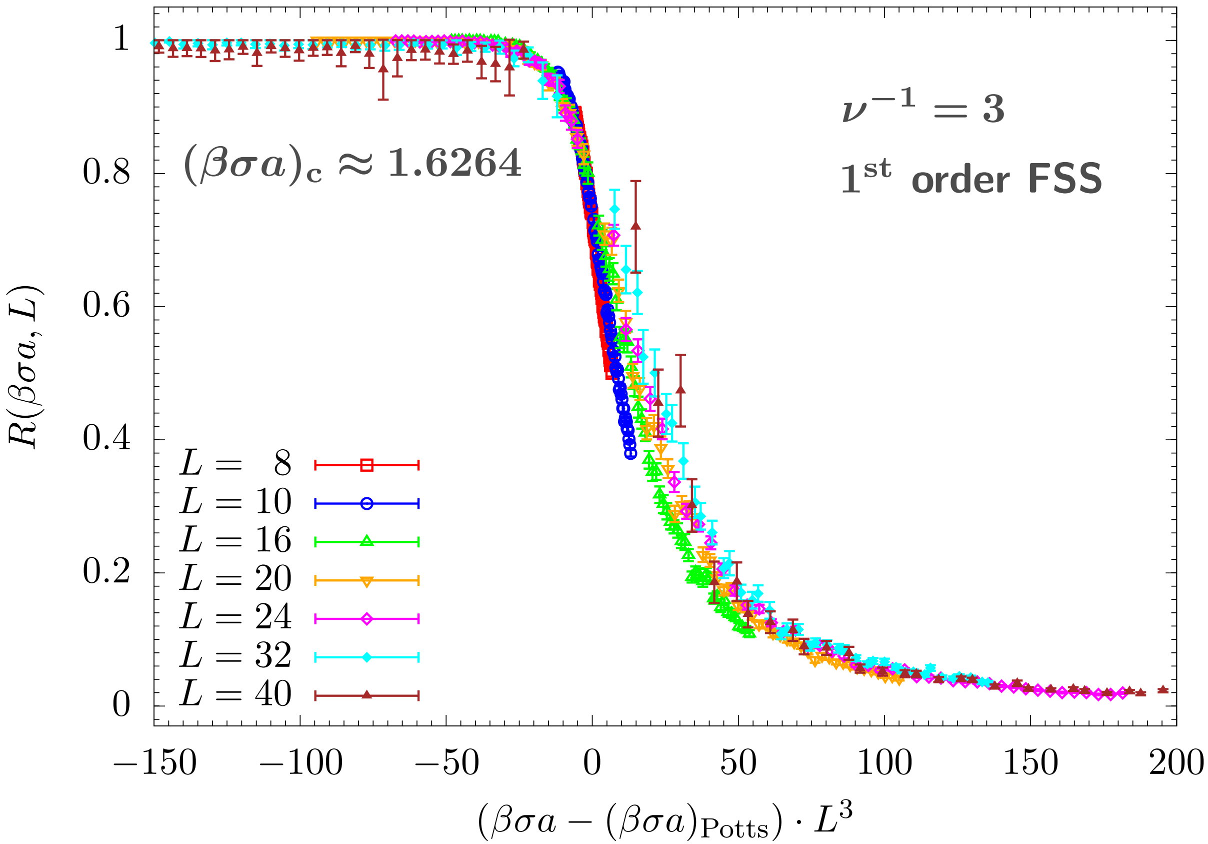
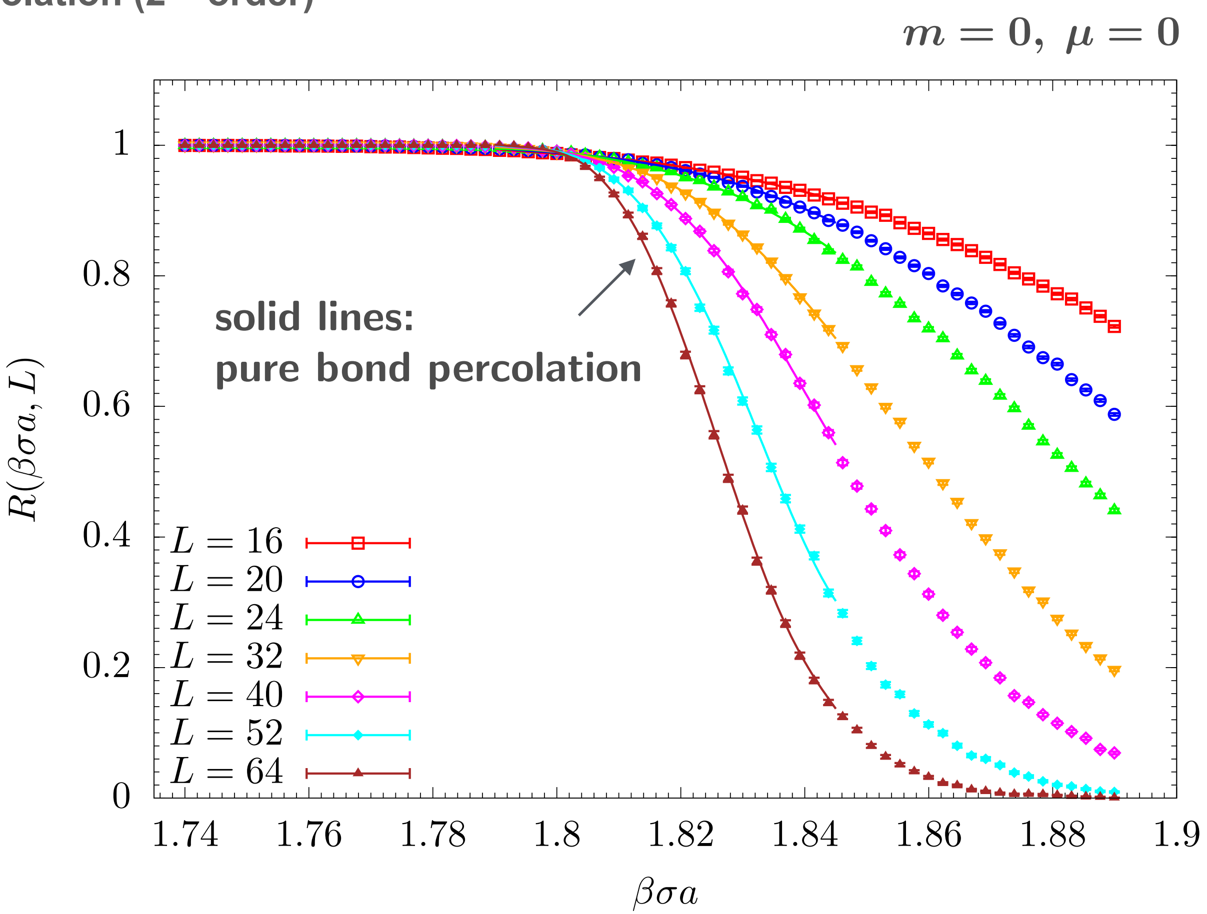


Percolation in Flux-Tube Model

- **massless limit**

  bond percolation (2nd order)



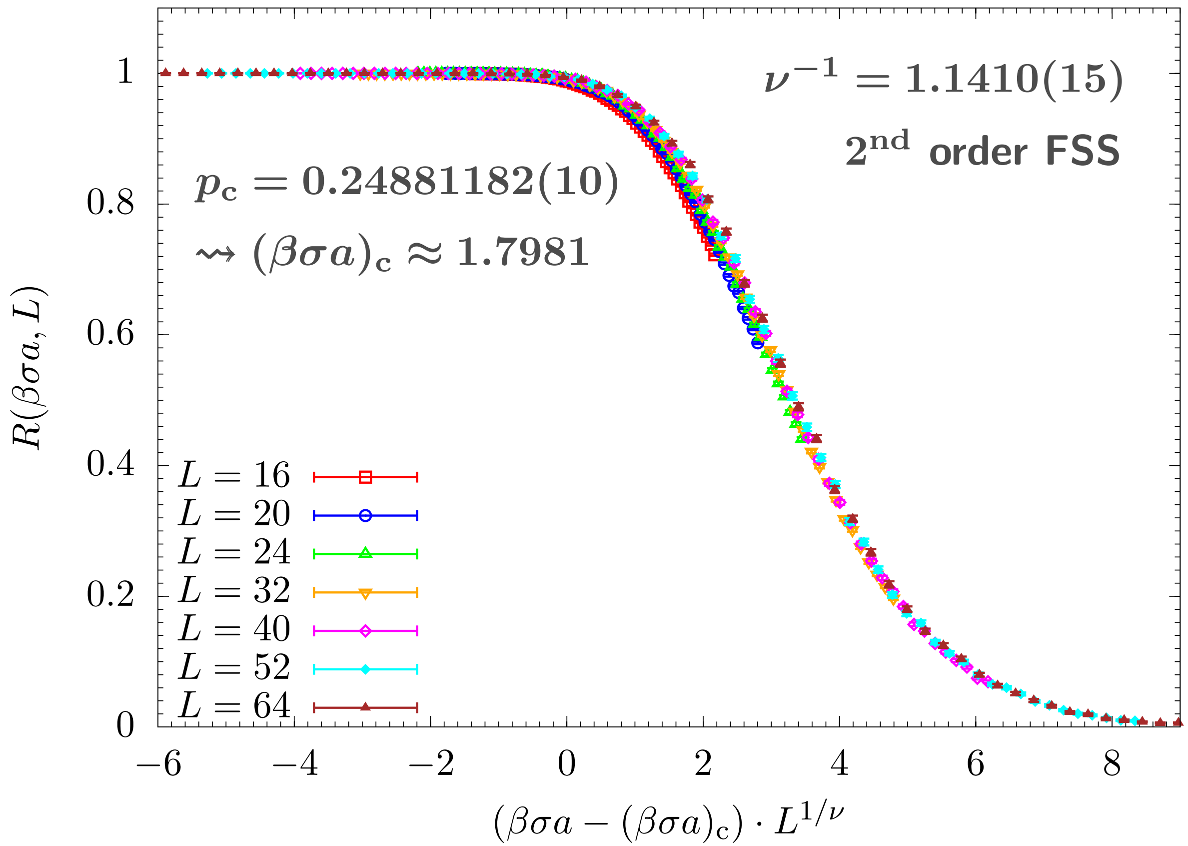
Percolation in Flux-Tube Model

- **massless limit**

  bond percolation (2nd order)

Wang, Zhou, Zhang et al., PRE 87 (2013) 052107

$m = 0,\ \mu = 0$

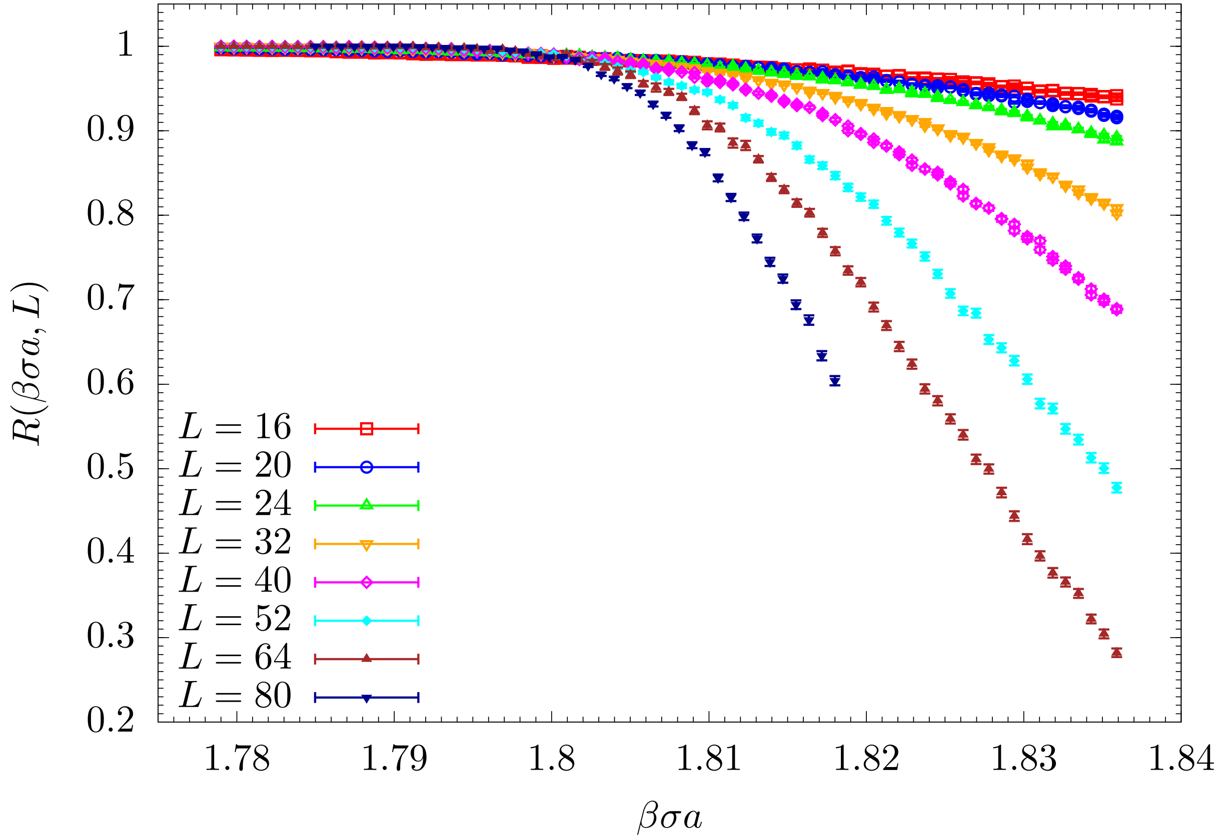


Percolation in Flux-Tube Model

- **fairly light quarks**

  smooth Z_3-Potts crossover

$m = \sigma a/6$



Percolation in Flux-Tube Model

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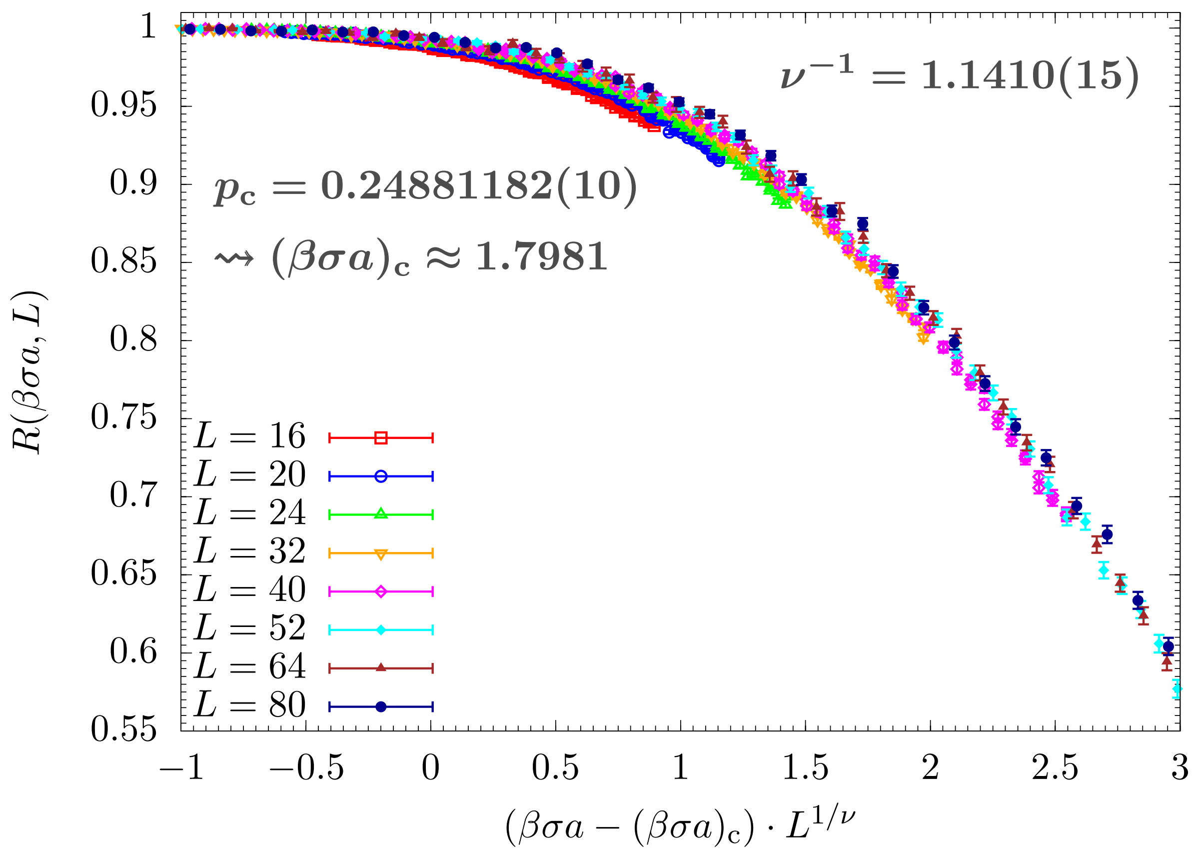
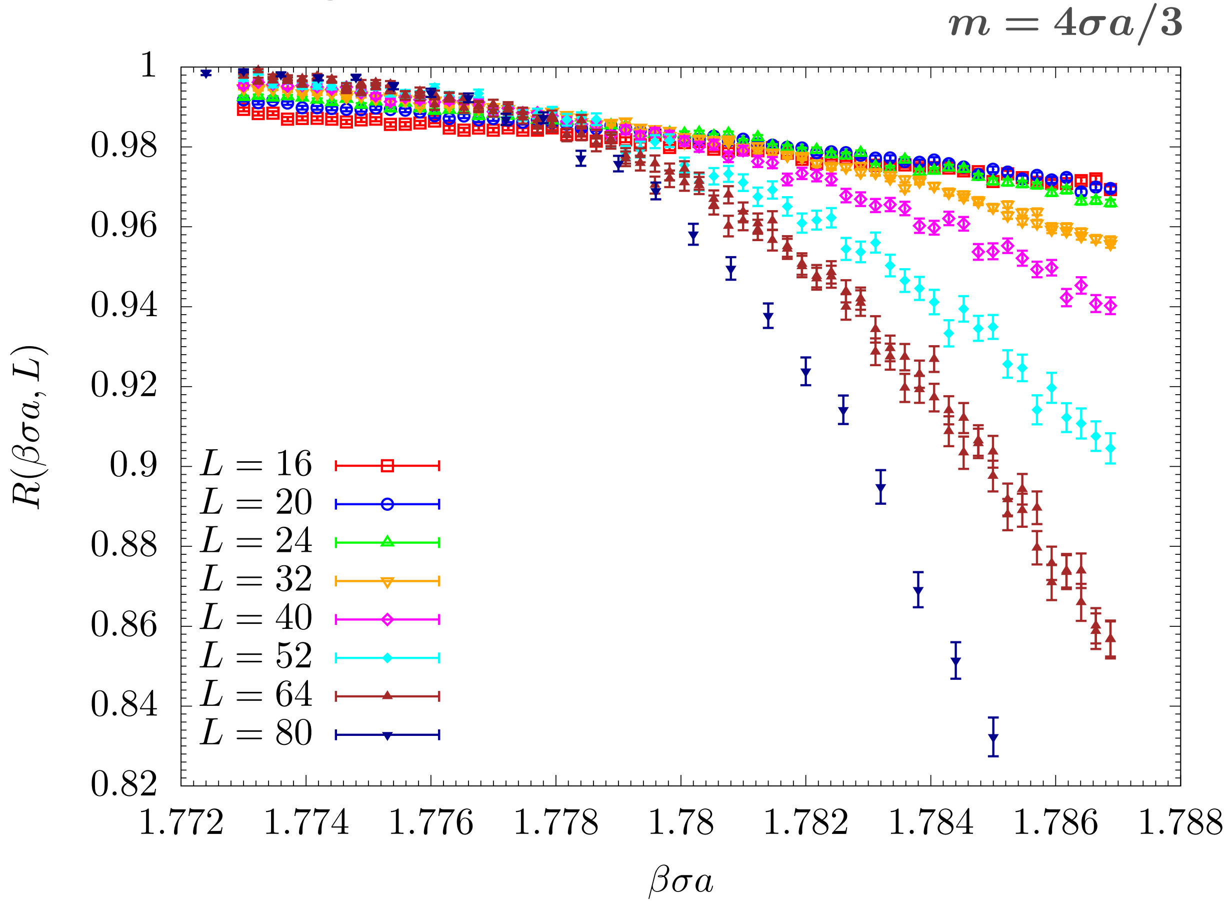


Percolation in Flux-Tube Model

- **medium heavy quarks**

  still in Z_3-Potts crossover region



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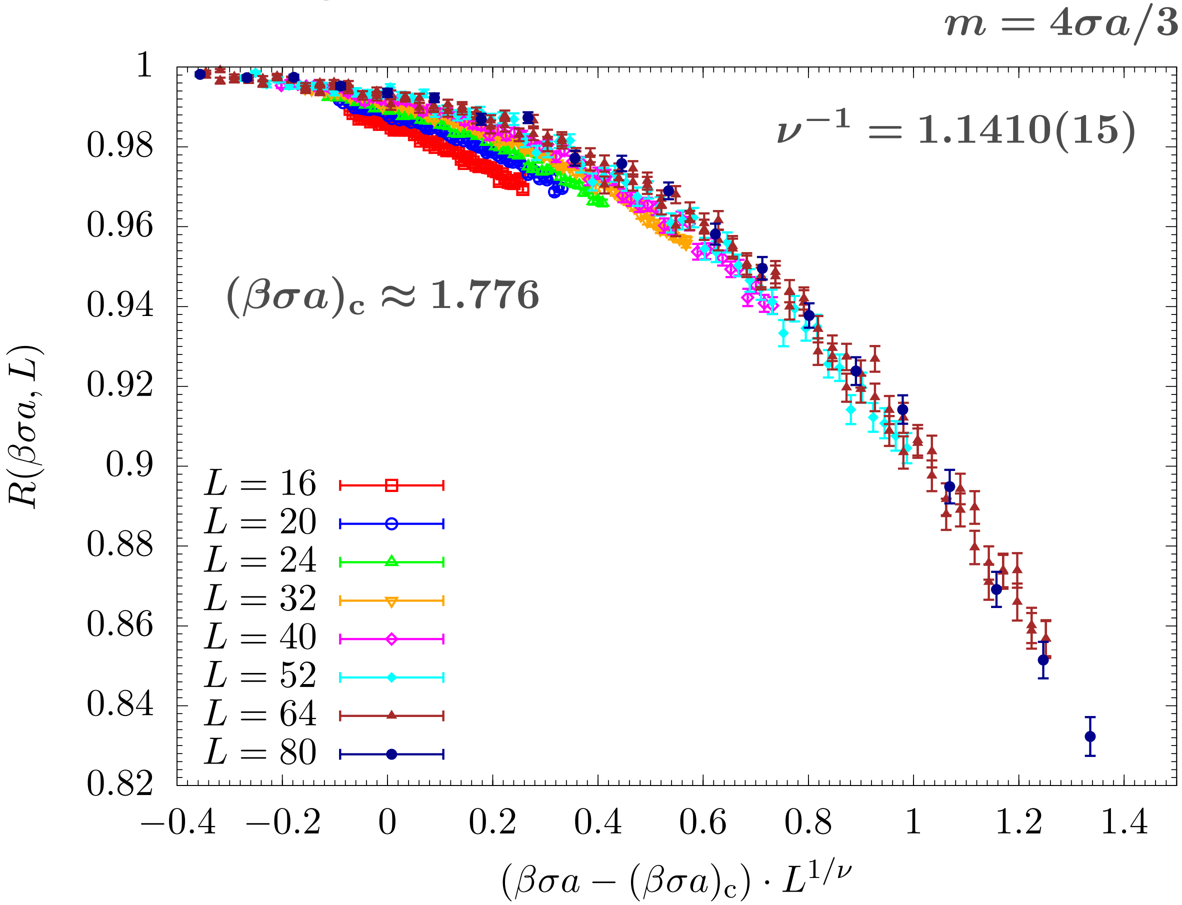
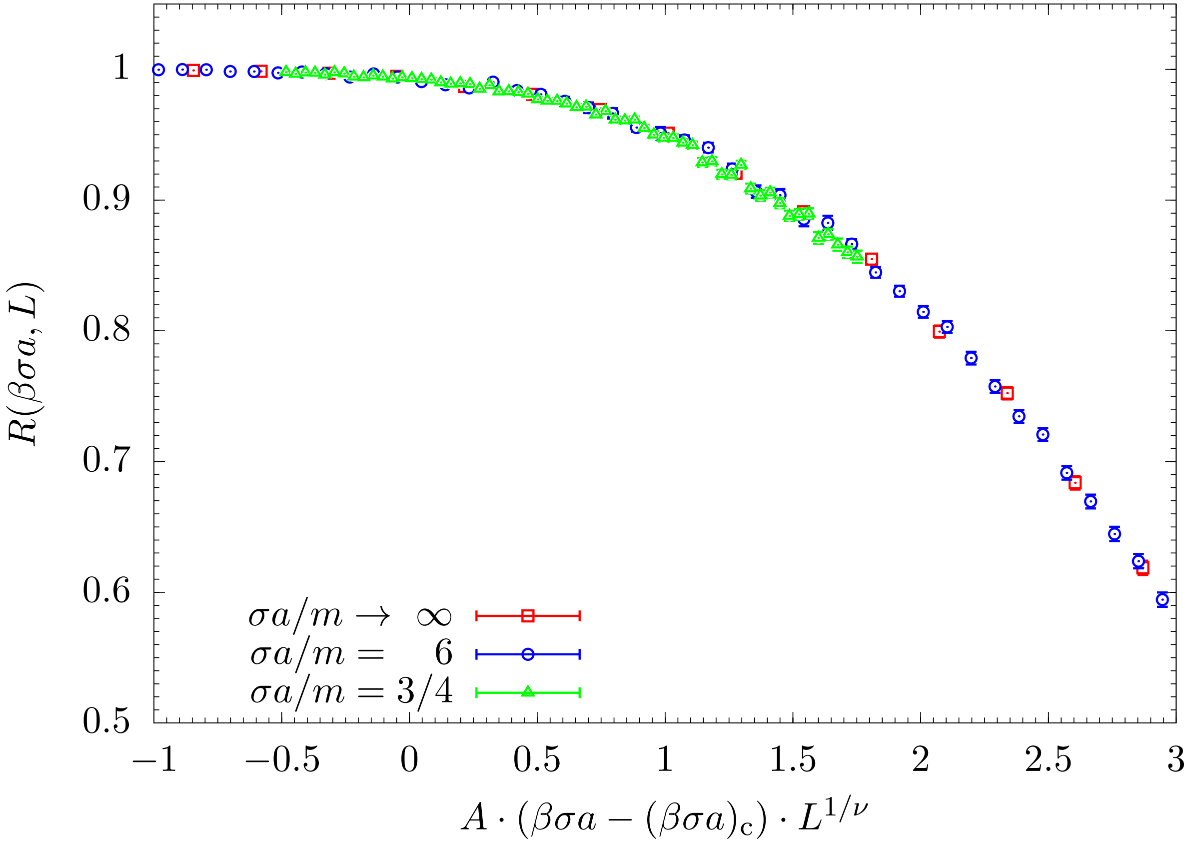


Percolation in Flux-Tube Model

- universal scaling function

combine $m = \{0,\, \sigma a/6,\, 4\,\sigma a/3\}$



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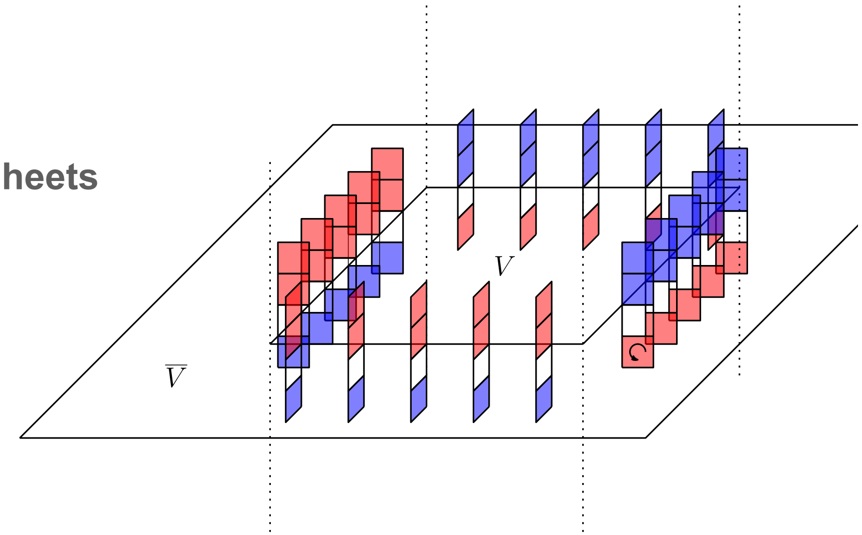
Summary

- **Quarks and triality in a finite volume**

  from FTs over stacks of closed center vortex sheets



- **Proof in two ways:**

  [see Ghanbarpour & LvS, PRD 106 (2022) 054513]

  1. **dualization of quark action**

     Gattringer & Marchis, NPB 916 (2017) 627

     Marchis & Gattringer, PRD 97 (2018) 034508

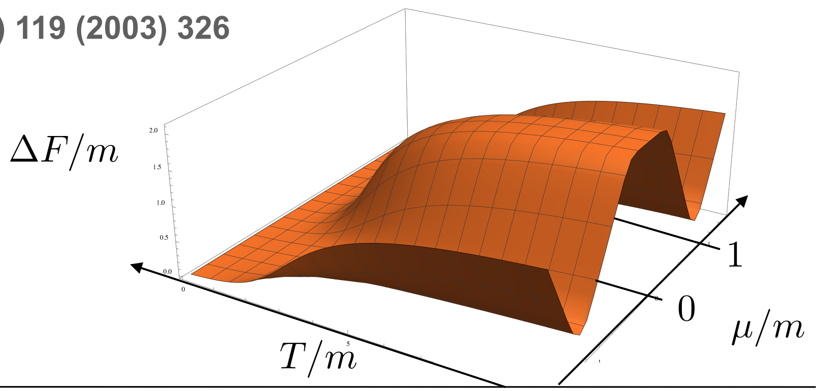
  2. **transfer matrix approach**

     Lüscher, Com. Math. Phys. 54 (1977) 283, Borgs & Seiler, Com. Math. Phys. 91 (1983) 329

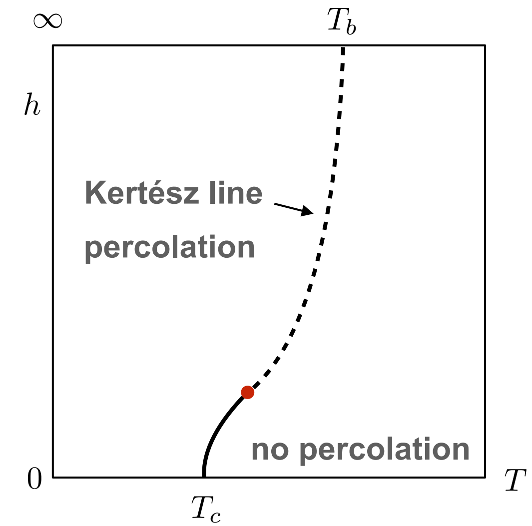
     Palumbo, NPB 645 (2002) 309, Mitrjushkin, NPB (PS) 119 (2003) 326

- **Illustration: heavy-dense QCD**

  effective theory dual to flux-tube model



Summary & Outlook



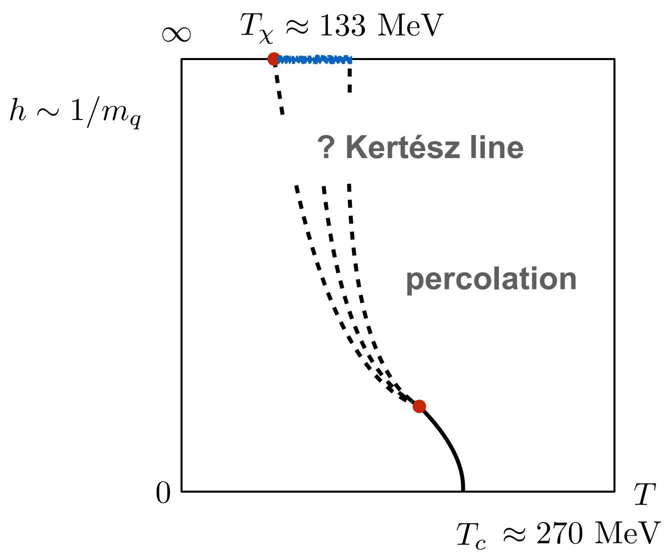
Potts model

- **Percolation of electric fluxes in effective theory**

  geometric deconfinement phase transition

  at strong coupling with static fermion determinant

- **Percolation of electric fluxes in QCD**



expect: geometric deconfinement phase transition

have: gauge invariant definition of fluxes and spanning probability

Thank you for your attention!