Non-perturbative determination of fermion condensates in large-N gauge theories

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Confinement and symmetry from vacuum to QCD phase diagram
9–15/02/2025 CCBPP Benasque

Based on:

The large-N limit of the chiral condensate from twisted reduced models
CB, P. Butti, M. García Pérez, A. González-Arroyo, K.-I. Ishikawa, M. Okawa
JHEP **12** (2023) 034 [arXiv:2309.15540]
$+$ new data (work in progress)

Large-N gauge theories

Pioneering works by 't Hooft and Witten about large-N QCD.
[NPB 72 (1974) 461 – NPB 160 (1979) 57]
$\to$ Factorization, suppression of non-planar diagrams, $1/N$ expansion.

Large-N $1/N$ expansion: non-perturbative tool allowing to unveil deep connections among chiral-symmetry breaking, chiral anomaly, confinement, topology [PRL 37, 8 (1976) – NPB 156, 269 (1979) – NPB 159, 213 (1979)]
(also, talk by T. Cohen tomorrow)

Large-N has also phenomenological predictive power for strong interactions: Witten–Veneziano formula for the η' mass [PRL 37, 8 (1976) – NPB 156, 269 (1979) – NPB 159, 213 (1979)], meson-meson scattering amplitudes [EPJC 80 (2020) 7, 638] and tetraquark state [JHEP 06 (2022) 049] . . .

Some models offer the possibility of analytically investigating specific non-perturbative properties of large-N gauge theories.

For the rest, we can rely on numerical Monte Carlo simulations of large-N lattice gauge theories (this talk).

Some features have been thoroughly investigated on the lattice at large-N.

• Mass spectrum: glueballs and strings

JHEP 06 (2004) 012 – JHEP 08 (2010) 119 – JHEP 12 (2021) 082 467

(also, morning talk by P. Bicudo)

• Confinement: Λ-parameter and critical deconfinement temperature

PLB 718 (2013) 1524-1528 – PLB 712 (2012) 279-283

• θ-dependence (vacuum energy): topological susceptibility, higher-order terms

PLB 762 (2016) 232-236 – PRD 94 (2016) 8, 085017 – JHEP 03 (2021) 111

• θ-dependence of deconfinement temperature, string tension, mass gap

PRL 109 (2012) 072001 – JHEP 05 (2024) 163 – JHEP 02 (2024) 156

This talk is about a less investigated topic in large-N lattice gauge theories: calculation of fermion condensates. We made significant progress recently:

• Quark condensate in large-N QCD *JHEP* **12** (2023) 034 [2309.15540]

• Gluino condensate in large-N SUSY YM *PRD* **110** (2024) 7 074507 [2406.08995]

This talk will present an updated study of the **quark chiral condensate** in large-N QCD including new simulations with N as large as **841**.

This was possible by virtue of **large-N volume independence**.

Large-N volume independence

Standard approach: extended lattice, periodic boundary conditions, extrapolation towards $1/N \to 0$. Typically $N \lesssim \mathcal{O}(10)$.
Our lattice calculation: **large-N twisted volume reduction**.

Large-N equivalence of space-time and color degrees of freedom

[Eguchi & Kawai PRL 48 (1982) 1063]

$V_{\text{eff}} = N^2 V \implies N \to \infty$ is a thermodynamic limit, finite-volume effects vanish.

This idea, taken to the extreme, suggests the possibility of studying the lattice SU(∞) Yang–Mills theory as a matrix model defined on single space-time point.
Reducing the volume allows to reach $N \sim \mathcal{O}(10^2\text{–}10^3)$.

Achtung! EK reduction holds if center-symmetry is not broken!
It is well established that it is spontaneously broken in certain regimes $\implies$ several proposals to enforce it (e.g., continuum reduction, trace deformation).

[NPB 696 (2004) 107-140 – PRD 78 (2008) 065035]

Our approach: large-N lattice gauge theories reduced on a one-point lattice with twisted boundary conditions $\implies$ **Twisted Eguchi–Kawai** (TEK) model

[A. González-Arroyo & M. Okawa PRD **27** (1983) 2397; JHEP**07** (2010) 043]

Status of large-N QCD calculations of quark condensate

Large-N QCD: $N \to \infty$ at fixed $\lambda = g^2 N$ and fixed N_f.

Quark chiral condensate

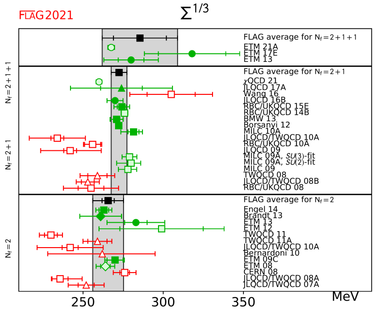
$$\Sigma \equiv -\lim_{m \to 0} \langle \overline{u} u \rangle \,, \qquad (m_u = m_d \equiv m)$$

$$\Sigma(N) = N \left[c_0 + c_1 \frac{N_\mathrm{f}}{N} + c_2 \frac{1}{N^2} + \mathcal{O}\left(\frac{1}{N^2}\right) \right]$$

Well-established quantity in QCD (plot on the right).

Very limited studies in large-N QCD.

- Narayanan, Neuberger (2004) [hep-lat/0405025]
  Exploratory study (1 lattice spacing, 1-loop perturbative renormalization).
- Hernández et al. (2019) [1907.11511]
  $N_\mathrm{f} = 4$, $3 \le N \le 6$, 1 latt. spacing, no renormalization.
- DeGrand, Wickenden (2023) [2309.12270]
  Appeared concurrently with our study.
  Preliminary continuum limit from coarse spacings, $N_\mathrm{f} = 2$, exploratory large-N limit ($3 \le N \le 5$)



TEK model of large-N QCD: gluons

Quarks sub-leading in $1/N \implies$ quarks are exactly quenched at large N.
Large-N QCD can be dynamically simulated as a pure Yang–Mills theory.

- $U_\mu(n) \longrightarrow U_\mu$ (one site $\implies$ only $d = 4$ links)
- $U_\mu(n + a\hat{\nu}) = \Gamma_\nu U_\mu \Gamma_\nu^\dagger$ (Twisted Boundary Conditions)
- $b = 1/\lambda$, with $\lambda = g^2 N$ the bare 't Hooft coupling $\quad (b = \frac{\beta}{2N^2})$

$$S_{\text{TEK}}[U] = -Nb \sum_{\nu \neq \mu} z_{\nu\mu} \text{Tr}\left\{U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger\right\} \qquad \text{(Wilson plaquette action)}$$

- Twist-eaters $\Gamma_\mu \Gamma_\nu = z_{\nu\mu}\Gamma_\nu\Gamma_\mu$ with twist factor $z_{\nu\mu} = \exp\left\{\frac{2\pi i k(N)}{\sqrt{N}}\varepsilon_{\nu\mu}\right\}$
- **Effective box size $L = \sqrt{N}$**
- L prime number, $k(N)$ co-prime with L and scaled with N
  This choices minimize non-planar finite-N corrections
  Chamizo & González-Arroyo, J. Phys. A 50 (2017) 26 265401 [1610.07972]
- Continuum limit achieved via $b \to \infty$, when lattice spacing $a(b) \to 0$

TEK model of large-N QCD: quarks

We adopt 2 degenerate **valence** Wilson quarks with mass m.

For fermions, reduction is a bit more involved. I just report the expression of the TEK Dirac–Wilson operator in the fundamental representation.

Details can be found here: González-Arroyo, Okawa (2015) [1510.05428]

$$D_{\text{TEK}} = \frac{1}{2\kappa} - \frac{1}{2}\sum_{\mu=0}^{d-1}\left[(\mathbb{I}+\gamma_\mu)\mathcal{W}_\mu + (\mathbb{I}-\gamma_\mu)\mathcal{W}_\mu^\dagger\right]$$

$\mathcal{W}_\mu = U_\mu \otimes \Gamma_\mu^*$ $\kappa \to$ usual Wilson hopping parameter

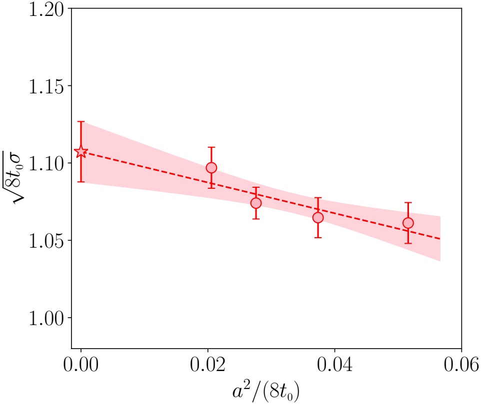
In momentum space:

$$\widetilde{D}_{\text{TEK}}(p) = \frac{1}{2\kappa} - \frac{1}{2}\sum_{\mu=0}^{d-1}\left[(\mathbb{I}+\gamma_\mu)\mathcal{W}_\mu e^{\mathrm{i}p_\mu} + (\mathbb{I}-\gamma_\mu)\mathcal{W}_\mu^\dagger e^{-\mathrm{i}p_\mu}\right]$$

- String tension $\sqrt{\sigma}$, obtained from smeared Creutz ratio
  González-Arroyo, Okawa (2013) [1206.0049]

- Gradient flow reference scale $\sqrt{8t_0}$, obtained from clover action density
  García Pérez et al. (2020) [2011.13061]

$$E(t) \equiv \langle \mathrm{Clover}(t) \rangle \qquad \frac{1}{N}\left[t^2 E(t)\right]\Big|_{t=t_0} = c = 0.1$$



Couplings:
$b = 0.355, 0.360, 0.365, 0.370$

- $\sqrt{\sigma}$: $N = 841$
- $\sqrt{8t_0}$: $169 \leq N \leq 361$

$a\sqrt{\sigma} \sim 0.241 - 0.157$
$\implies a \sim 0.107 - 0.069$ fm
$(\sqrt{\sigma} = 445$ MeV$)$

Range of a similar to QCD simulations.

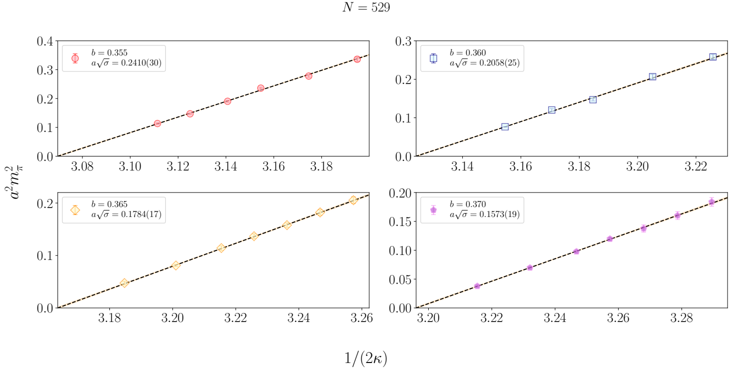
Chiral condensate from pion mass

$$m_\pi^2 = 2\frac{\Sigma}{F_\pi^2}m = 2\frac{\Sigma_{\rm R}}{F_\pi^2}m_{\rm R} \equiv 2B_{\rm R}m_{\rm R} \qquad \text{(Gell-Mann–Oakes–Renner)}$$

Slope m_π^2 vs $1/(2\kappa) \to B_{\rm R}/Z_{\rm S}$

Pion correlator: anti-transform $\vec{p}=\vec{0}$ correlation function in momentum space:

$$C(\tau) = \sum_{p_0} \exp\left\{\frac{i\pi\tau p_0}{\sqrt{N}}\right\} C(p_0) \qquad C(p_0) = \langle \gamma_5 \widetilde{D}^{-1}_{\rm TEK}(0)\gamma_5\widetilde{D}^{-1}_{\rm TEK}(p_0)\rangle$$



Chiral condensate from the mode number

Banks–Casher relates the chiral condensate with spectral density in the origin:

$$\frac{\Sigma}{\pi} = \lim_{\lambda\to 0}\lim_{m\to 0}\lim_{V\to\infty}\rho(\lambda, m)$$

Mode number is the integral of ρ.
More amenable to be computed on the lattice:

$$\langle \nu(M)\rangle \equiv \langle \#\ |i\lambda + m| \leq M\rangle = V\int_{-\Lambda}^{\Lambda}\rho(\lambda, m)d\lambda, \qquad \Lambda^2 \equiv M^2 - m^2.$$

- Banks–Casher implies linear rise of $\langle \nu(M)\rangle$ close to $M = m$:

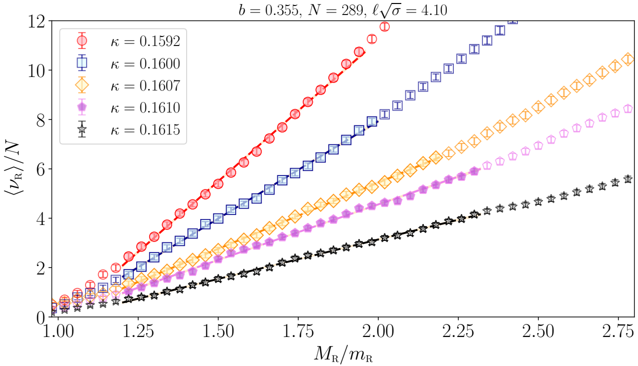
$$\langle \nu(M)\rangle = \frac{2}{\pi}V\Sigma\Lambda + \mathcal{O}(\Lambda^2) \qquad V = V_{\text{eff}} = a^4 N^2$$

- **Giusti–Lüscher method** [JHEP 03 (2009) 013 – 0812.3638]

$$\Sigma^{(\text{eff})}(m) = \frac{\pi}{2V}\sqrt{1-\frac{m^2}{M^2}}\left[\frac{\partial\langle\nu(M)\rangle}{\partial M}\right] \longleftarrow \text{slope of } \langle\nu(M)\rangle \text{ vs } M$$

$$\Sigma^{(\text{eff})}(m) = \Sigma\left[1+\mathcal{O}(m)\right] \qquad \Longrightarrow \qquad \Sigma = \lim_{m\to 0}\Sigma^{(\text{eff})}(m)$$

- Solve numerically $[\gamma_5 D_{\rm TEK}]\, u_\lambda = \lambda\, u_\lambda$ for the lowest $\sim \mathcal{O}(100)$ eigenmodes
- Count modes below treshold M to obtain $\langle \nu(M) \rangle$
- Slope: linear best fit of $\langle \nu(M) \rangle$ vs M close to $M \simeq m \to \Sigma^{(\rm eff)}(m)$



- $\langle \nu \rangle = \langle \nu_{\rm R} \rangle \qquad M = Z_{\rm P} M_{\rm R} \qquad Z_{\rm P} m_{\rm R} = \frac{Z_{\rm A} Z_{\rm S}}{Z_{\rm P}} \times \frac{Z_{\rm P}}{Z_{\rm S}} \times m_{\rm PCAC}$
  - $m_{\rm PCAC}$ from usual Ward identity
  - $\frac{Z_{\rm P}}{Z_{\rm S} Z_{\rm A}}$ from slope of $m_{\rm PCAC}$ vs $1/(2\kappa)$
- $\frac{Z_{\rm P}}{Z_{\rm S}}$ from eigenvectors u_λ [Giusti–Lüscher (2009) – 0812.3638]

Finite-N effects in the TEK model

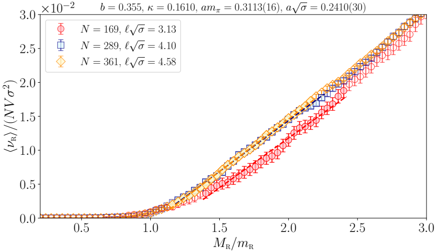
TEK model: finite-N effects $\to$ finite (effective) volume effects.
This is a consequence of Eguchi–Kawai reduction.

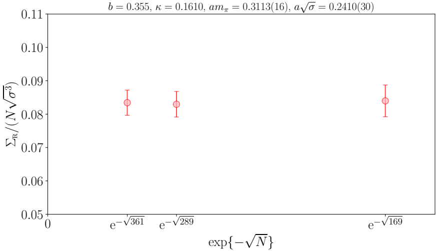
Finite-N effects expected to be the same of a periodic box with effective size $L = \sqrt{N}$: exponentially small when $\ell = a\sqrt{N} \gtrsim 1/\Lambda$.

We observe no finite-N effects in mode number **slope** when:

$$\ell\sqrt{\sigma} = \sqrt{N} \times a\sqrt{\sigma} \gtrsim 3 \qquad \Longrightarrow \qquad \ell \gtrsim 1.4 \text{ fm}$$

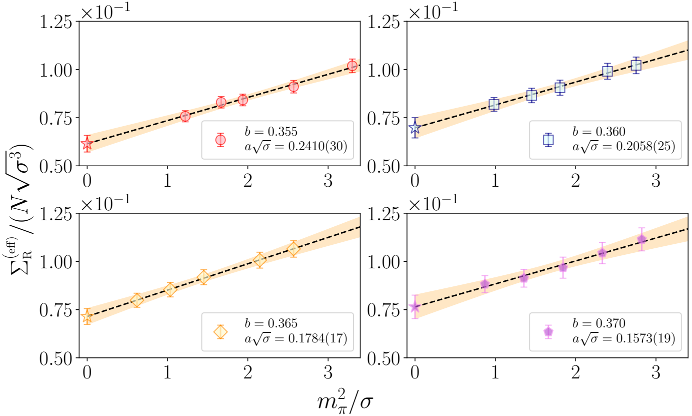
Much like what people see in standard simulations.





Chiral behavior of spectral determination

From mode number fit $\rightarrow \Sigma_{\mathrm{R}} m_{\mathrm{R}}$. Since we know $Z_{\mathrm{P}} m_{\mathrm{R}} \implies \Sigma_{\mathrm{R}}/Z_{\mathrm{P}}$.
Renormalized via non-perturbative large-N determinations of Z_{P} in $\overline{\mathrm{MS}}$ at $\mu = 2$ GeV [L. Castagnini (2015) – inspire:1411974]



$b = 0.355,\ N = 289$
$\rightarrow \ell\sqrt{\sigma} = 4.10$

$b = 0.360,\ N = 361$
$\rightarrow \ell\sqrt{\sigma} = 3.91$

$b = 0.365,\ N = 529$
$\rightarrow \ell\sqrt{\sigma} = 4.10$

$b = 0.370,\ N = 841$
$\rightarrow \ell\sqrt{\sigma} = 4.56$

Data support Chiral Perturbation Theory prediction (no chiral logs at large-N)
[Giusti–Lüscher JHEP 03 (2009) 013]

Continuum limit

Wilson fermions: $\sim \mathcal{O}(a)$ lattice artifacts $\implies$ Need continuum limit

We have 3 data sets:

- $\Sigma_{\rm R}$ from mode number
- $\dfrac{B_{\rm R}}{Z_{\rm S}}$ from pion mass $\to$ non-pert. renorm. via $\dfrac{Z_{\rm S}}{Z_{\rm P}} \times Z_{\rm P}$
- $\dfrac{F_\pi}{Z_{\rm A}}$ from $\dfrac{1}{m_\pi^2} \langle 0 | A_0(0) | \pi(\vec{p} = \vec{0}) \rangle \to$ non-pert. renorm. via $\dfrac{Z_{\rm P}}{Z_{\rm S}} \times \dfrac{Z_{\rm S} Z_{\rm A}}{Z_{\rm P}}$
- $F_\pi \to$ chiral limit $\qquad F_\pi^{(\rm phys)} \simeq 92$ MeV (2+1 QCD)

We can perform a combined fit of these 3 data sets imposing they are described by 2 ChiPT Low-Energy Constants (LECs):

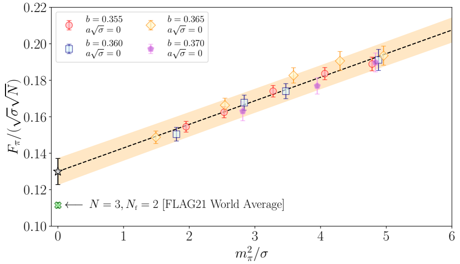
$$\frac{\Sigma_{\rm R}}{N} \sim \mathcal{O}(N^0) \text{ and } \frac{F_\pi}{\sqrt{N}} \sim \mathcal{O}(N^0)$$

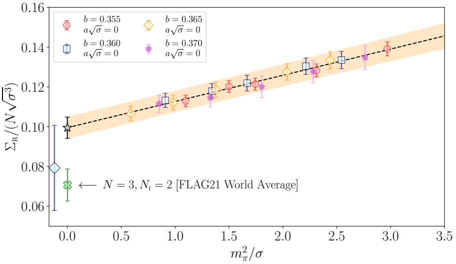
$$\text{with } B_{\rm R} \equiv \frac{\Sigma_{\rm R}}{N}\frac{N}{F_\pi^2} = \frac{\Sigma_{\rm R}}{F_\pi^2} \sim \mathcal{O}(N^0)$$

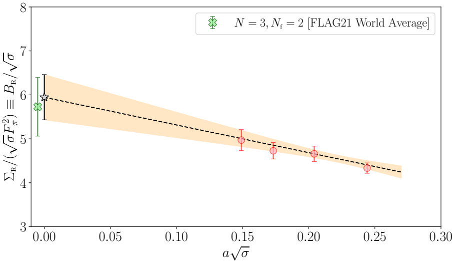
Combined chiral-continuum fit: $\mathcal{O}(a, m_\pi) = \mathcal{O} + k_1 \frac{a}{\sqrt{\sigma}} + k_2 \frac{m_\pi^2}{\sigma}$

Conversion to physical units of $N = \infty$ results: $\sqrt{\sigma} = 445(7)$ MeV

[Most recent 2+1 QCD lattice result: PLB 854 (2024) 138754 – arXiv:2403.00754]







- $N = \infty$

$$\Sigma_R/N = [206(4)\ \text{MeV}]^3$$
$$F_\pi/\sqrt{N} = 58(3)\ \text{MeV}$$
$$B_R = 2.65(23)\ \text{GeV}$$

- $N = 3,\ N_f = 2$ [FLAG21 World Average]

$$\Sigma_R/3 = [184(7)\ \text{MeV}]^3$$
$$F_\pi/\sqrt{3} = 49.6(7)\ \text{MeV}$$
$$B_R = 2.53(30)\ \text{GeV}$$

Conclusions and future outlooks

TEK model allows to efficiently address the non-perturbative lattice investigation of large-N gauge theories.

The techniques presented in this talk where also applied to determine the gluino condensate in large-N SUSY Yang–Mills.
CB, Butti, García Pérez, González-Arroyo, Ishikawa, Okawa, PRD 110 (2024) 7 074507 [2406.08995]

We recently also had a paper on the calculation of the mass of the lightest gluino-gluon bound state in large-N SUSY Yang–Mills (mass gap).
CB, García Pérez, González-Arroyo, Ishikawa, Okawa [2412.02348]

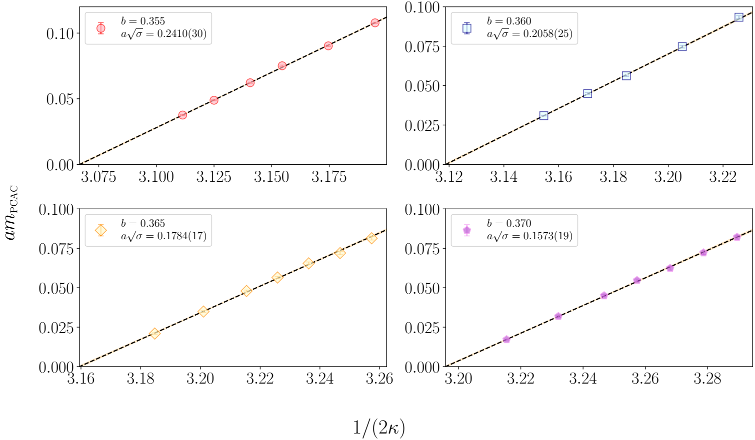
We are currently working to compute the large-N meson spectrum of up to $N = 841$ including excited states. We will have a 5^{th} lattice spacing.

Me and Margarita are collaborating with M. D'Elia (Pisa) to investigate the running coupling and Λ-parameter via twisted volume independence.
CB, Dasilva Golán, García Pérez, D'Elia, Giorgieri, EPJC 84 (2024) 9 916 [2403.13607]
PoS LATTICE2024 (2025) 404 [2501.18449]

Future outloooks: finite T? Glueball/string spectrum? Overlap quarks?

Back-up slides

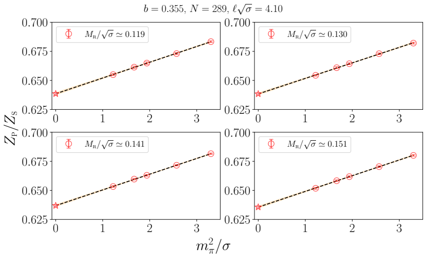
PCAC mass

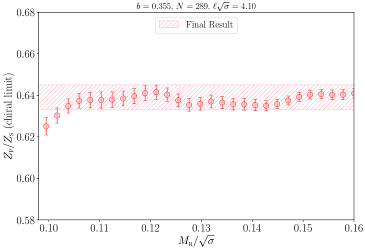


Slope m_{PCAC} vs $1/(2\kappa) \to Z_{\text{P}}/(Z_{\text{S}}Z_{\text{A}})$

Chiral limit achieved when $\kappa \to \kappa_c$.
Determinations of κ_c from m_π and m_{PCAC} agree

Calculation of $Z_{\mathrm{P}}/Z_{\mathrm{S}}$





From the same eigenproblem solved to obtain the mode number $\langle \nu(M) \rangle$ we also obtained $Z_{\mathrm{P}}/Z_{\mathrm{S}}$ non-perturbatively [Giusti–Lüscher JHEP **03** (2009) 013]

$$\left(\frac{Z_{\mathrm{P}}}{Z_{\mathrm{S}}}\right)^2 = \frac{\langle s_{\mathrm{P}}(M) \rangle}{\langle \nu(M) \rangle} \qquad s_{\mathrm{P}}(M) \equiv \sum_{|\lambda|,|\lambda'| \leq M} \left| u_\lambda^\dagger \gamma_5 u_{\lambda'} \right|^2,$$