The helium-4 abundance BSM: with solutions

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I. THE EXERCISE: CALCULATING THE HELIUM-4 ABUNDANCE AFTER THE BIG BANG

The neutron abundance in the early Universe is governed by the following processes:

$$n \leftrightarrow p + e^- + \bar{\nu}_e \tag{1a}$$
$$n + \nu_e \leftrightarrow p + e^- \tag{1b}$$
$$n + e^+ \leftrightarrow p + \bar{\nu}_e \tag{1c}$$

Based on dimensional analysis, the rate of these interactions roughly scale as $\Gamma \simeq G_F^2 T^5$ and hence they will freeze-out at $T \sim 1\,\mathrm{MeV}$. In the Standard Model, effectively all of the neutrons in the plasma end up forming ^{4}He because it is the most tightly bounded light nuclei. This means that if we are able to track the neutron abundance of the Universe we will know how much ^{4}He should have been generated after the Big Bang. By contrasting this prediction with observations of the helium mass fraction in the Universe, $Y_p \equiv \rho_{^4\mathrm{He}}/\rho_B = 0.245 \pm 0.003$ [PDG-24], we can set bounds on an array of BSM models, including to the parameter $N_{\rm eff}$. This exercise will guide you through the basics of the calculation.

1. Assuming that there are only neutrons and protons in the plasma, write down the Boltzmann equation for the neutron and proton number densities given processes that interconvert them. Write these equations as a function of the rate of neutron conversion λ_{np} and proton conversion λ_{pn}.

   Solution: The equations are as follows:

$$\frac{dn_n}{dt} + 3Hn_n = -\lambda_{np} n_n + \lambda_{pn} n_p\,, \tag{2a}$$
$$\frac{dn_p}{dt} + 3Hn_p = +\lambda_{np} n_n - \lambda_{pn} n_p\,. \tag{2b}$$

2. Given these expressions write the equation for the fraction of neutrons in the plasma, i.e. $X_n \equiv n_n/(n_n+n_p)$.

   Solution:

$$\frac{dX_n}{dt} = -\lambda_{np} X_n + \lambda_{pn}(1-X_n)\,. \tag{3}$$

   the nice thing is that now the expansion term has disappeared.

3. Assuming that the rates of proton-to-neutron and neutron-to-proton conversion are very efficient, this is, we are in thermal and chemical equilibrium, obtain the relationship between the λ_{pn} and λ_{np} reaction rates. *Tip:* For this you want to use chemical equilibrium of neutron-proton interactions (i.e. $\mu_n = \mu_p$) and the formula for Maxwell-Boltzmann number densities.

   Solution: Assuming Maxwell-Boltzmann statistics we can easily relate the number densities of protons and neutrons. The first step is to note that if processes interconverting protons and neutrons are active then $\mu_n + \mu_{\nu_e} = \mu_p + \mu_e$. In addition, due to charge conservation $\mu_e \simeq 10^{-9}\,T$ and hence can be neglected. Furthermore, in most BSM scenarios for baryogenesis $L \simeq -B$ and therefore $\mu_{\nu_e} \simeq 10^{-9}\,T$ is small as well. That means we can neglect both and find

$$\mu_n = \mu_p\,. \tag{4}$$

   With this, we simply need to consider the number density ratio of neutrons and protons in thermal equilibrium which will simply be given by:

$$n_n/n_p = e^{-\Delta m/T}\,. \tag{5}$$

This means that

$$X_n^{\rm eq} = 1/(1 + e^{\Delta m/T})\,, \tag{6}$$

where $\Delta m \equiv m_n - m_p = 1.29333\,{\rm MeV}$.

Finally, using the detailed balance condition, i.e. that $dX_n/dt = 0$ if the rates are efficient, relates the two reaction rates as follows:

$$\lambda_{pn} = \lambda_{np}\left[\frac{X_n^{\rm eq}}{1-X_n^{\rm eq}}\right] = e^{-\Delta m/T}\lambda_{np} \tag{7}$$

4. The neutron-to-proton interaction rates require a phase space integration over the distribution functions of e and ν_e. A simplified (but still rather accurate) formula for it is given by:

$$\lambda_{n+\nu_e\to p+e^-} = \lambda_{n+e^+\to p+\bar{\nu}_e} \simeq 15\frac{1}{\tau_n}\frac{T^4\Delta m}{m_e^5}\,, \tag{8}$$

where $\tau_n = 878.4 \pm 0.5\,{\rm s}$ is the neutron lifetime, $\simeq 15\,{\rm mins}$. Note that at $T \sim {\rm MeV}$ these rates are $\sim 10^3$ times faster than neutron decay $\Gamma_n = 1/\tau_n$.

Remember that we can write the Hubble parameter as

$$H = 1.66\sqrt{g_\star}\frac{T^2}{M_{\rm Pl}}\,. \tag{9}$$

With this, pose the differential equation for X_n as a function of T as the time variable. One can use the simplification that $dT/dt = -HT$.

Solution: By simply rewriting the differential equation as a function of T by simply using the chain rule and one finds:

$$\frac{dX_n}{dT} = \frac{dX_n}{dt}\frac{1}{dT/dt}\,. \tag{10}$$

Approximating $dT/dt = -HT$ (as it would happen in an adiabatically expanding Universe) it is pretty easy to write down the full differential equation. It simply reads:

$$\frac{dX_n}{dT} = \frac{1}{HT}\left[-\lambda_{np}X_n + \lambda_{pn}(1-X_n)\right] \tag{11}$$

5. Numerically solve this system using $g_\star = 10.75 + 2\frac{7}{8}\Delta N_{\rm eff}$ and find the resulting solution for X_n without taking into account for the moment n decay. What do you find for $\Delta N_{\rm eff} = -1, 0, +1$? What is the fractional change to this neutron fraction?

Solution:

$$X_n(T\to 0) = 0.1556 \qquad [\Delta N_{\rm eff} = -1] \tag{12a}$$

$$X_n(T\to 0) = 0.1629 \qquad [\Delta N_{\rm eff} = 0] \tag{12b}$$

$$X_n(T\to 0) = 0.1691 \qquad [\Delta N_{\rm eff} = +1] \tag{12c}$$

I.e. one finds:

$$\frac{\Delta X_n}{X_n} \simeq 0.04\,\Delta N_{\rm eff}\,, \tag{13}$$

6. Finally, we need to relate the neutron abundance in the Universe to the abundance of Helium-4. In order to form Helium-4, a substantial amount of D, T and ^{3}He needs to be synthesized before hand. Critically, any "heavy" element formation requires the presence of substantial amounts of deuterium and this means that BBN will happen quickly once the abundance of deuterium is $\mathcal{O}(1)$. We can then approximate the abundance of helium-4 as given by the abundance of all the neutrons in the plasma once there is a large number of deuterium in the plasma and BBN starts. This is:

$$n_{^4{\rm He}} = n_n/2\,, \tag{14}$$

since what is typically quoted is the mass fraction of helium then one finds:

$$Y_P = 4\frac{n_{^4\mathrm{He}}}{n_B} = 2X_n|_{T=T_{\mathrm{BBN}}}\,. \tag{15}$$

The goal now is to estimate this temperature, T_{BBN}. In the plasma there are processes $n+p \leftrightarrow D+\gamma$ occurring efficiently and this tells us that $\mu_n + \mu_p = \mu_D$. Again, by using the Maxwell-Boltzmann number density formula we can then relate the number densities of n, p, γ and D. Find this relationship in terms of the deuterium fraction, i.e. $X_D = n_D/n_B$ where n_B is the number of baryons. Note that $g_D = 3$ (because Deuterium is a vector state). Once you have that formula solve for when $X_D \simeq 10^{-2}$ to find the temperature at which BBN starts. You should find $T_{\mathrm{BBN}} \simeq 0.072\,\mathrm{MeV}$.

Solution:

$$X_n^{\mathrm{NSE}} = \frac{3\sqrt{\frac{2}{\pi}}\zeta(3)(m_D T)^{3/2}}{m_n^{3/2} m_p^{3/2}}\eta_B X_n X_p e^{\frac{B_D}{T}} \tag{16}$$

where here $B_D \simeq 2.22\,\mathrm{MeV}$ is the binding energy of deuterium, $\eta_B = n_B/n_\gamma$ is the baryon to photon ratio $\eta_b \simeq 6\times 10^{-10}$. Solving this explicitly with $X_n \simeq 0.15$, $Y_p \simeq 1-0.15$ and $\eta_b = 6\times 10^{-10}$ one finds:

$$X_n^{\mathrm{NSE}} = 10^{-2} \qquad \text{for} \qquad T_\gamma \simeq 0.072\,\mathrm{MeV} \tag{17}$$

and this tells us that $T_{\mathrm{BBN}} = 0.072\,\mathrm{MeV}$ and this is the temperature we need to evaluate the lifetime of the neutron for to get the helium abundance.

7. Given that we know the temperature at which BBN starts we can calculate the final helium abundance using Eq. (7). The only thing that changes between neutron-proton freeze-out is that the neutrons are decaying. This can be easily incorporated by taking:

$$Y_P = 2X_n(T\to 0)\times e^{-t_{\mathrm{BBN}}/\tau_n} \tag{18}$$

The only piece we have left is simply to get the time at which this corresponds. Remember that in a radiation dominated Universe we have $t \simeq 1/(2H)$. Taking into account entropy conservation find that at $T_{\mathrm{BBN}} \simeq 0.072\,\mathrm{MeV}$ almost all e^+e^- have annihilated and therefore $g_\star \simeq 3.36 + 0.46\Delta N_{\mathrm{eff}}$. With this, calculate Y_P for $\Delta N_{\mathrm{eff}} = -1, 0, +1$. What is the fractional change to this neutron fraction? Given the observed value of $Y_p = 0.245 \pm 0.003$ what bound can be derived on ΔN_{eff}?.

Solution: With this we then find:

$$Y_P = 2\times 0.1556 \times 0.739 = 0.230 \qquad [\Delta N_{\mathrm{eff}} = -1] \tag{19a}$$
$$Y_P = 2\times 0.1629 \times 0.755 = 0.246 \qquad [\Delta N_{\mathrm{eff}} = 0] \tag{19b}$$
$$Y_P = 2\times 0.1691 \times 0.769 = 0.260 \qquad [\Delta N_{\mathrm{eff}} = +1] \tag{19c}$$

which can be summarized as:

$$\frac{\Delta Y_{\mathrm{P}}}{Y_{\mathrm{P}}} \simeq 0.056\times \Delta N_{\mathrm{eff}} \tag{20}$$

This allows us to understand that 1), since the helium abundance has been measured with $\simeq 1\%$ precision and its measurements agree with the SM prediction one can bound $\Delta N_{\mathrm{eff}} \simeq \pm 0.2$. And that the effect of the modified expansion history affects both neutron-proton freeze-out as well as the time of BBN and that the two effects roughly contribute to $2/3$ and $1/3$ of the effect, respectively.