When entanglement entropy tends (*not*) to diverge

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Entanglement entropy $S = -\mathrm{Tr}\,\rho \ln \rho$

Keywords

(1) **Tensor-Network** studies of spin systems in the thermodynamic limit.

(2) **Phase-transition** analysis using ground-state properties.

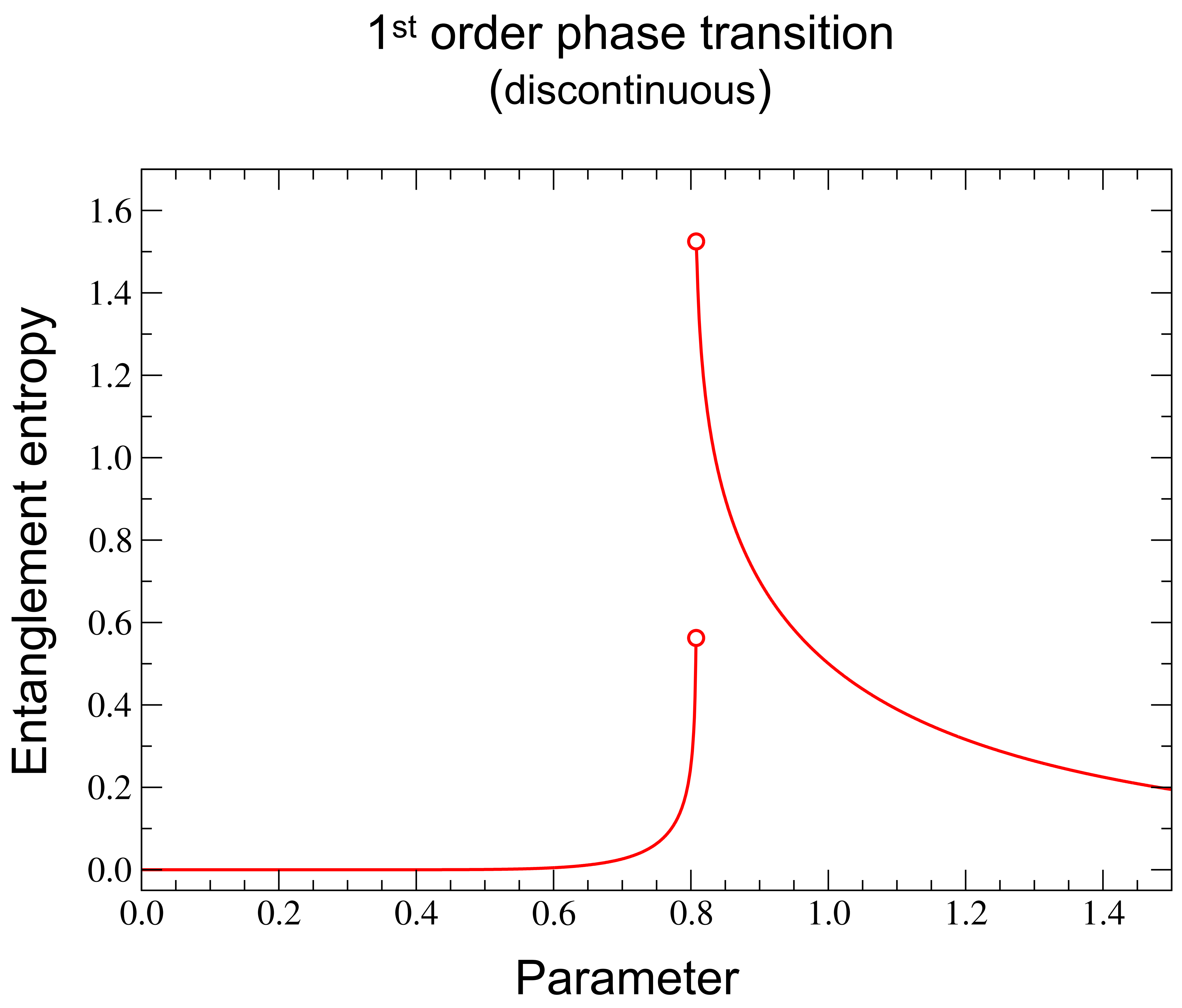
(3) **Fractals** and negative (**hyperbolic**) curvature.

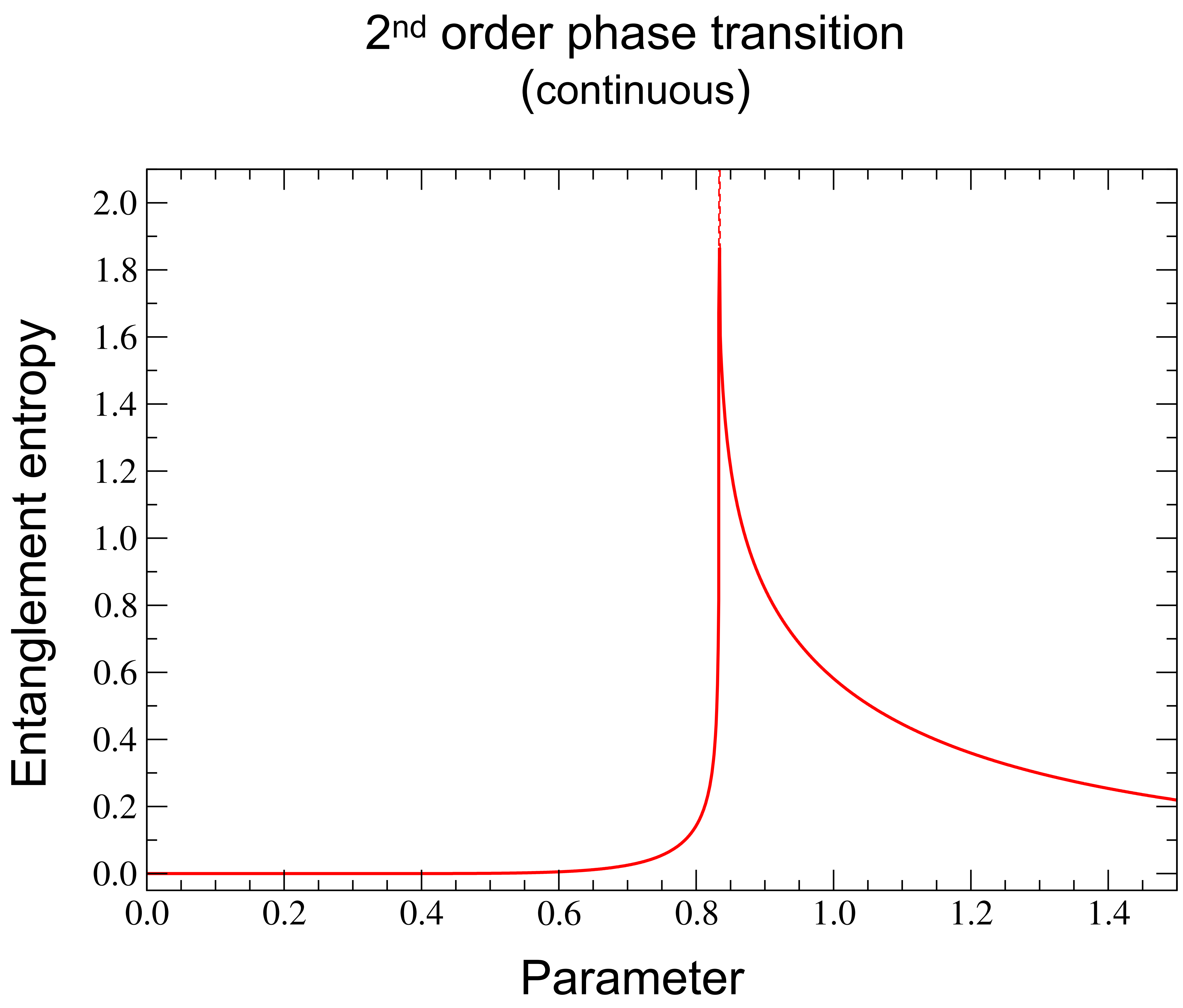
Part I

Motivation

- Physics at maximum entanglement entropy
- Criticality and phase transitions

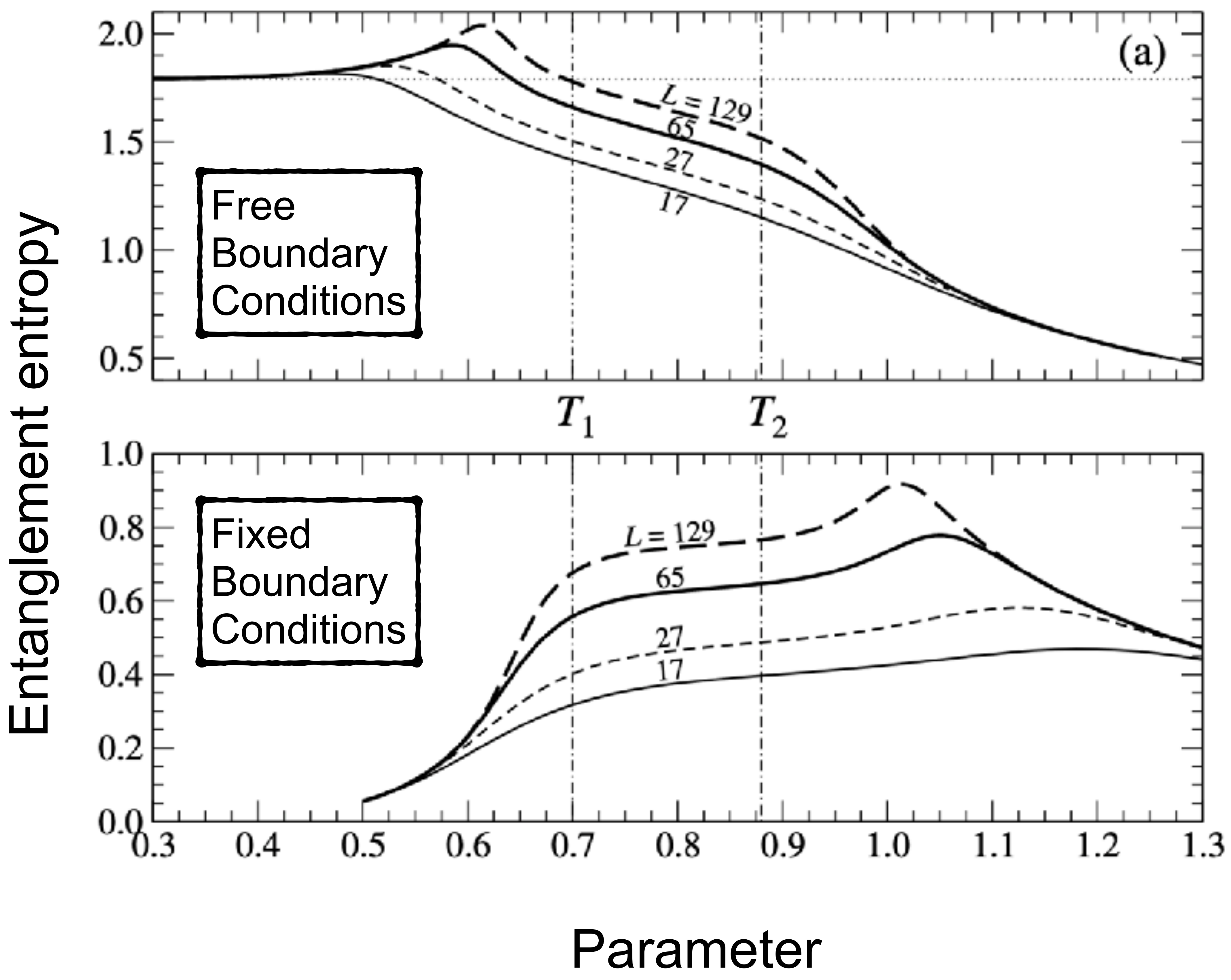
Discontinuous and continuous phase transitions



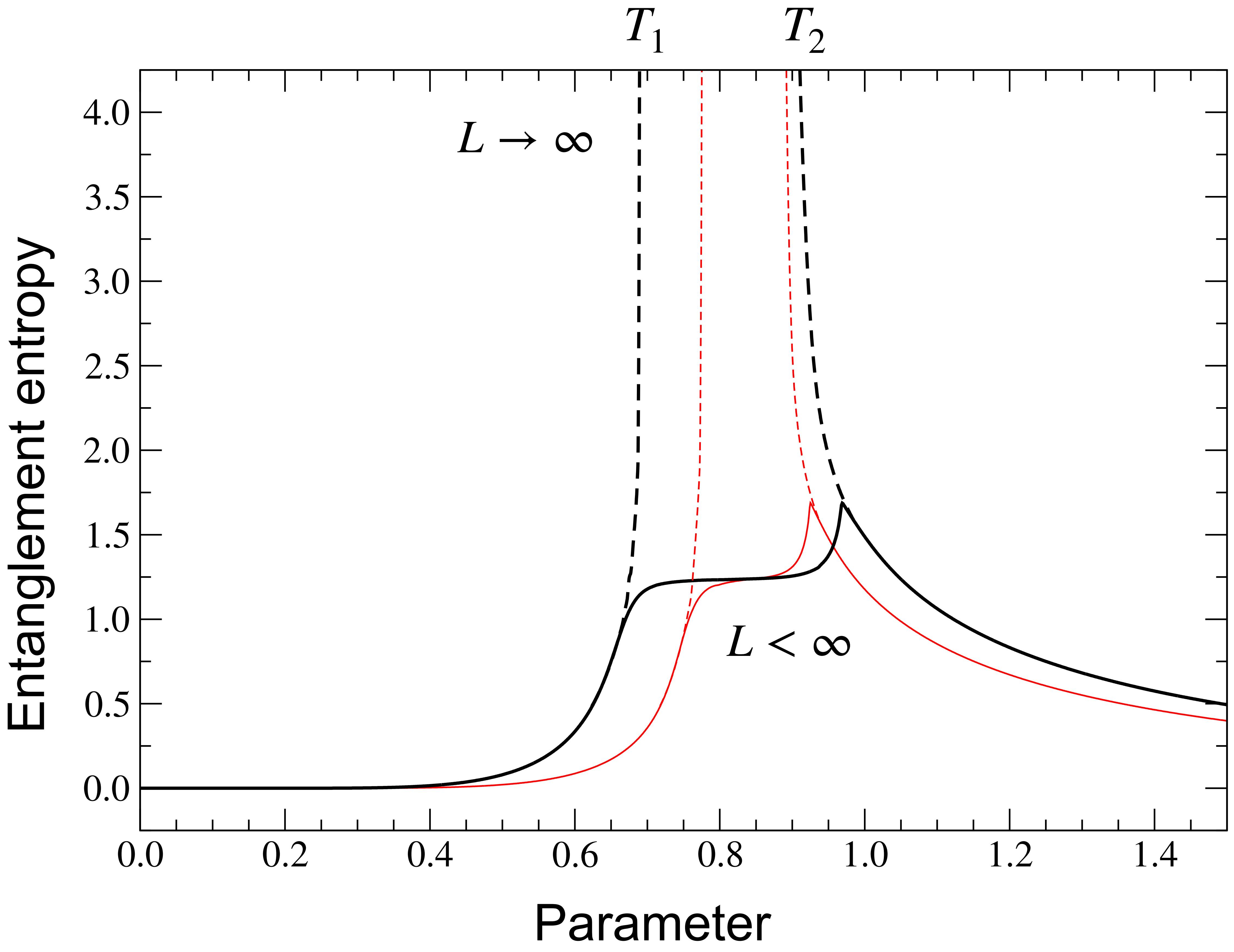


Berezinskii-Kosterlitz-Thouless phase transition

∞^{th} order phase transition



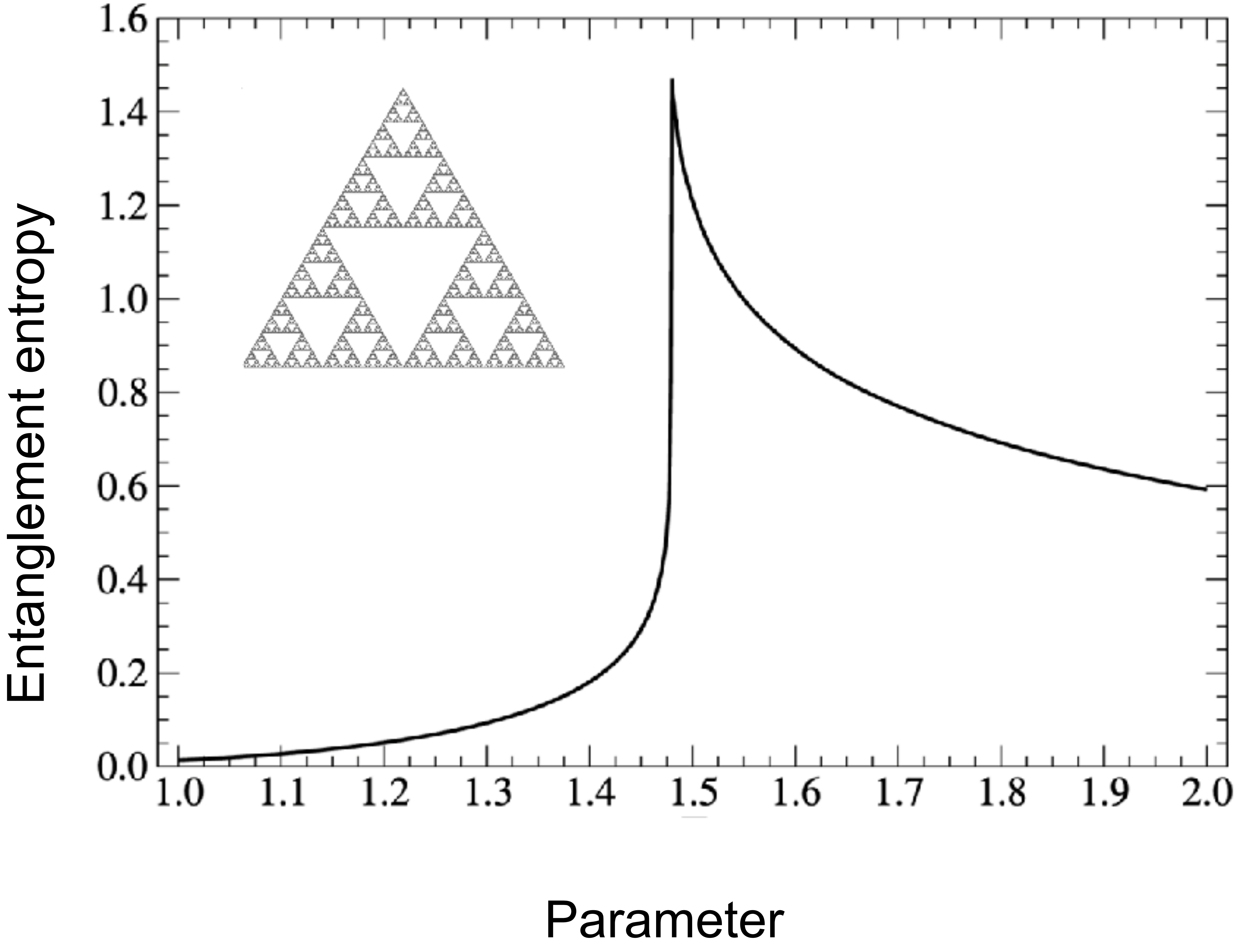
∞^{th} order phase transition



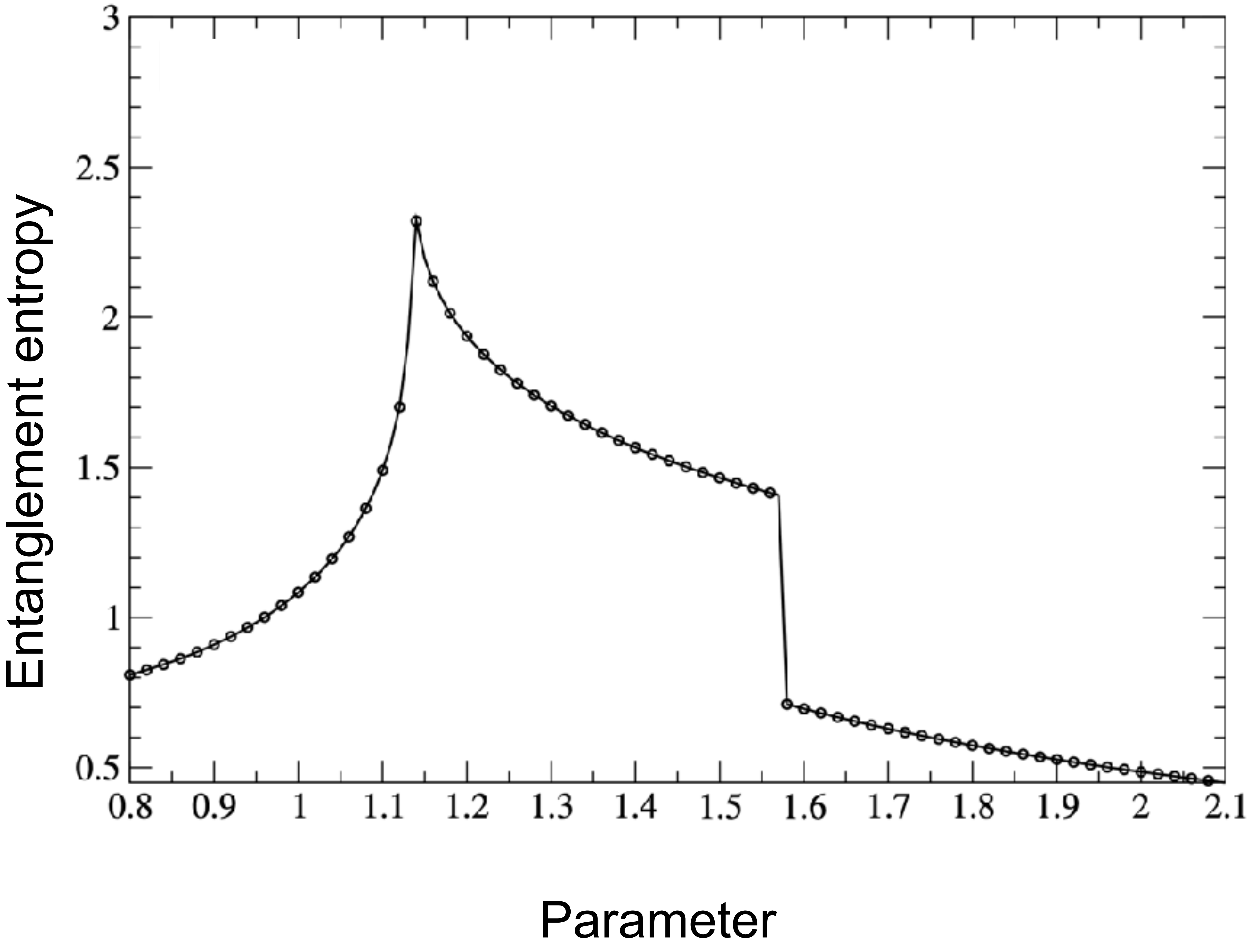
H. Ueda, K. Okunishi, K. Harada, R. Krčmár, A. Gendiar, S. Yunoki, and T. Nishino, Phys. Rev. E **101**, 062111 (2020)

*Continuous phase transition on **fractals***

2nd order phase transition

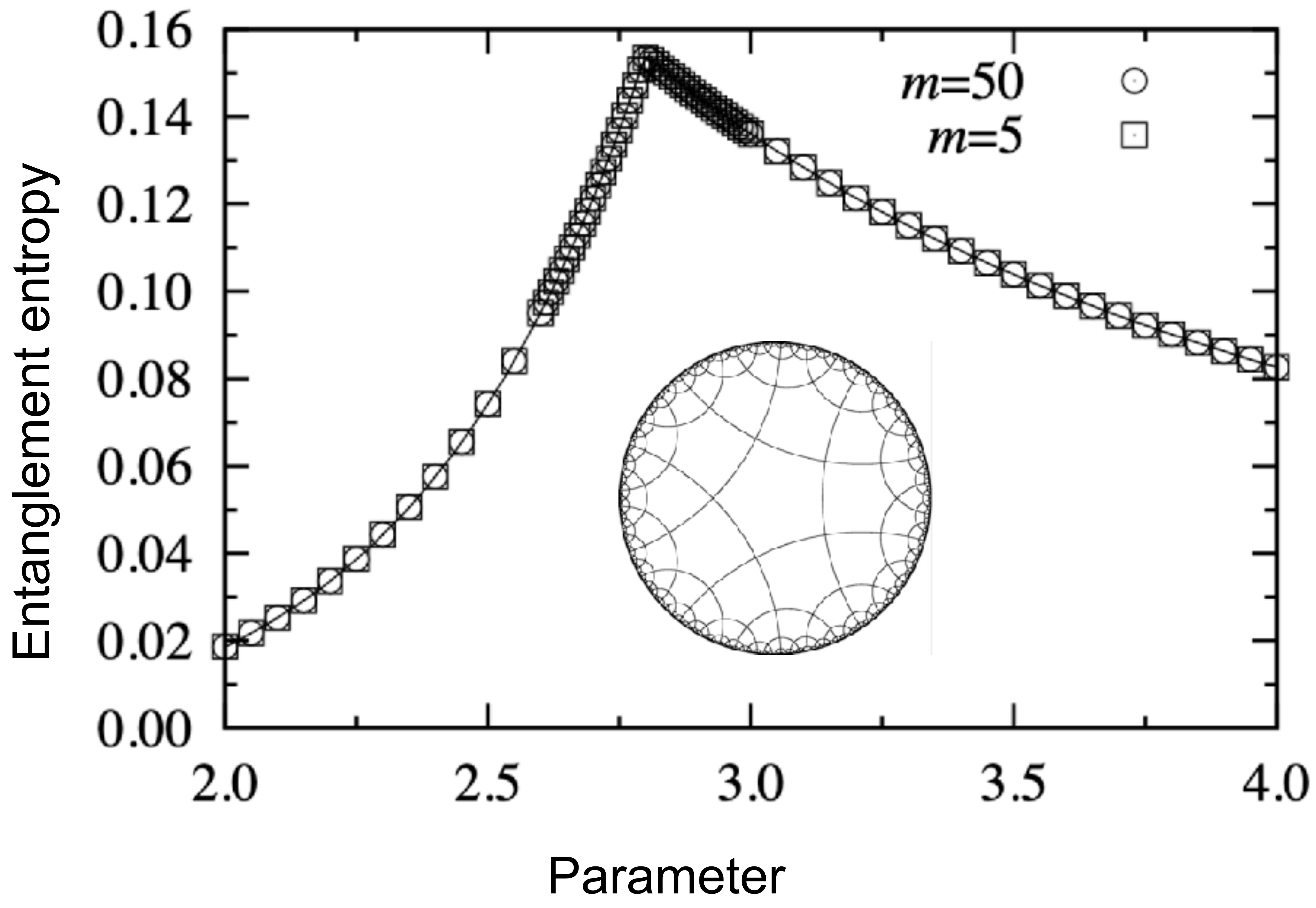


2nd order phase transition

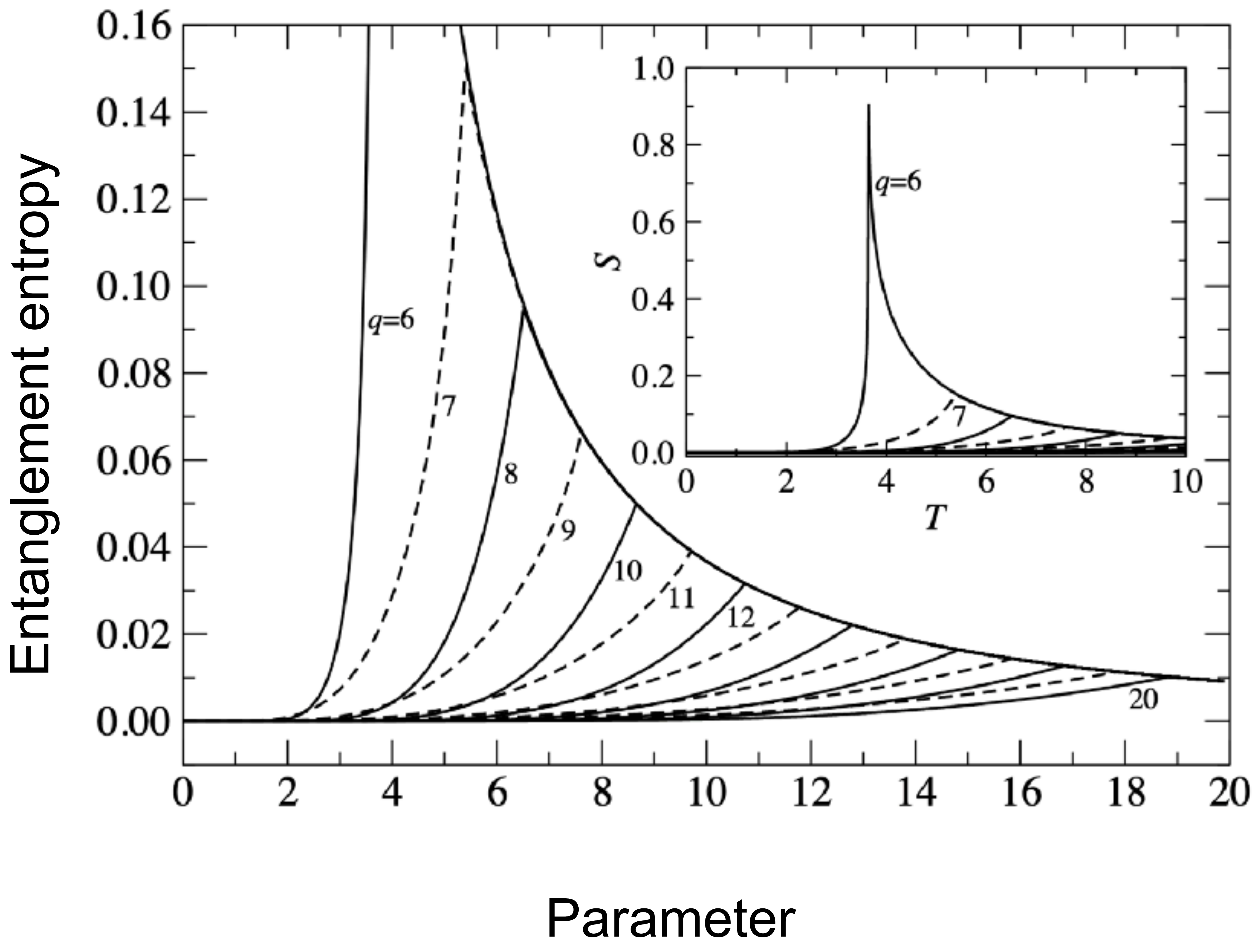


*Continuous phase transition in **hyperbolic** space*

2nd order phase transition



2nd order phase transition

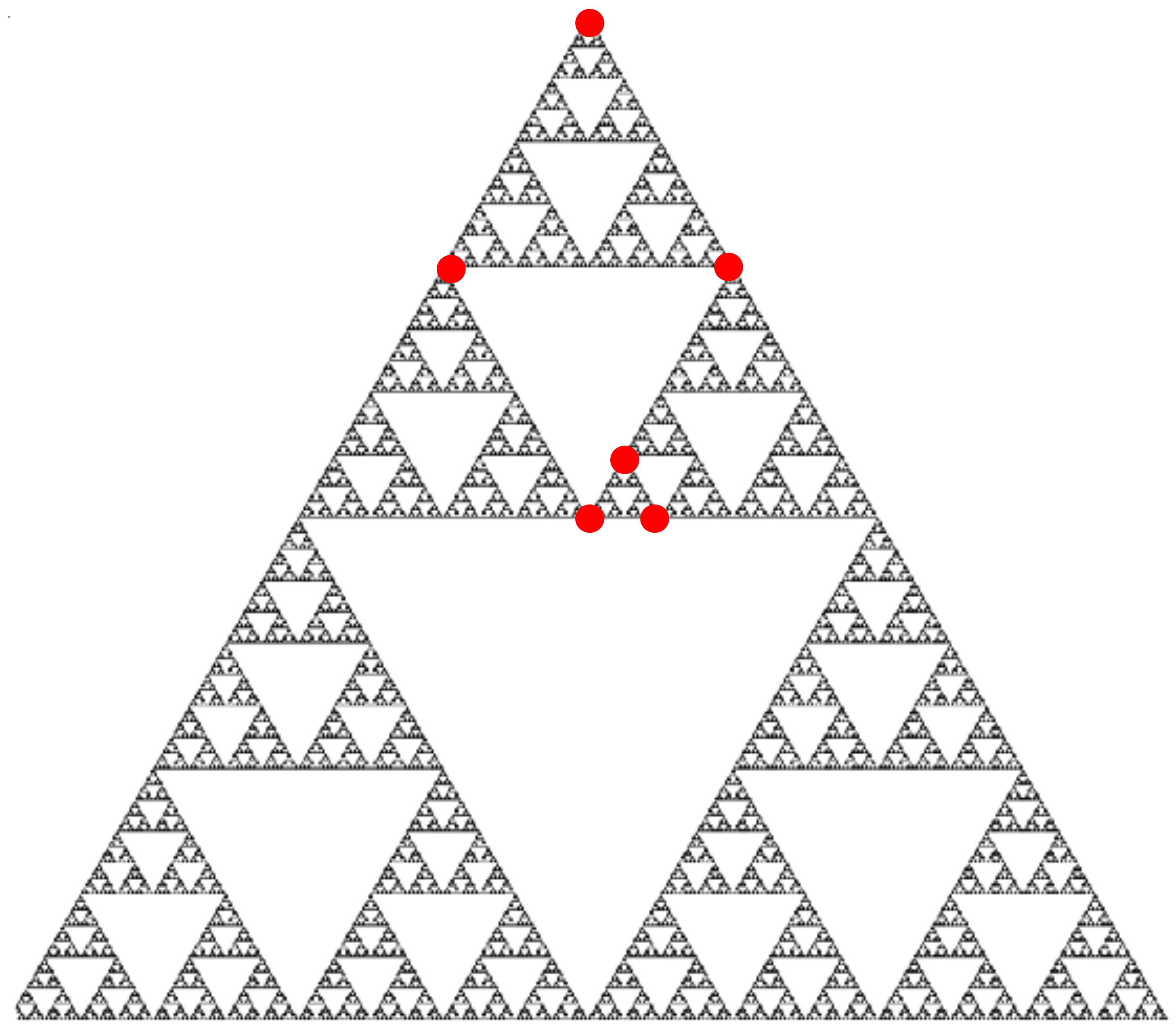


Entanglement entropy of

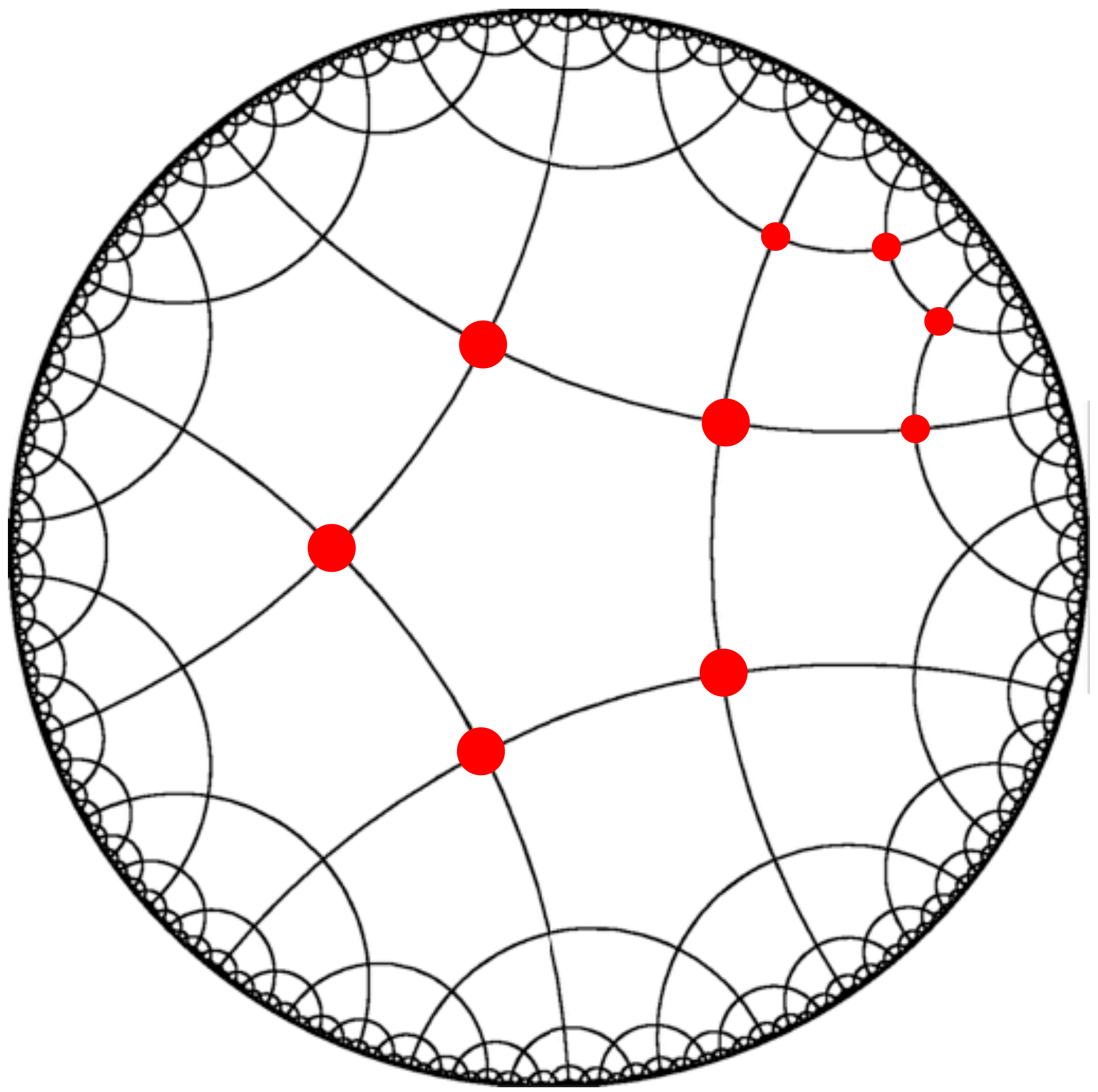
- **Part II** fractal lattices
- **Part III** hyperbolic lattices

- Focus on **two** types of lattice systems, where spins can interact.
- Classical and quantum spin models (example are shown in 2D).

Infinite fractal lattices



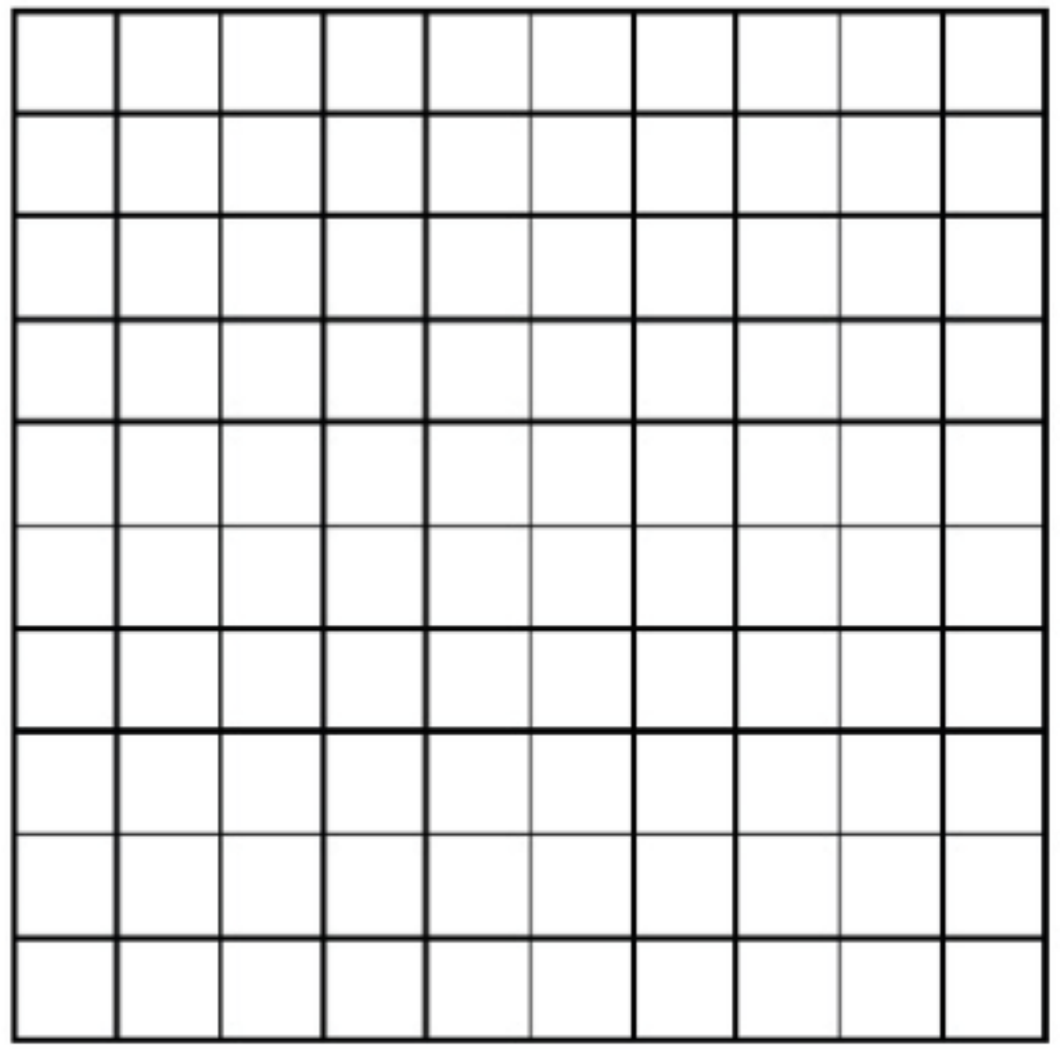
Infinite hyperbolic lattices



Tensor-Network analysis provides access to more complex systems.

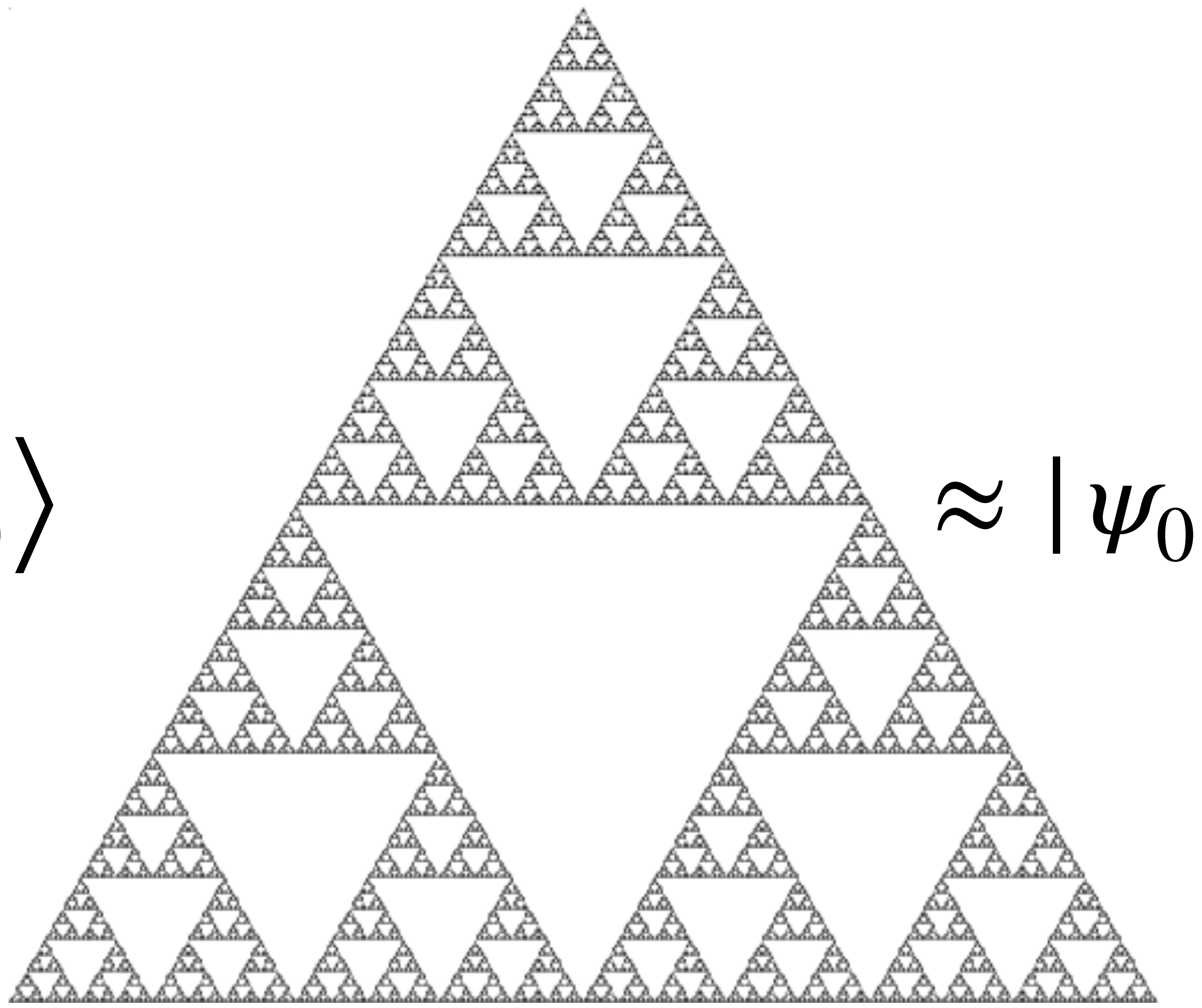
We use it as a robust **tool** for developing novel concepts for physics.

Let us restrict to ground state $|\psi_0\rangle$ and the strongest entanglement.



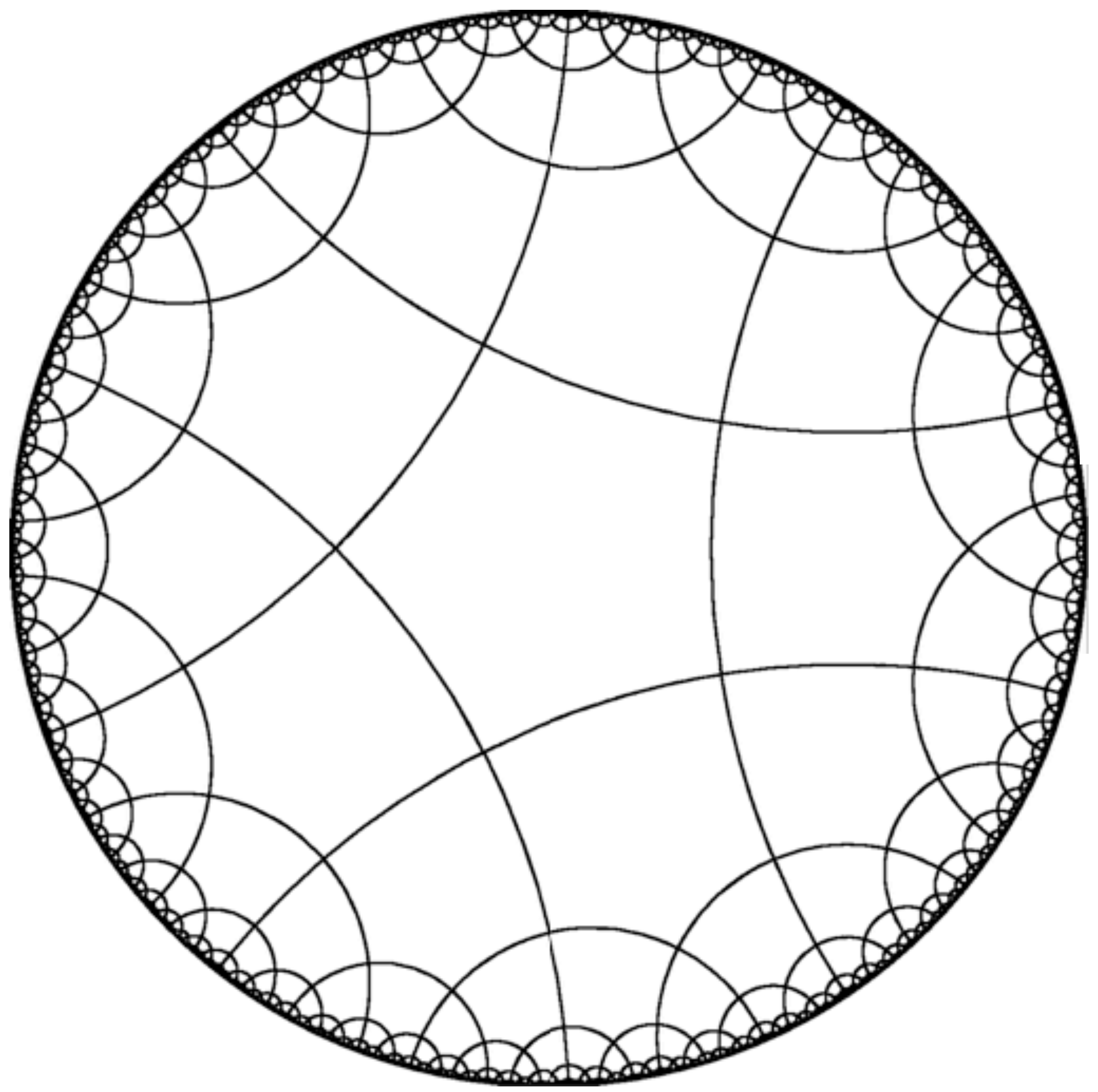
$\approx |\psi_0\rangle$

Square lattice
(Euclidean space)
$d_{\mathrm{H}} = 2$



$\approx |\psi_0\rangle$

Fractal lattice
(Self-similarity)
$d_{\mathrm{H}} = 1.585...$

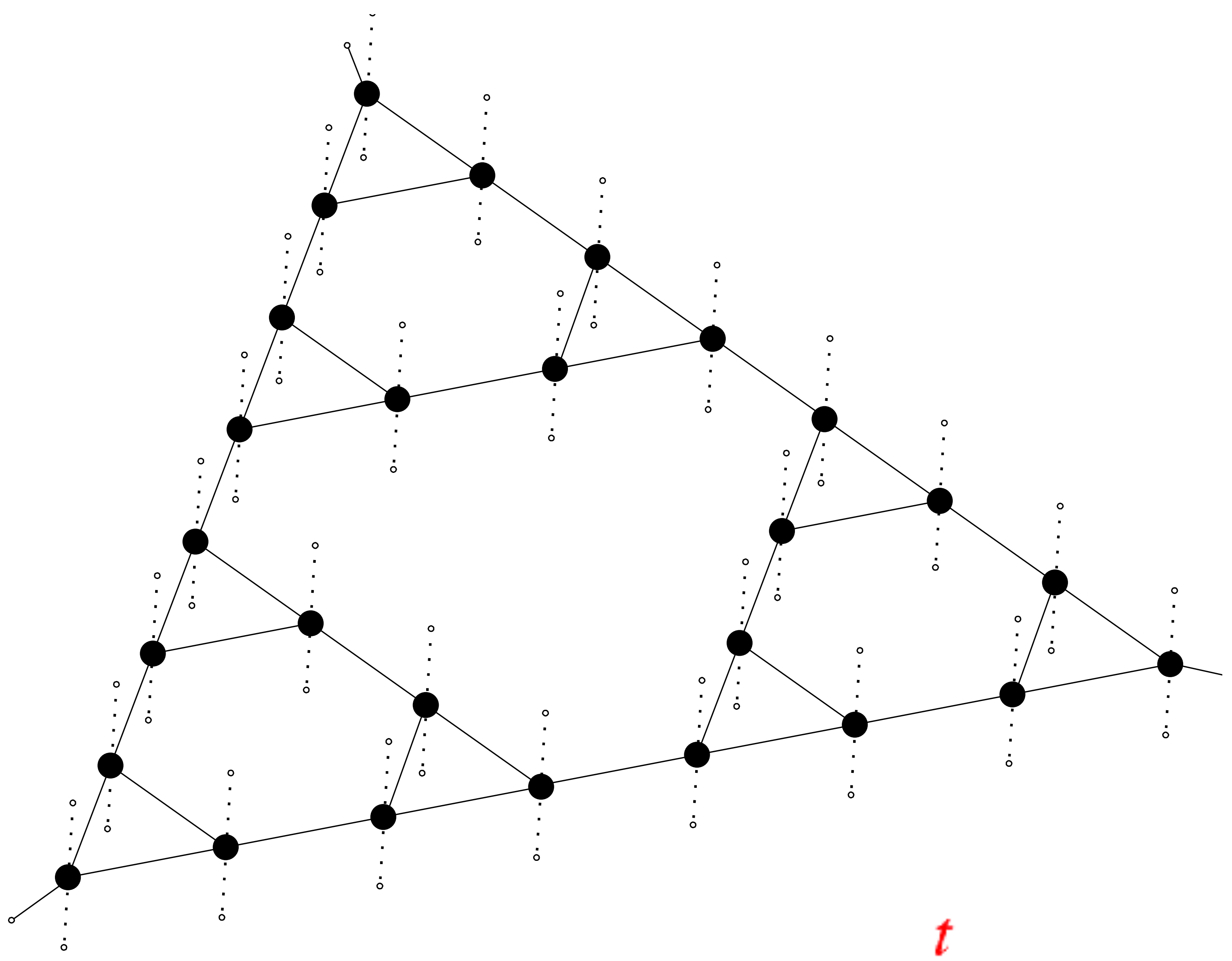


$\approx |\psi_0\rangle$

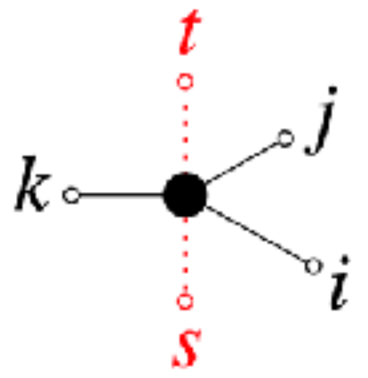
Hyperbolic lattice
(non-Euclidean space)
$d_{\mathrm{H}} \to \infty$

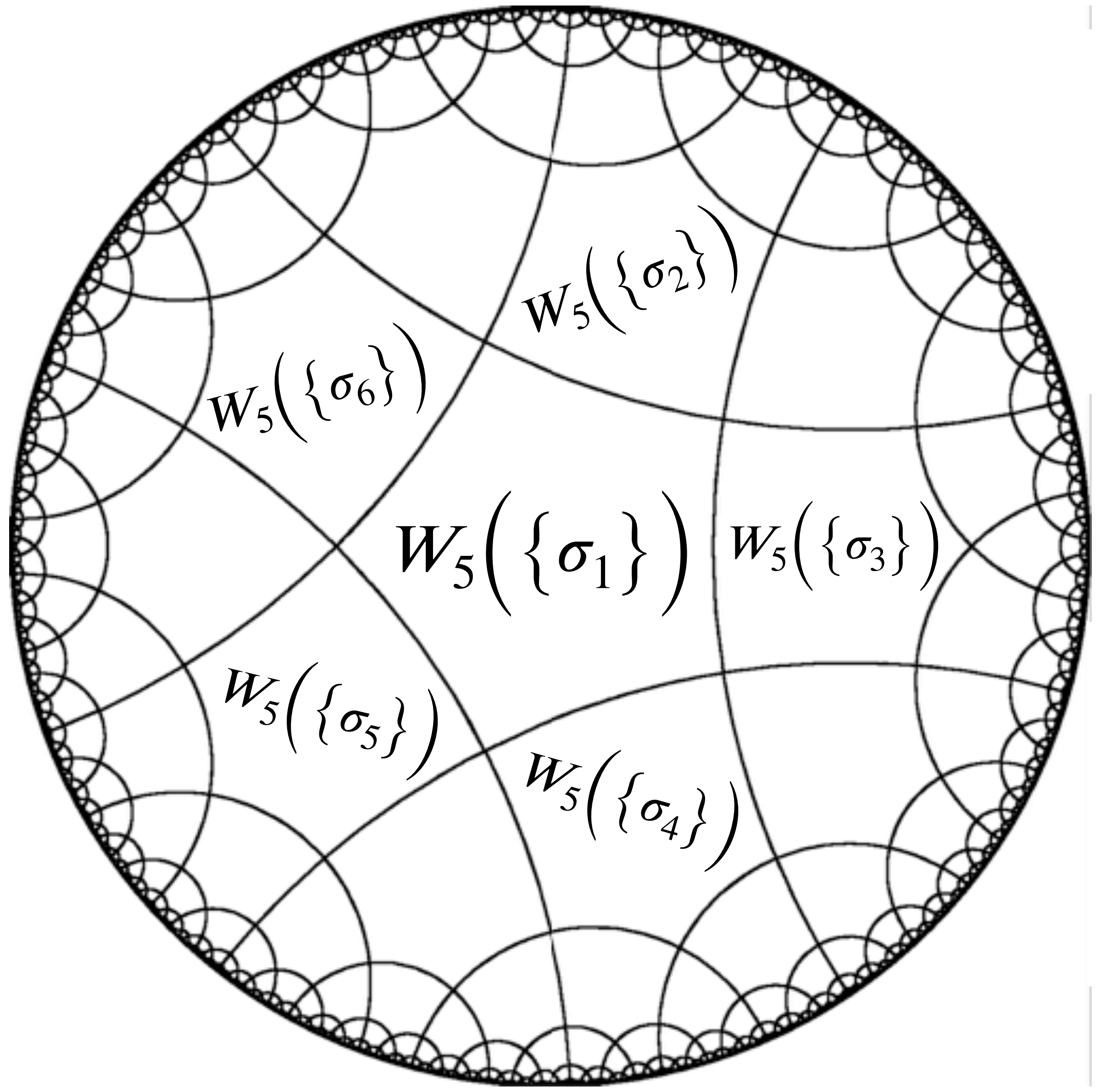
***Three** generalized Tensor-network algorithms used:*

1. Tensor Product Variational Approximation (TPVA)
2. Higher-Order Tensor Renormalization Group (HOTRG)
3. Corner Transfer Matrix Renormalization Group (CTMRG)



$W_{ijk,ts}$ =





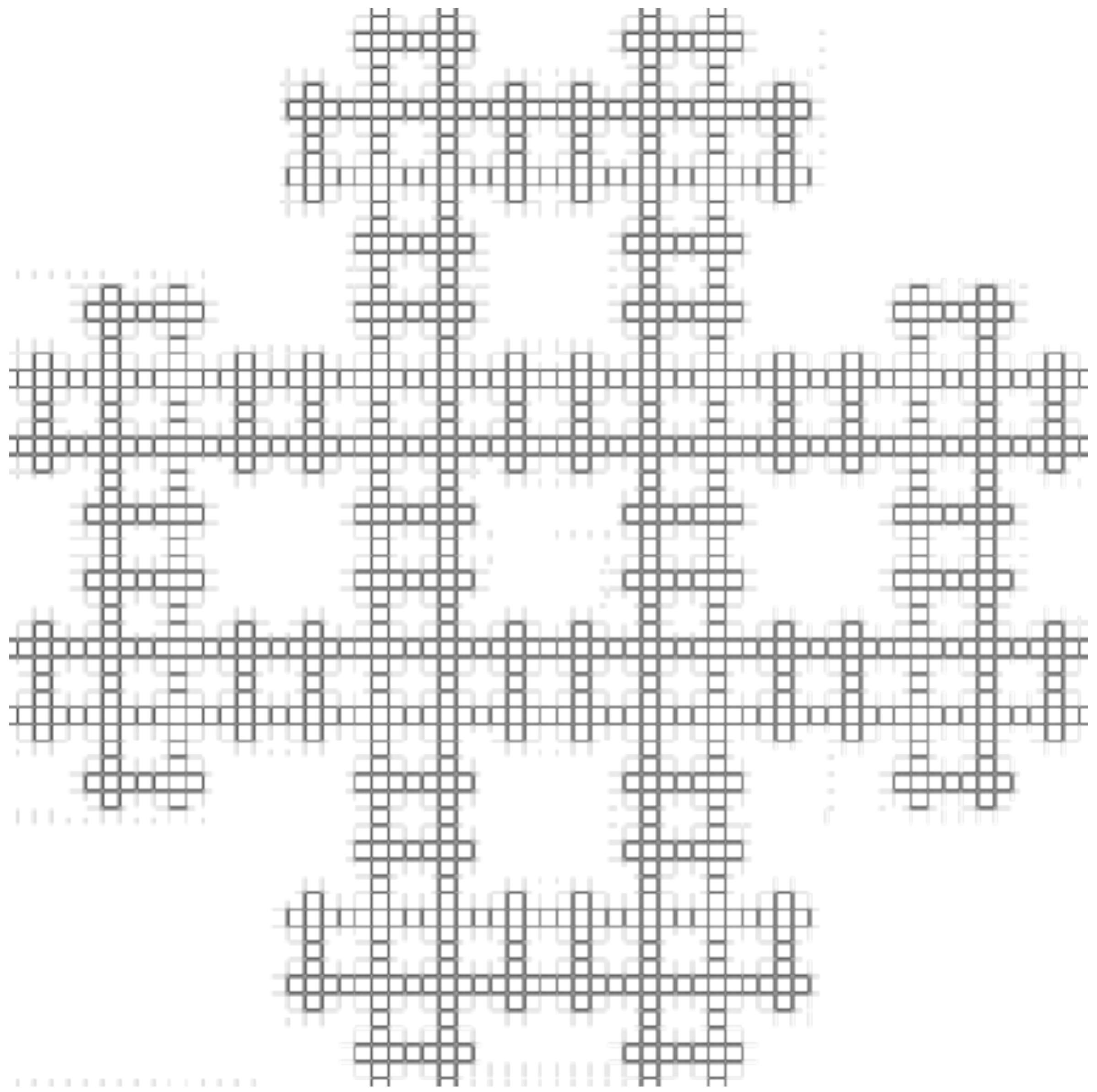
$$|\psi_{(p,q)}\rangle := |\psi_p\rangle = \lim_{N\to\infty} \sum_{\{\sigma_1\},\{\sigma_2\}\ldots\{\sigma_N\}} \prod_{\langle k\rangle_p} W_p\left(\{\sigma_k\}\right) |\sigma_1\sigma_2\ldots\sigma_N\rangle$$

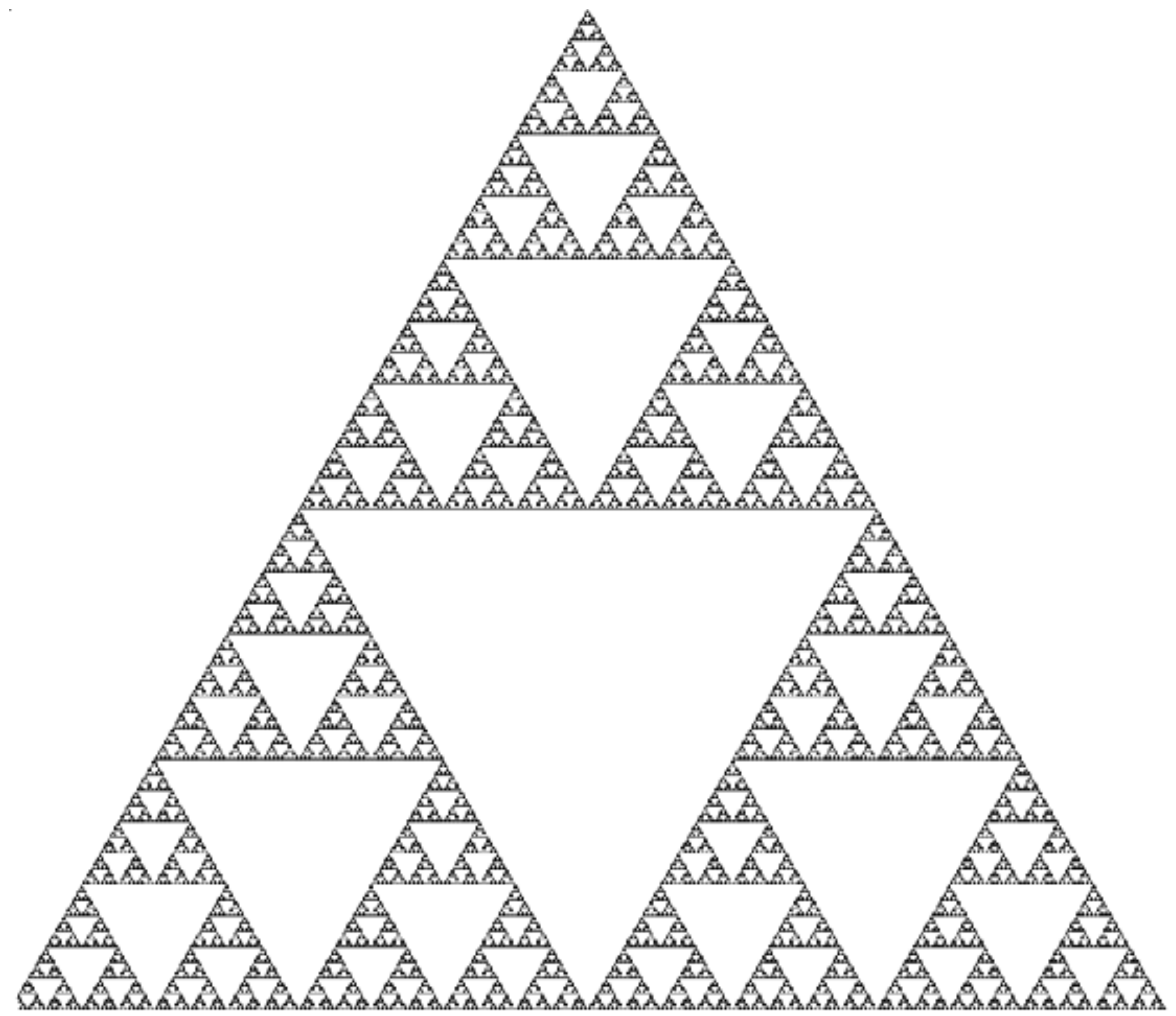
Part II

Phase transitions on ***fractals***

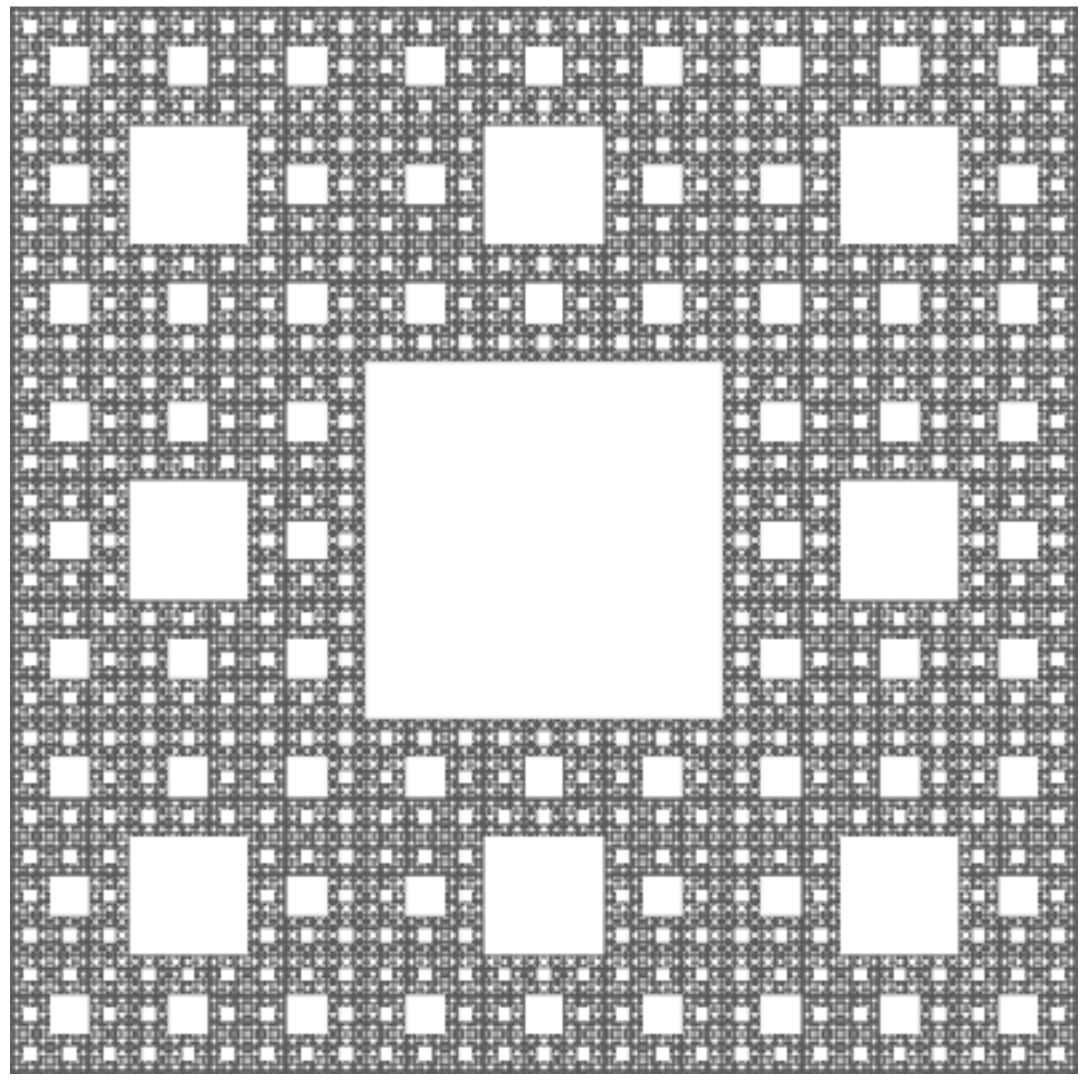
- classical and quantum spin models

Tensor networks designed to study fractals





No phase transition
(for classical spins)

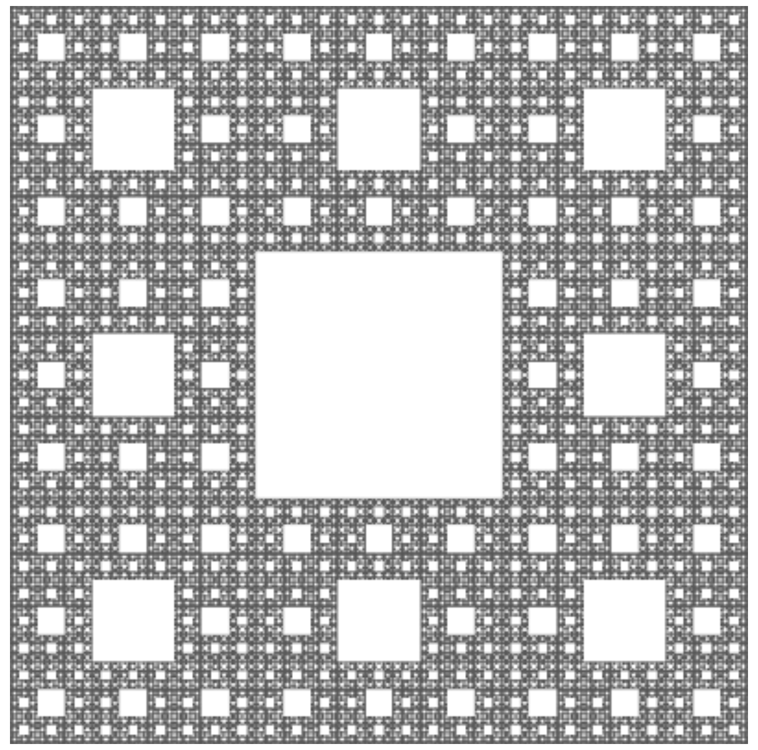


Fractal Dimension

What happens as we increase the linear-size (diameter) of the system?

L: linear-size magnification factor

N: volume-size multiplication factor

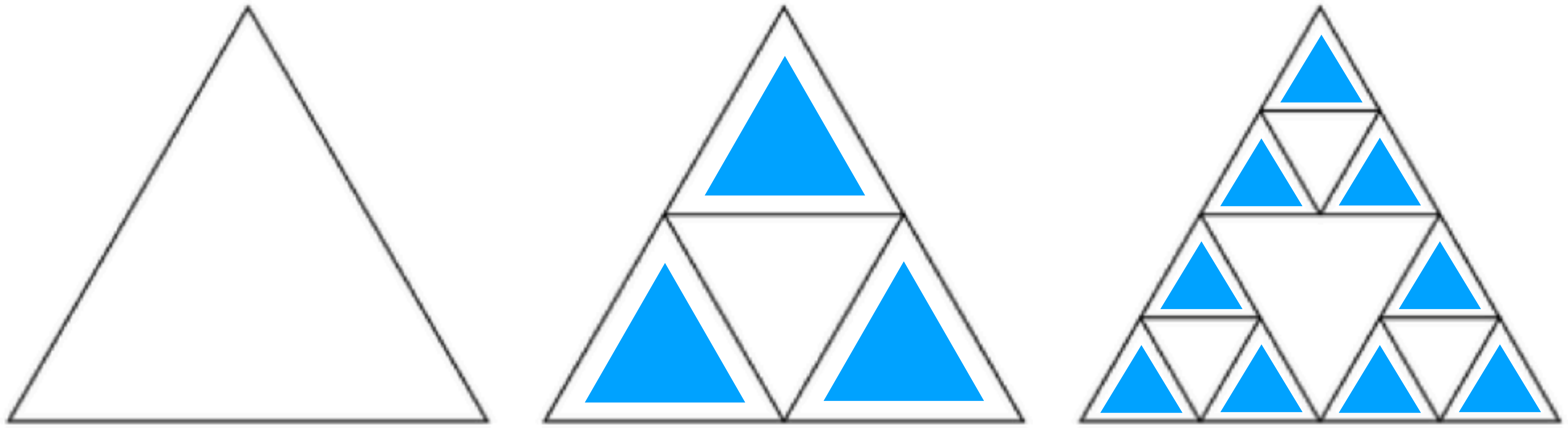


$$d_H = \frac{\ln 8}{\ln 3} \approx 1.893$$

Scaling of the volume (*Hausdorff* dimension): d_H

$$N = L^{d_H}$$

$$d_H = \frac{\ln N}{\ln L}$$

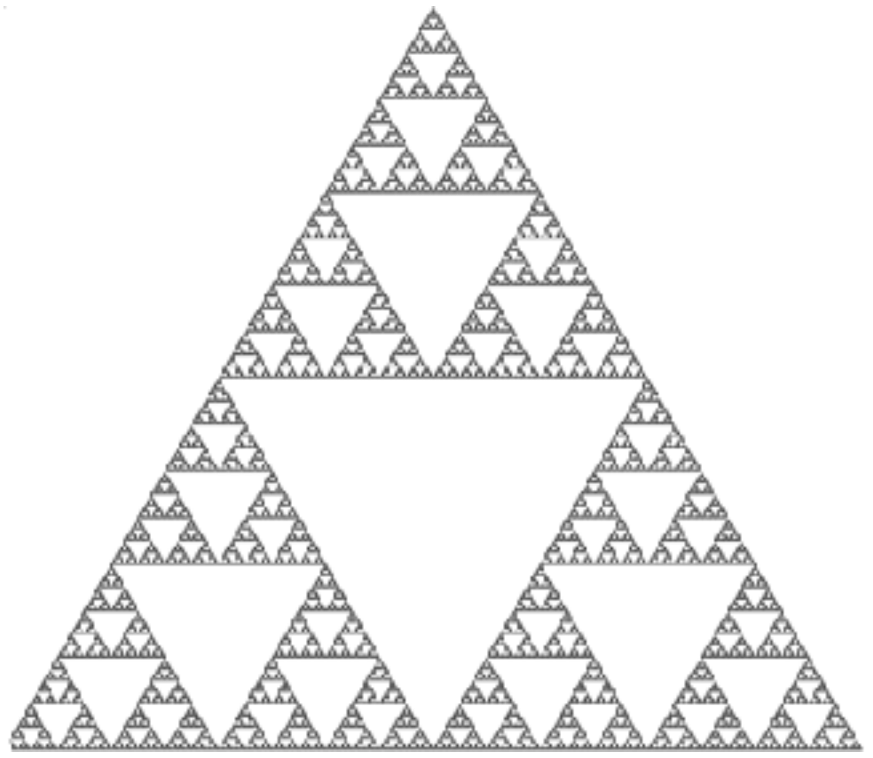


$N = 3^0 = 1$, $L = 2^0 = 1$

$N = 3^1$, $L = 2^1$

$N = 3^2$, $L = 2^2$

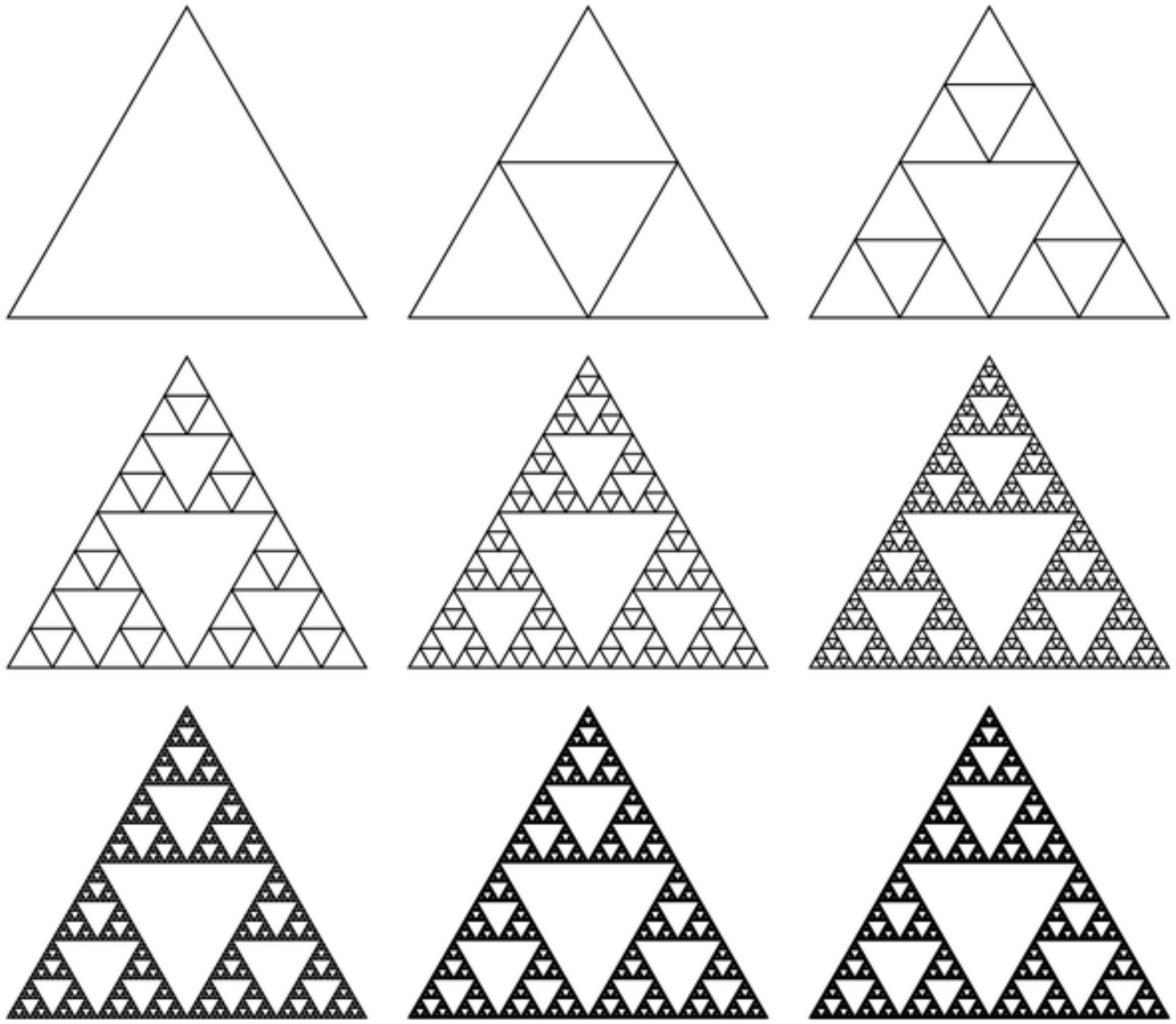
…



$$d_H = \frac{\ln 3^N}{\ln 2^N} \approx 1.585$$

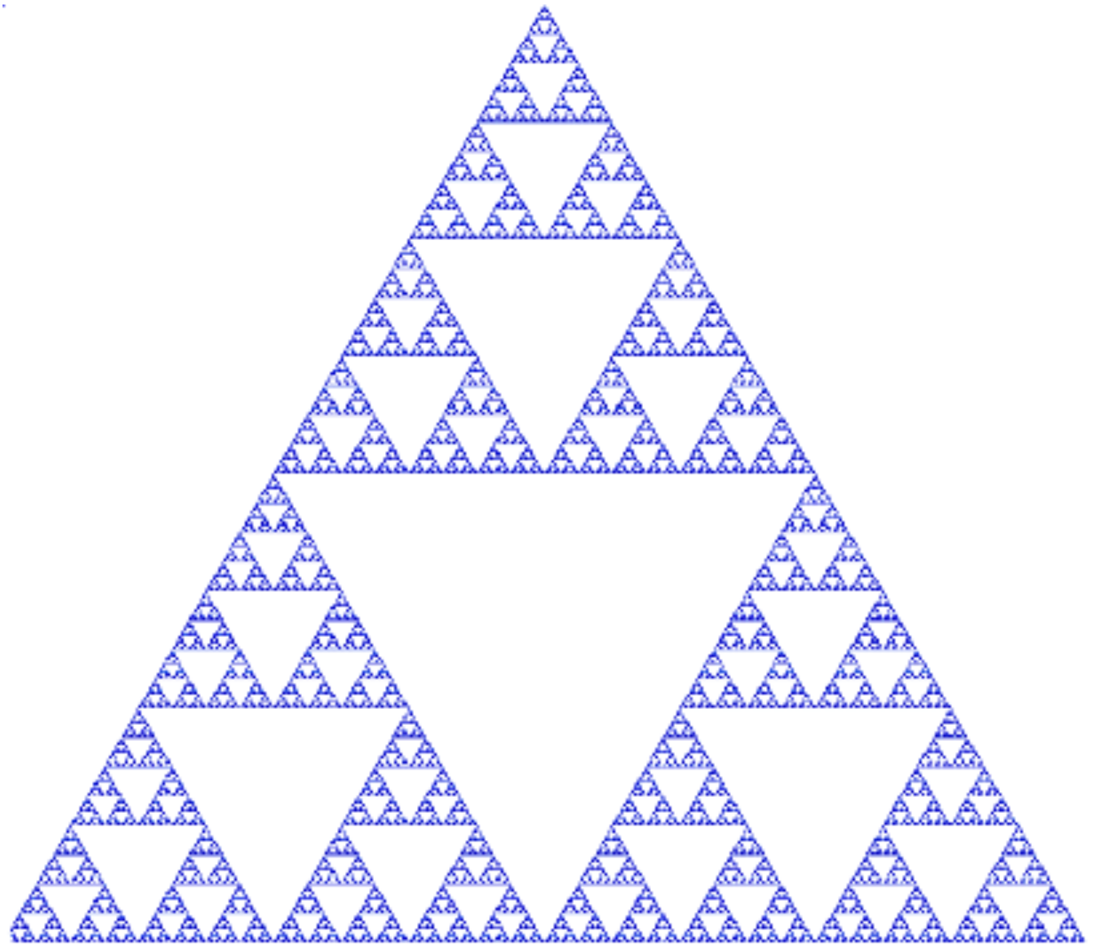
Self-similarity on an infinite lattice with spacings fixed to be finite and identical.

No phase transition exists for the **classical** Ising model on the **Sierpinski triangle** (gasket).
However, there is a phase transition for the quantum Ising model on this fractal.

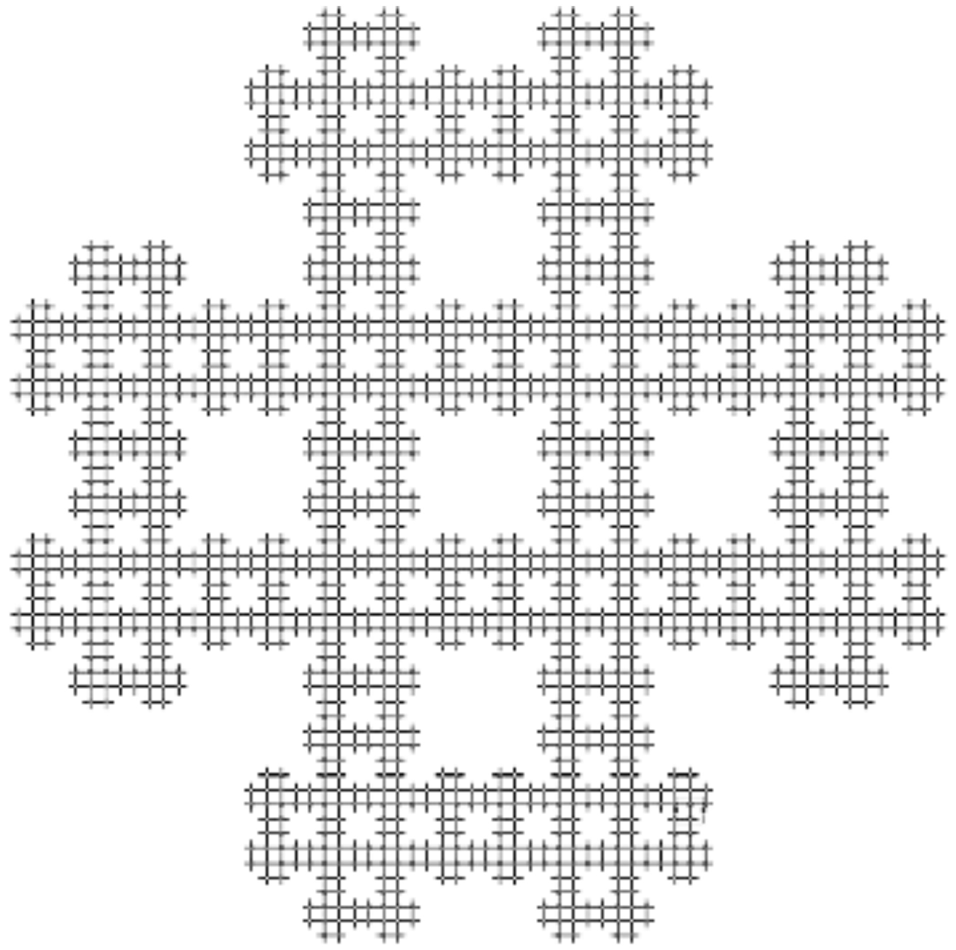


Fractal (Hausdorff) dimension: $d_H = \log_2 3 \approx 1.585$

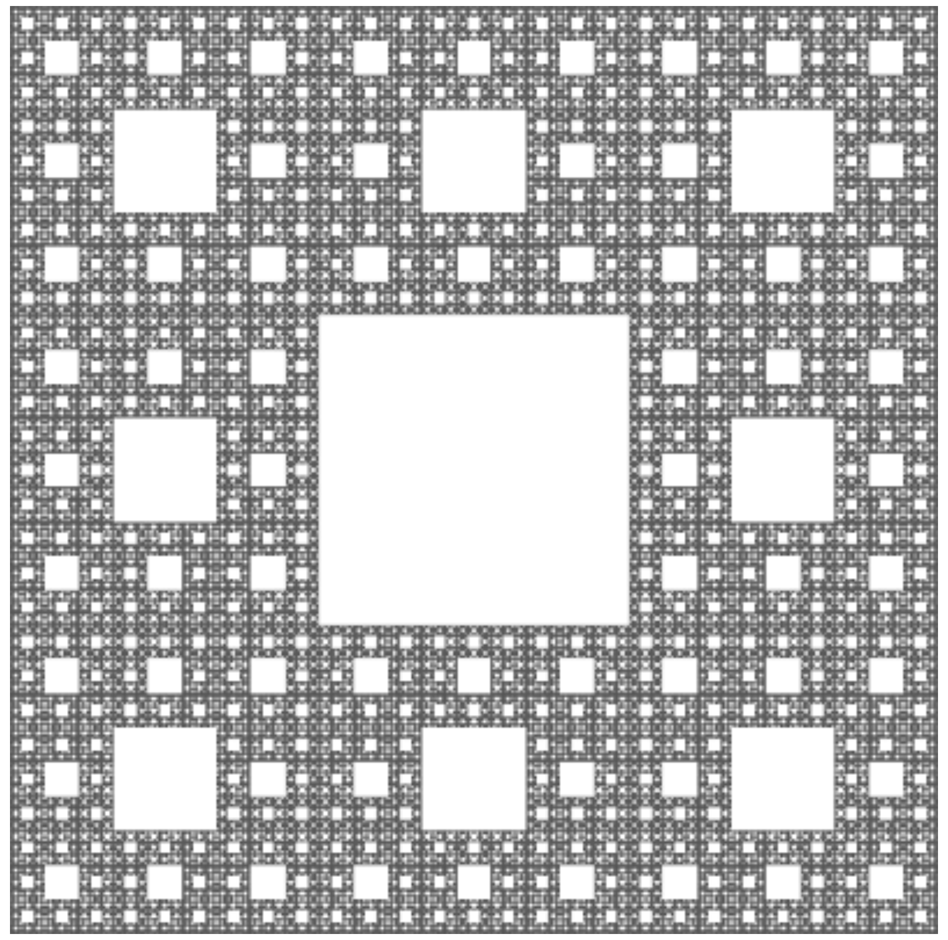
List of basic fractals we have successfully studied by Tensor Networks



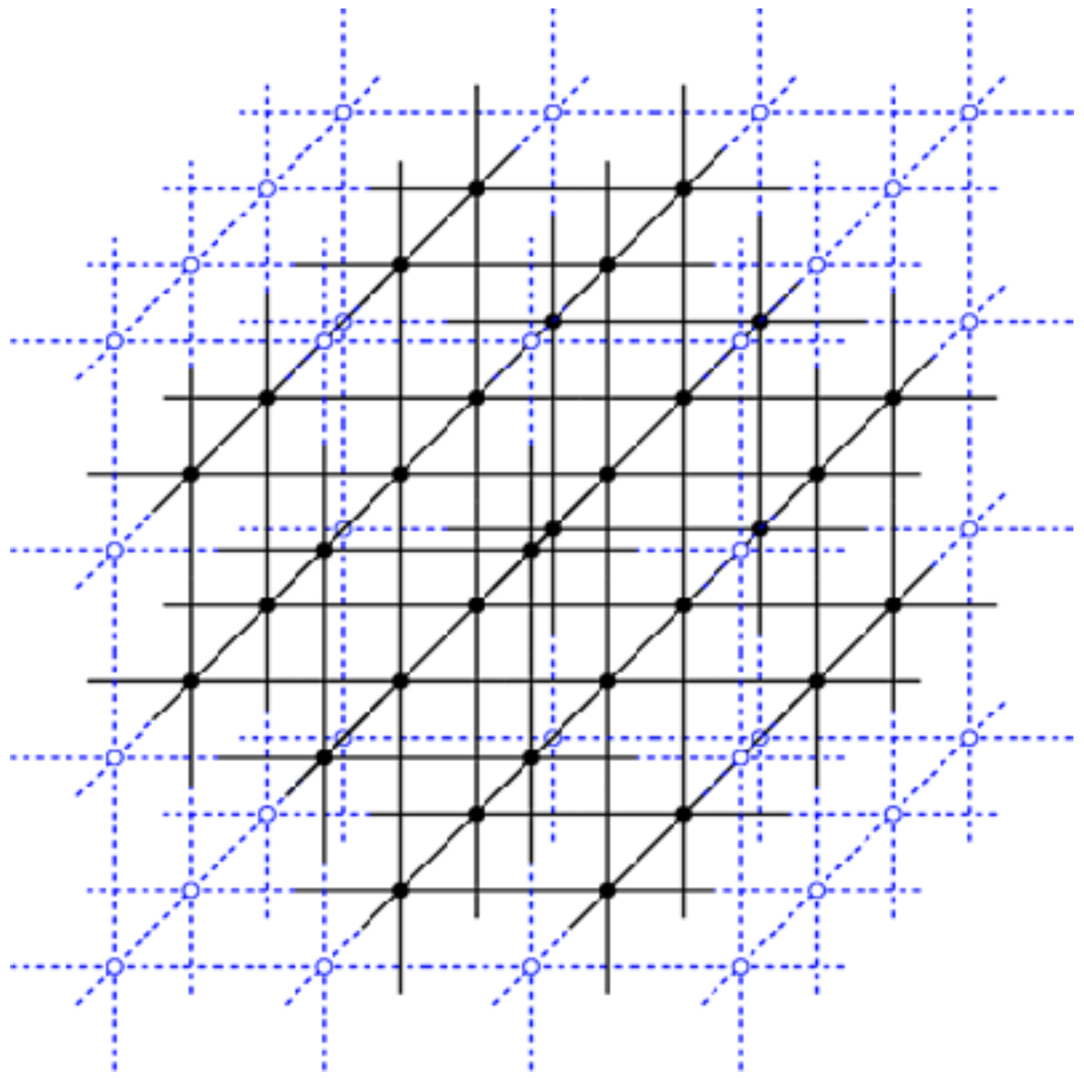
$d_H = \log_2 3 \approx 1.585$

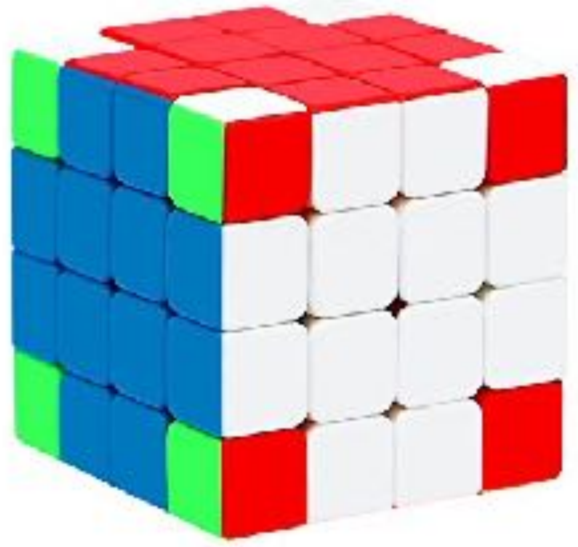


$d_H = \log_4 12 \approx 1.792$

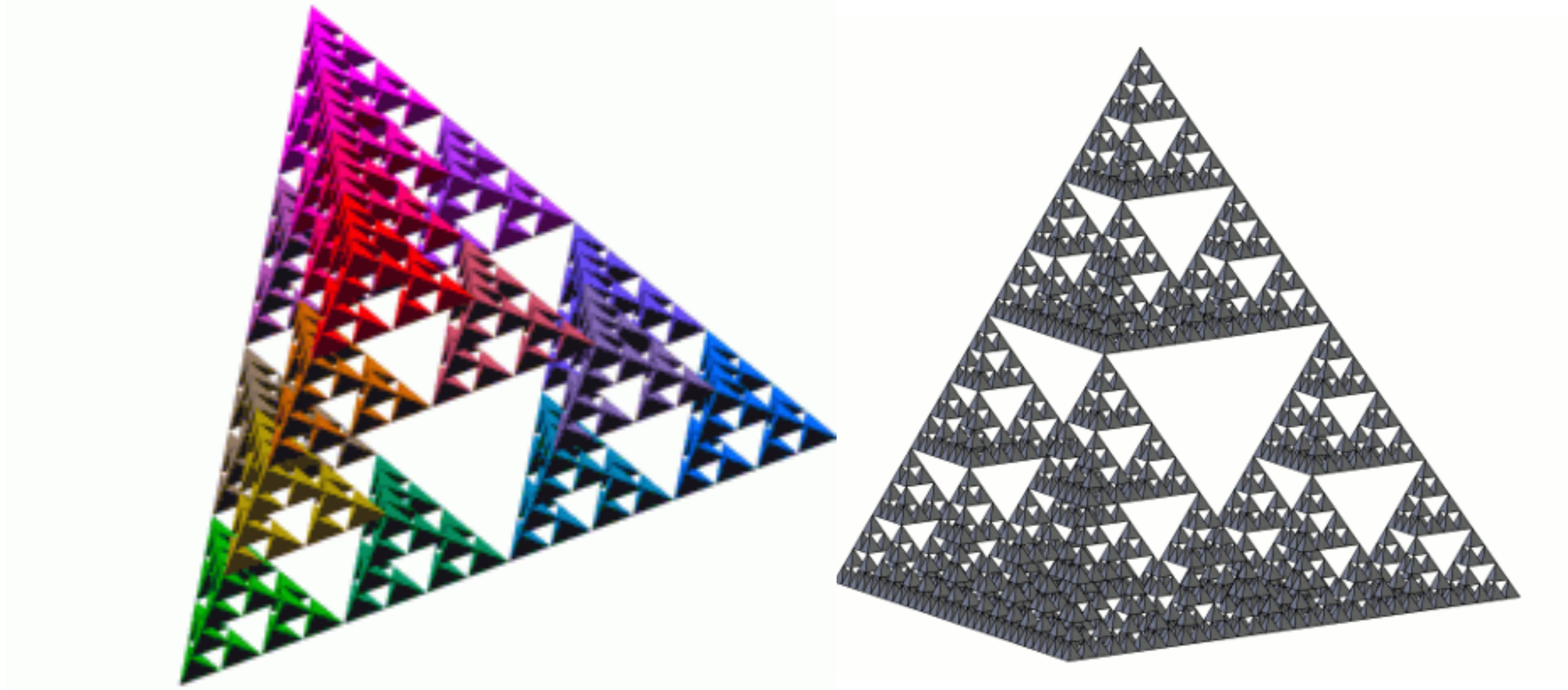


$d_H = \log_3 8 \approx 1.893$





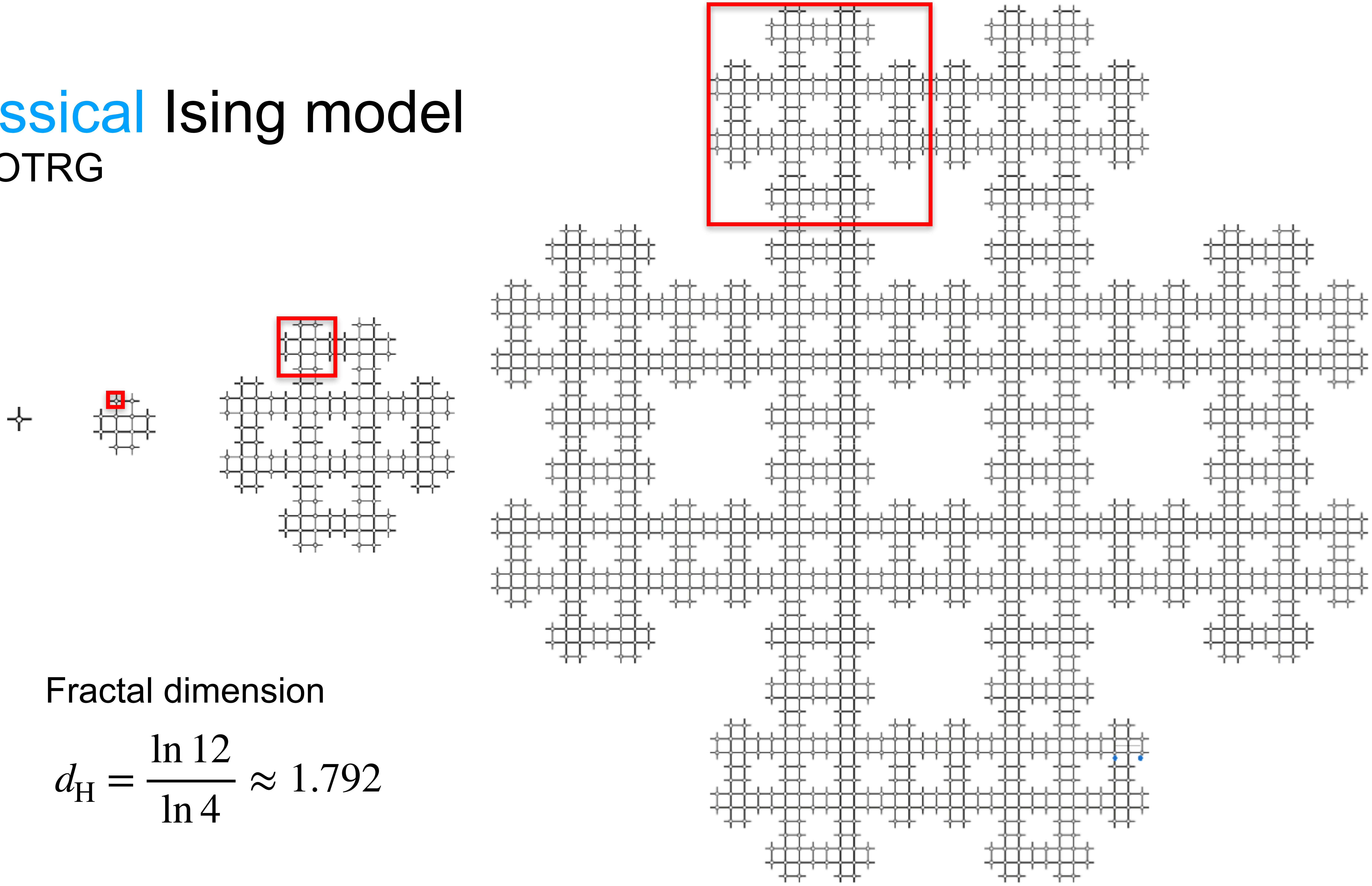
$d_H = \log_4 32 = 2.5$



$d_H = \log_2 4 = 2$ (!)

Classical Ising model

by HOTRG



Fractal dimension

$$d_{\mathrm{H}} = \frac{\ln 12}{\ln 4} \approx 1.792$$

J. Genzor, A. Gendiar, T. Nishino, *Phys. Rev. E* **93**, 012141 (2016)

A weak 2nd order phase transition is detected in the classical Ising model

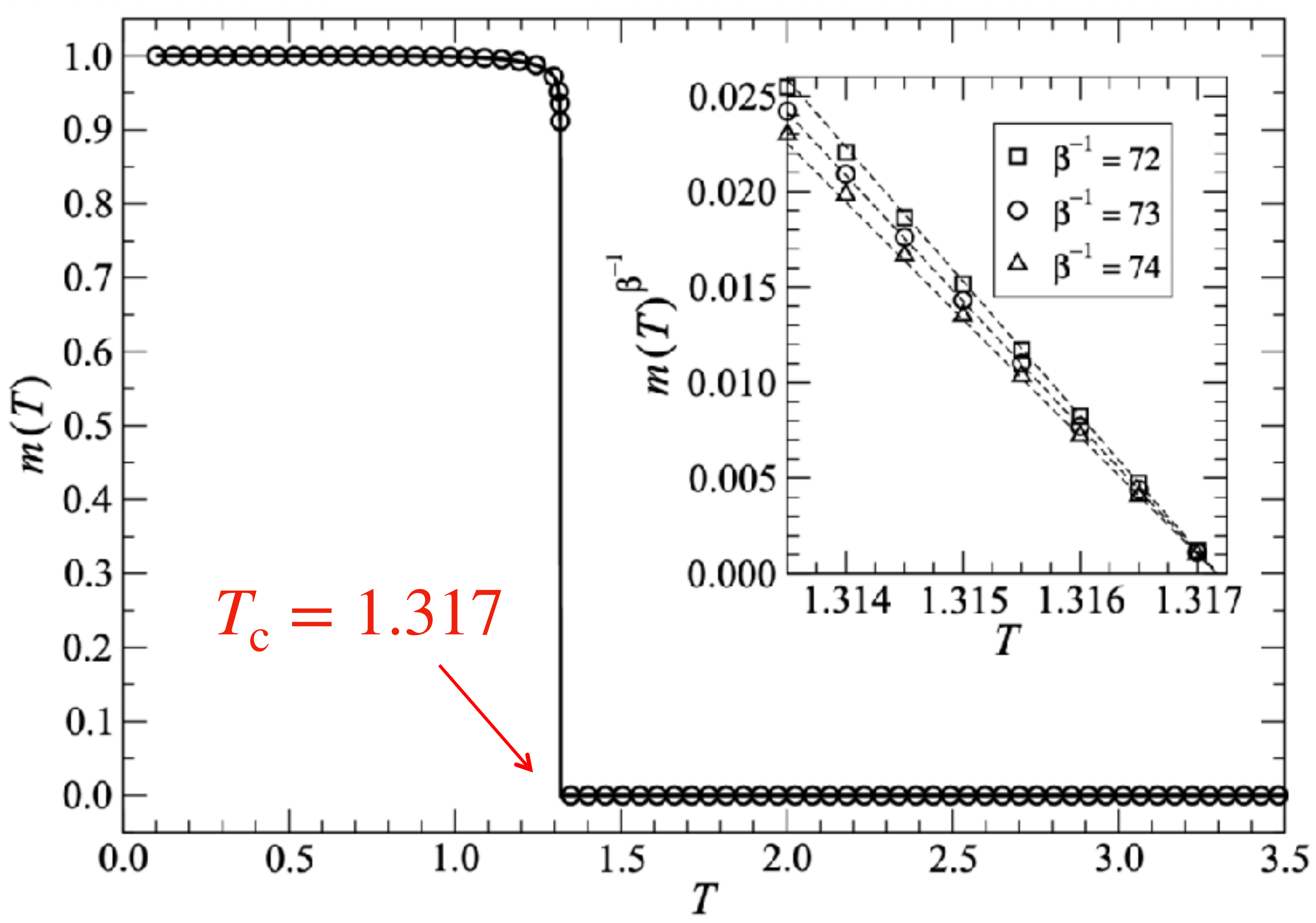
$T_c = 1.317$

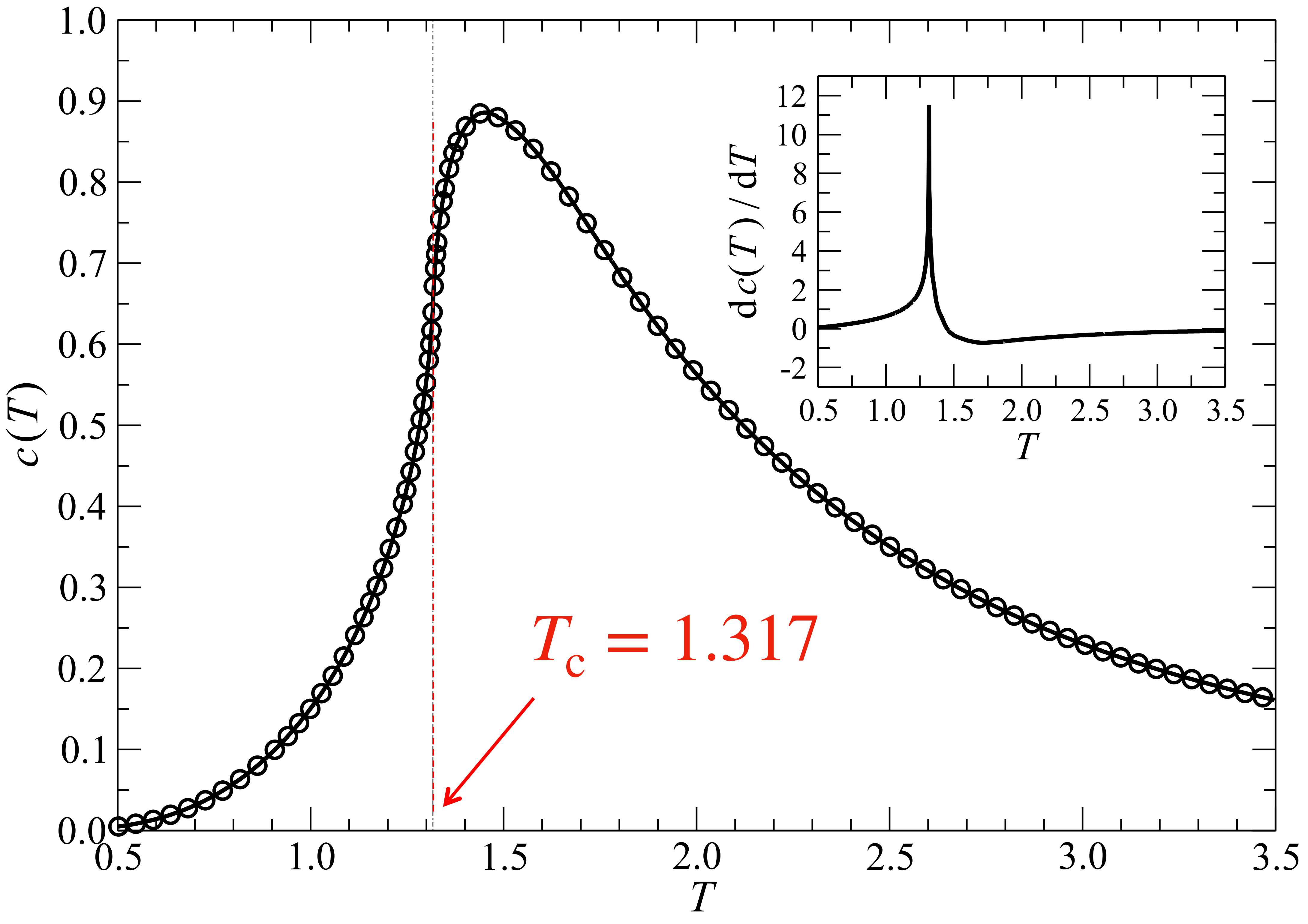
$\beta = 0.0137$

$$c(T) = -T\frac{d^2F}{dT^2}$$

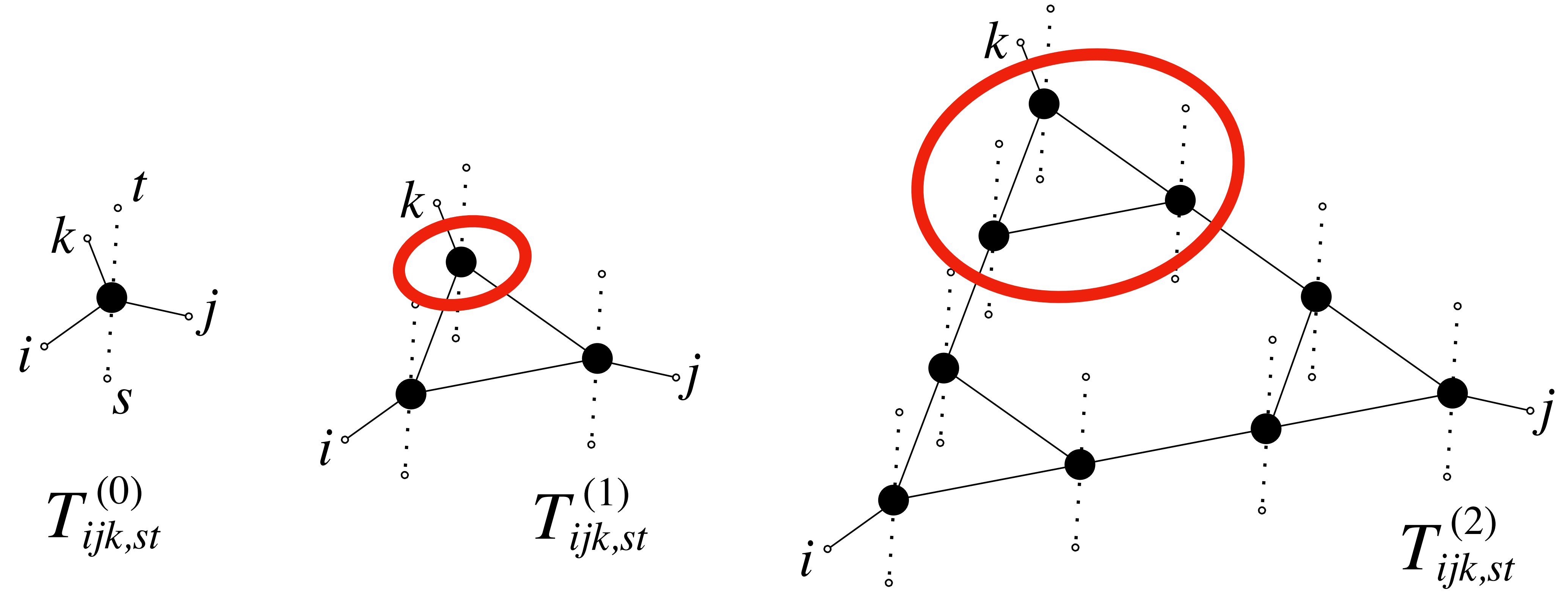
Non-diverging specific heat?!

(logarithmic corrections)



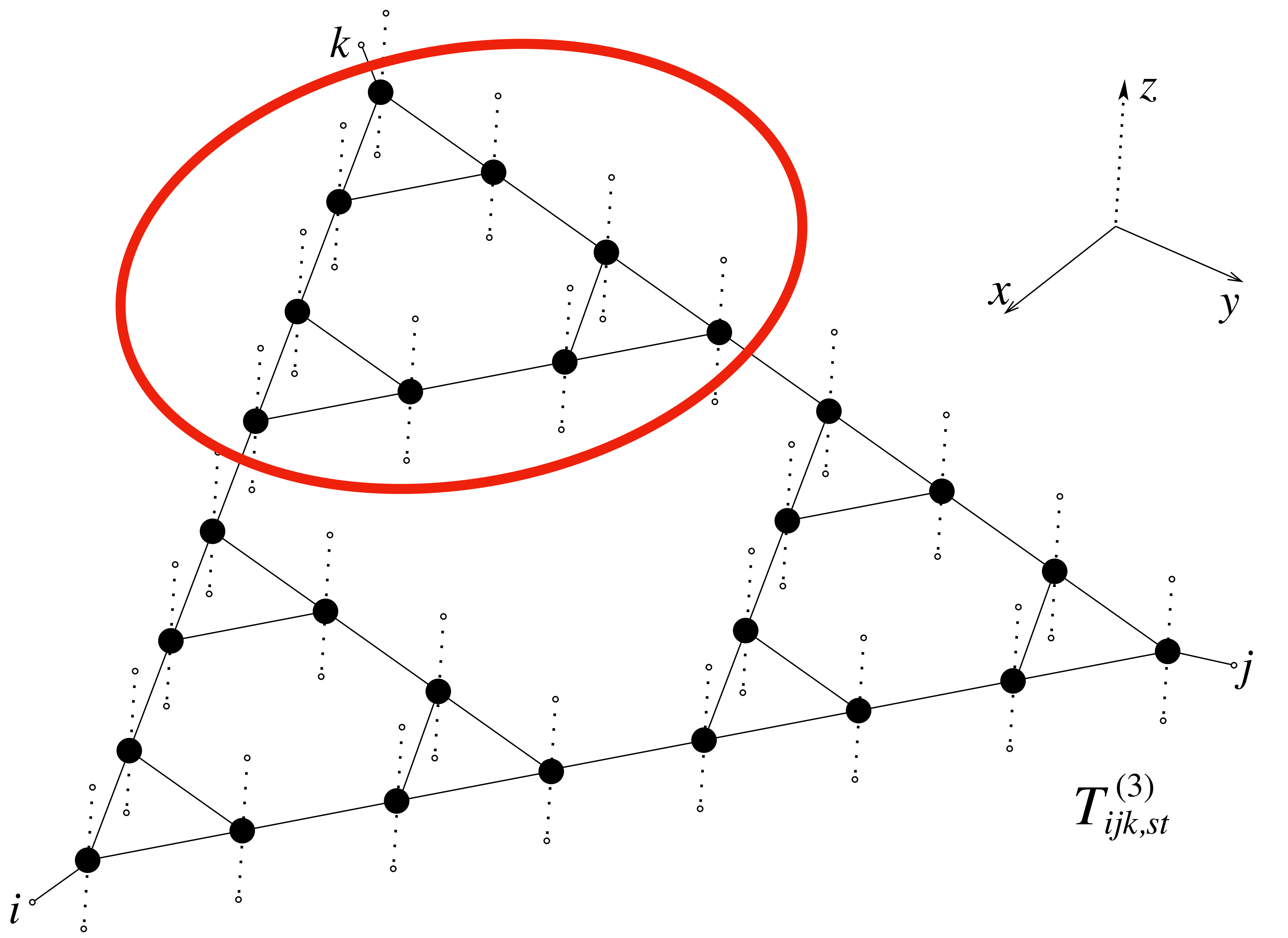


Quantum Ising model
by HOTRG



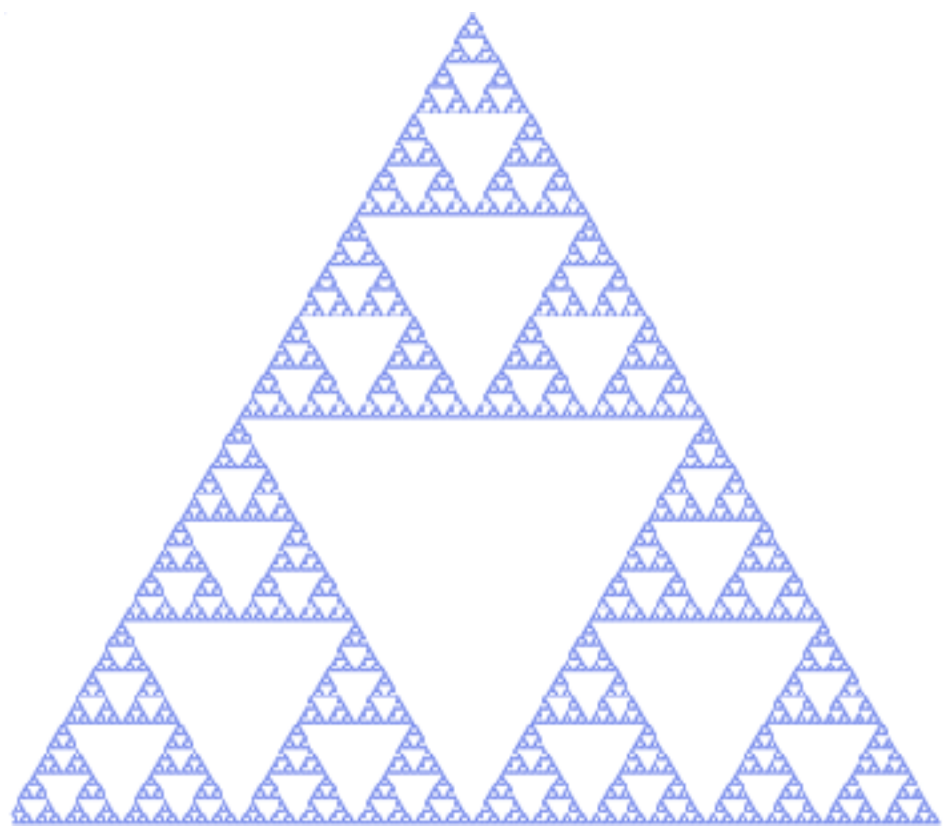
Fractal dimension

$$d_{\rm H} = \frac{\ln 3}{\ln 2} \approx 1.585$$



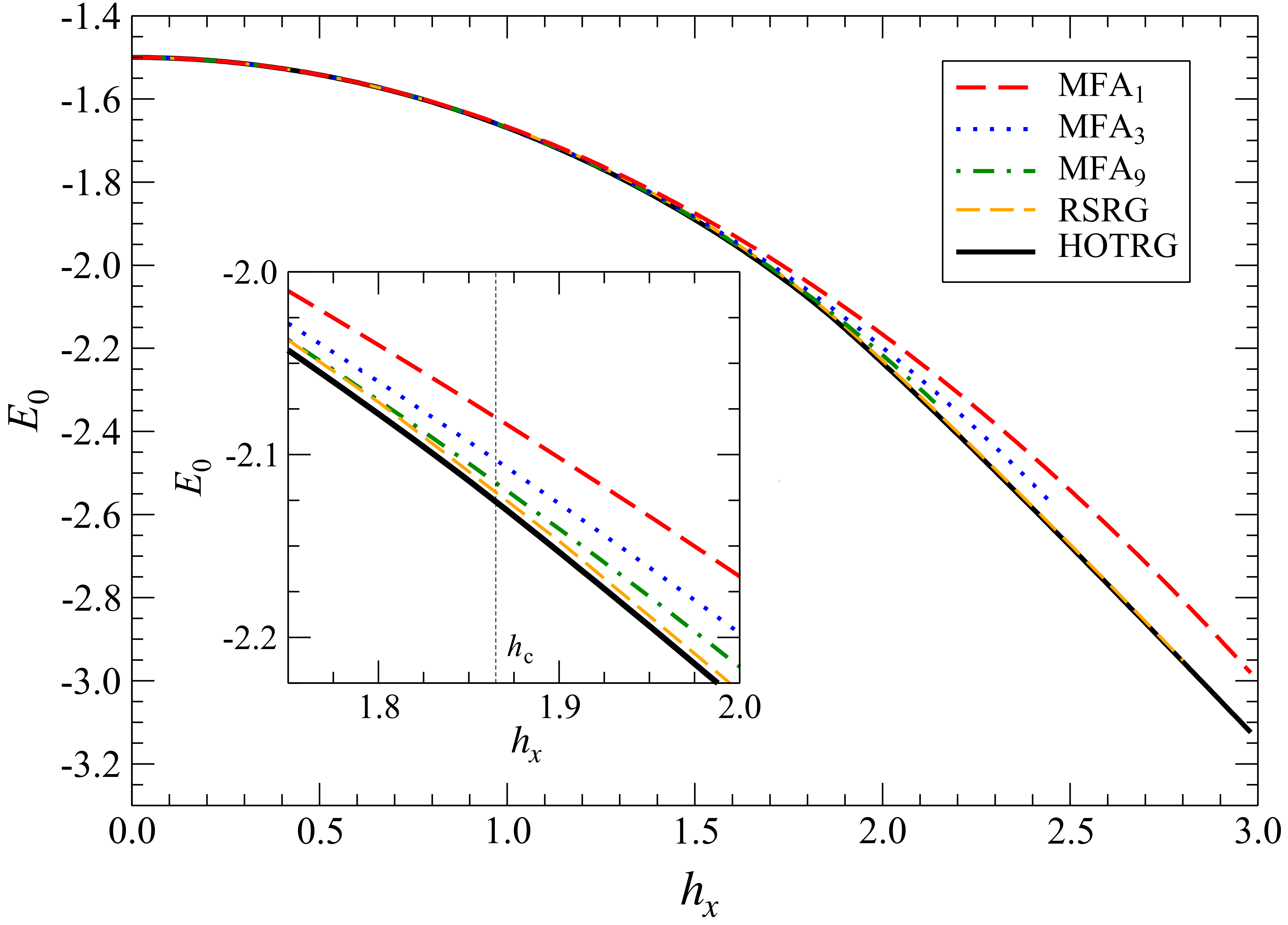
R. Krčmár, J. Genzor, Y. Lee, H. Čenčariková, T. Nishino, A. Gendiar, *Phys. Rev. E* **98**, 062114 (2018)

Ground-state energy E_0 of the **quantum Ising model** on Sierpiński triangle



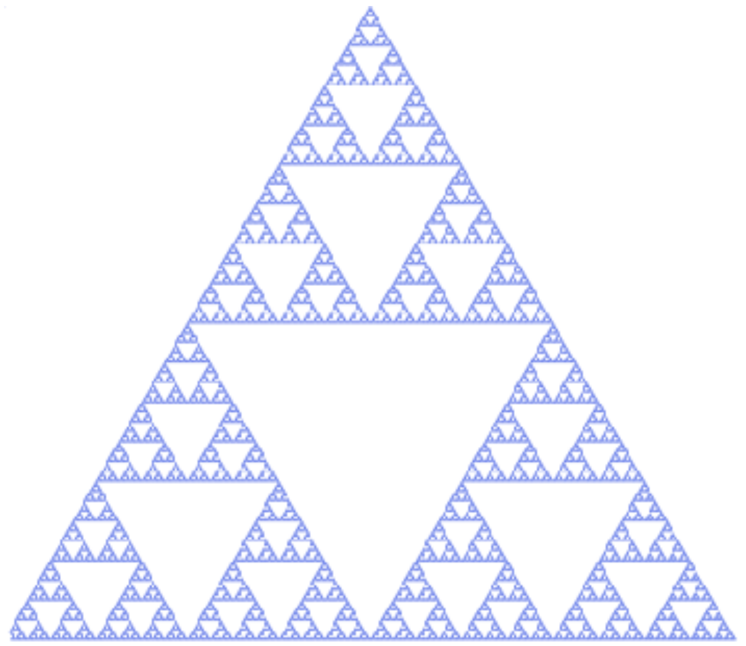
$$\mathcal{H}|\Psi_0\rangle = E_0|\Psi_0\rangle$$

$$\mathscr{H} = -J\sum_{\{i,j\}}\sigma_i^z\sigma_j^z - h_x\sum_i\sigma_i^x$$



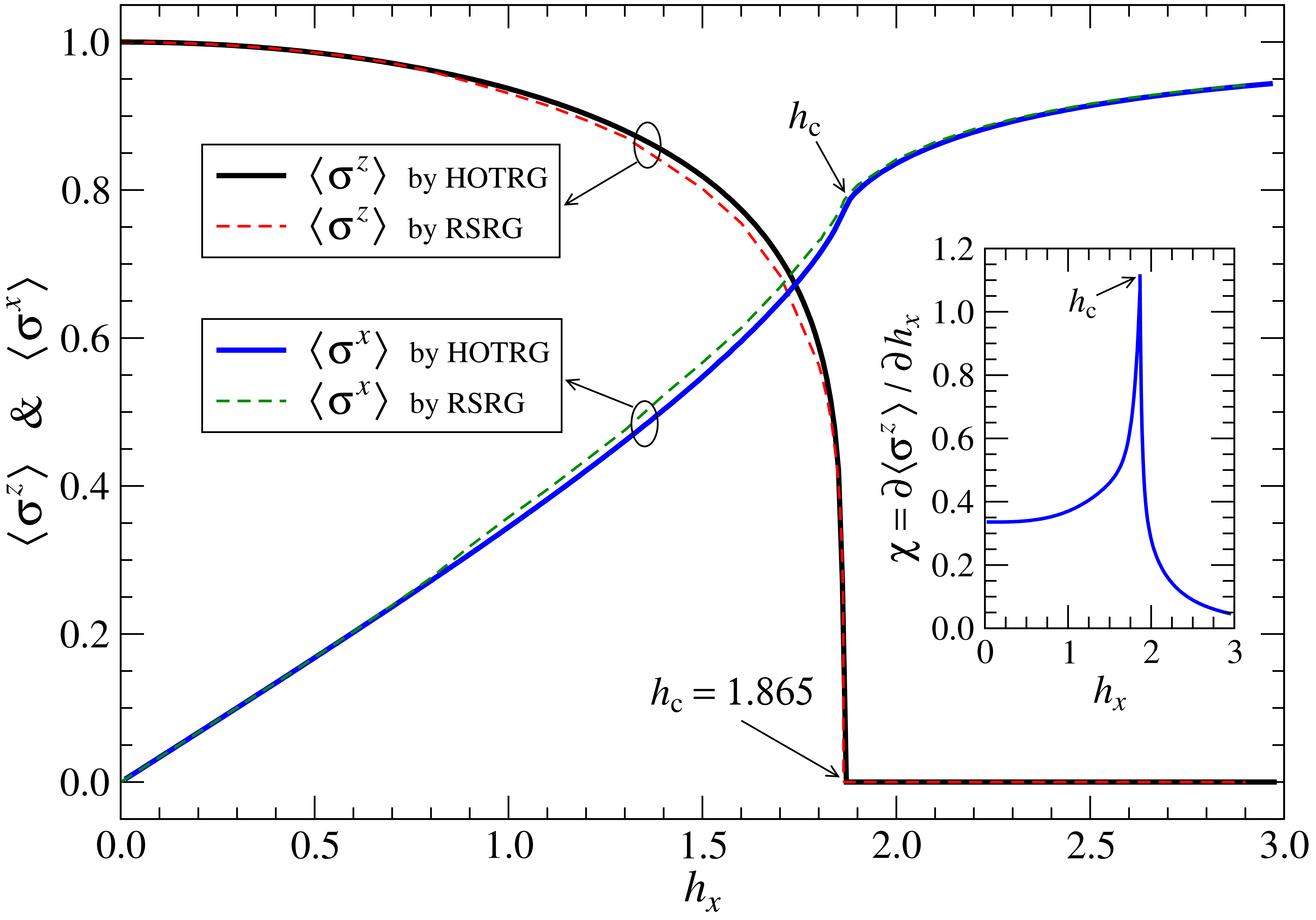
Magnetization $\langle\sigma^z\rangle$ and $\langle\sigma^x\rangle$ on Sierpiński triangle

$$\mathscr{H} = -J \sum_{\{i,j\}} \sigma_i^z \sigma_j^z - h_x \sum_i \sigma_i^x$$

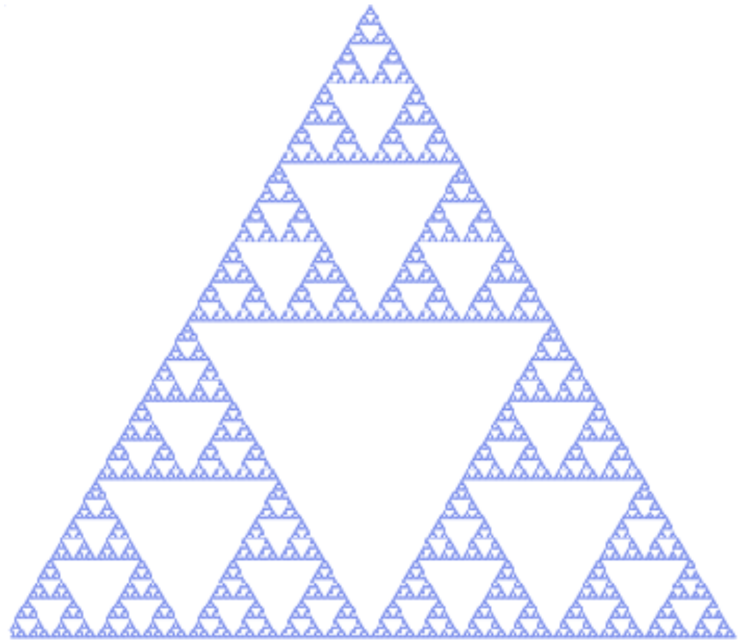


$$\langle\sigma^z\rangle = \langle\psi_0|\sigma_z|\psi_0\rangle$$

$$\langle\sigma^x\rangle = -\frac{\mathrm{d}E_0}{\mathrm{d}h_x}$$



Critical exponent β cand δ on Sierpiński triangle



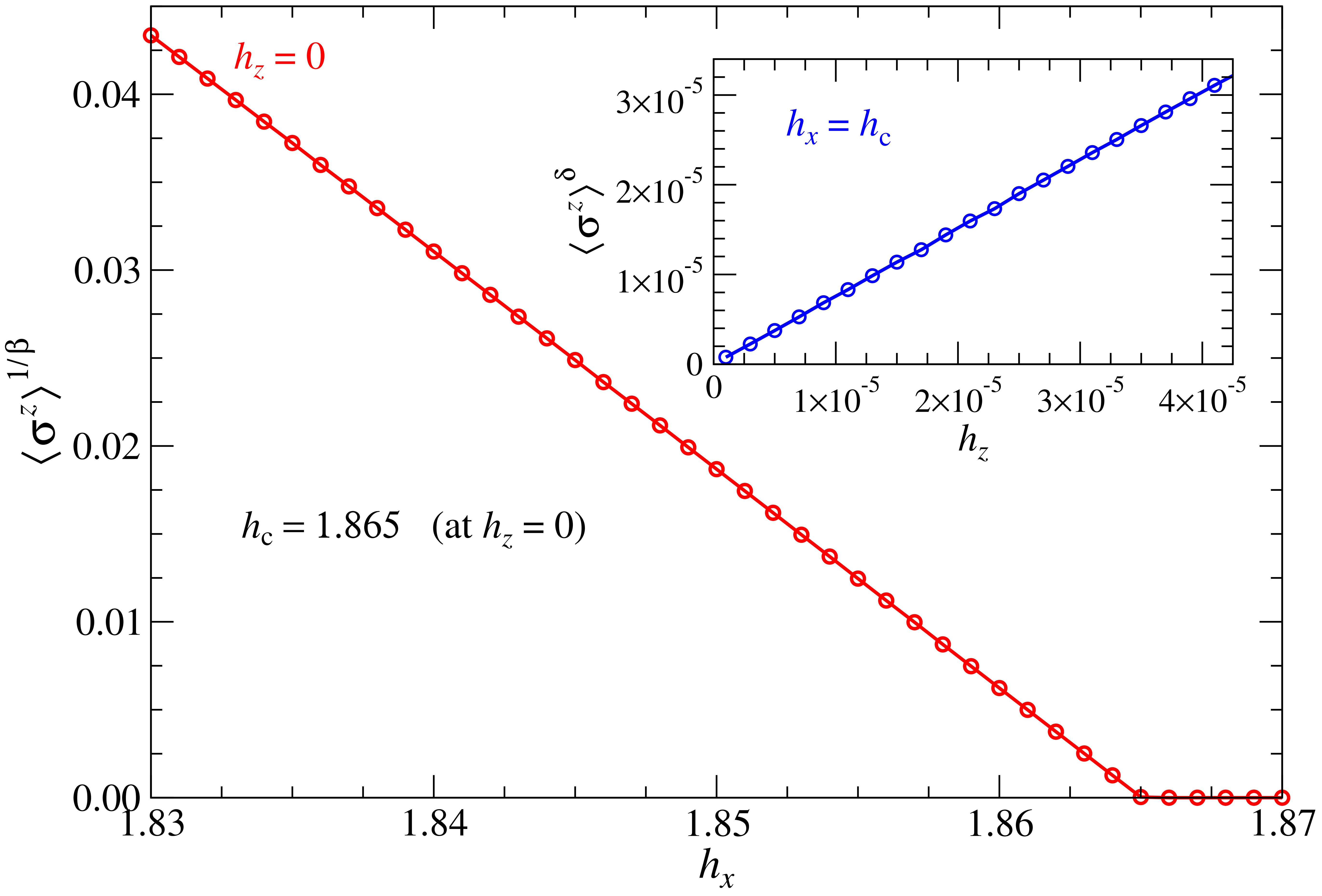
$$\mathscr{H} = -J \sum_{\{i,j\}} \sigma_i^z \sigma_j^z - h_x \sum_i \sigma_i^x - h_z \sum_i \sigma_i^z$$

$$\langle \sigma^z \rangle \propto (h_c - h_x)^{\beta}$$

$$\langle \sigma^x \rangle \propto h_z^{\,1/\delta} \quad \text{at} \quad h_x = h_c$$

$\beta \approx 1/5$

$\delta \approx 8.7$



Summary: Transverse-field Ising model on the Sierpiński triangle

$$\mathcal{H} = -J \sum_{\langle a,b \rangle} S_a^z S_b^z - h \sum_a S_a^x$$

We applied **three** independent methods.

1. The improved Mean-Field Approximations (**MFA$_n$**): h_c = 3.00 (MFA$_1$)
   h_c = 2.46 (MFA$_3$)
   h_c = 2.17 (MFA$_9$)

2. Real-Space Renormalization Group (**RSRG**): h_c = 1.864

3. Higher-Order Tensor Renormalization Group (**HOTRG**): h_c = **1.86497**

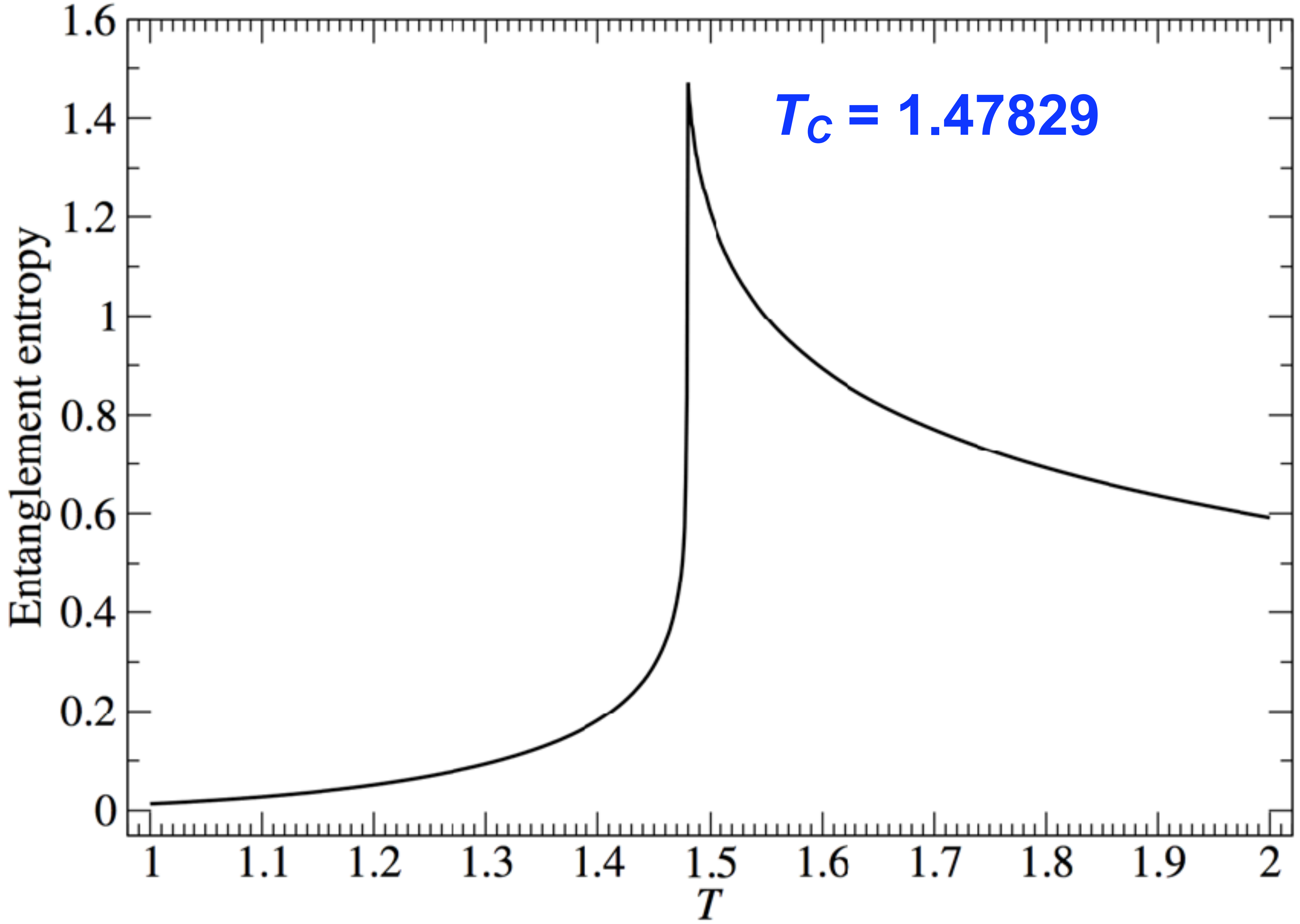
d_H	h_c	β	δ	method used
$\log_2 2 = 1$	1	0.125	15	exact solution
$\log_2 3 \approx 1.585$	1.865	0.20	8.7	HOTRG, MC
$\log_2 4 = 2$	3.0439	0.3295	4.8	HOTRG, CAM

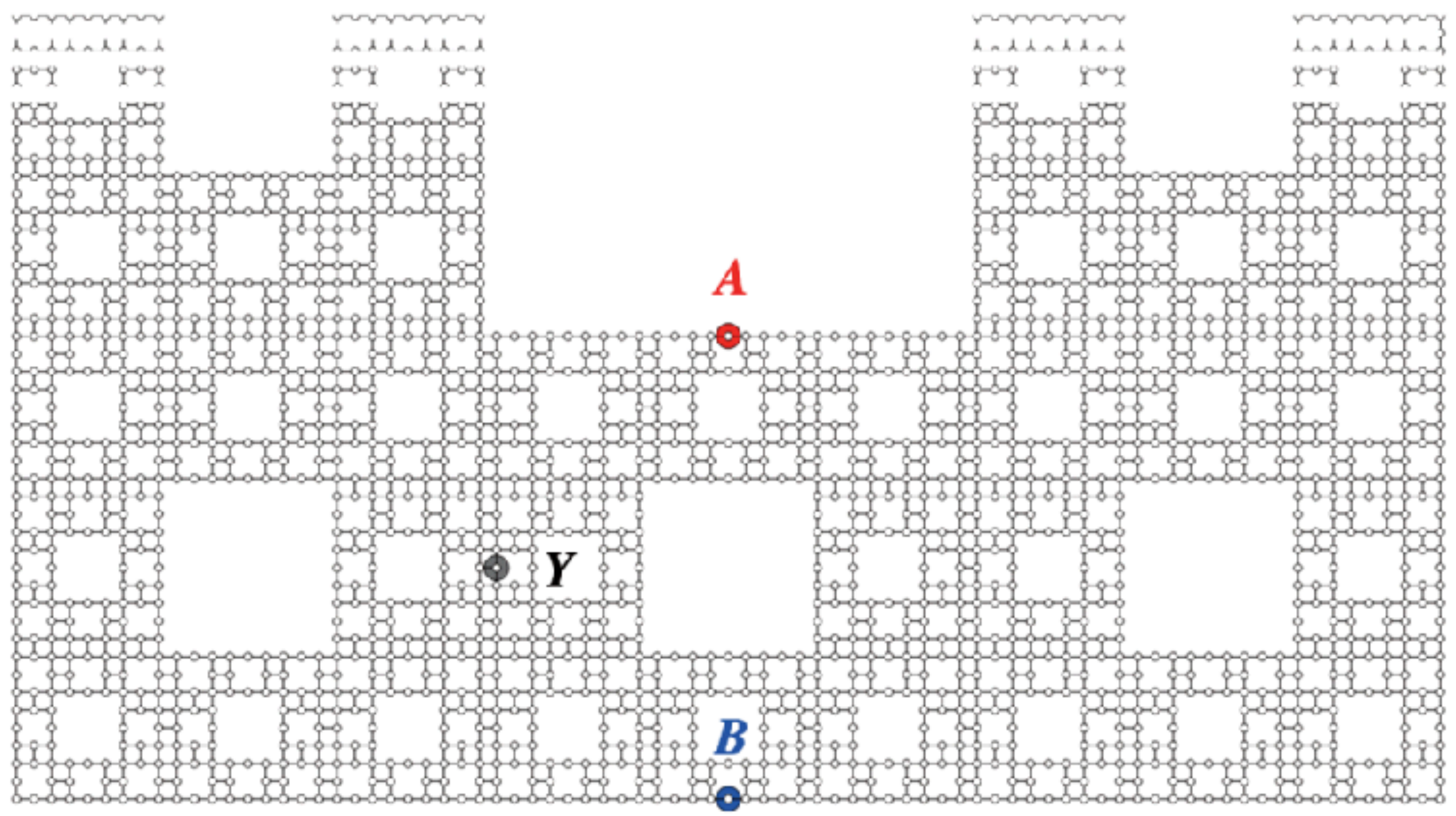
R. Krčmár, J. Genzor, Y. Lee, H. Čenčariková, T. Nishino, A. Gendiar, *Phys. Rev. E* **98**, 062114 (2018)

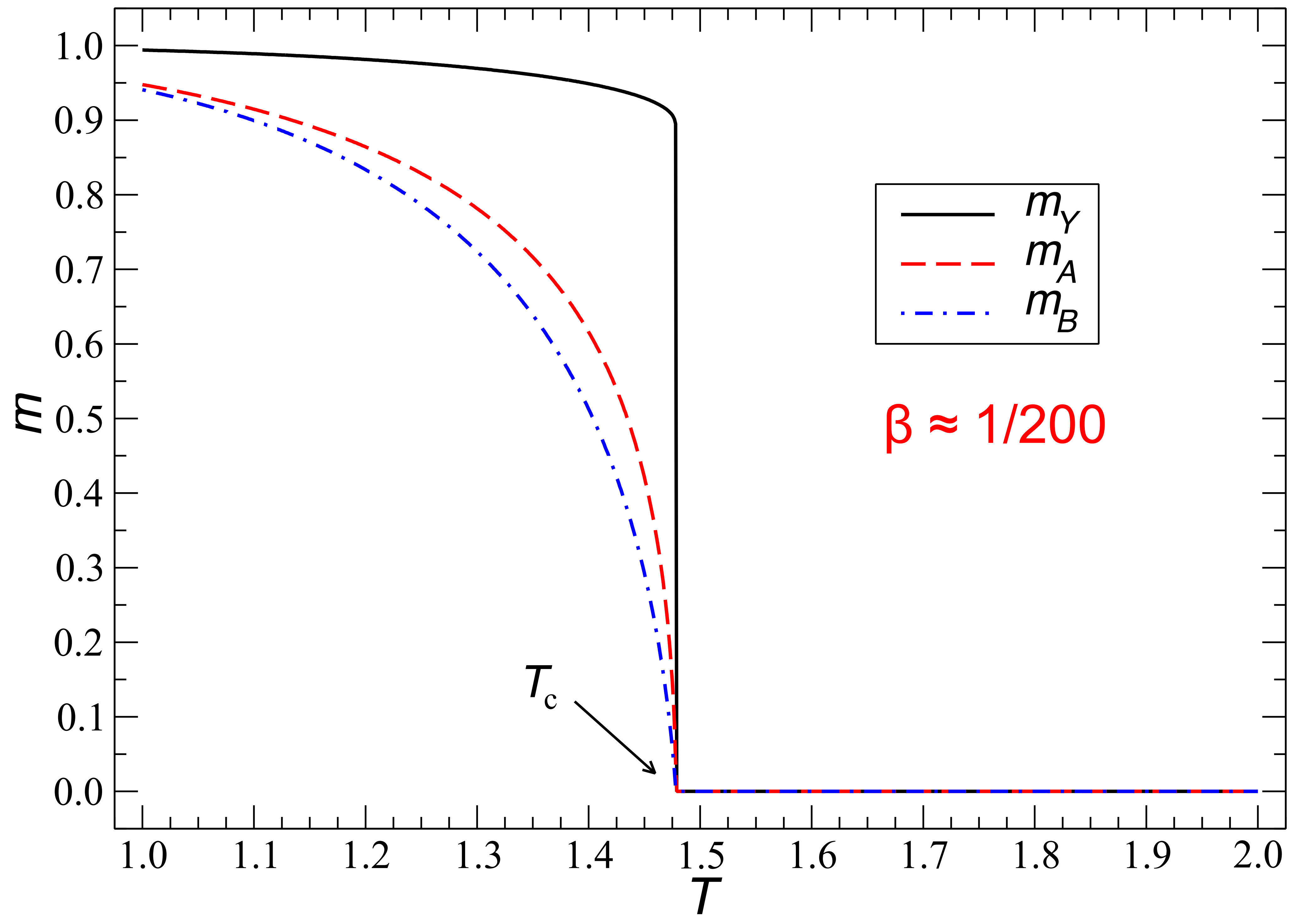
Boundaries in fractals might not be trivial

Classical Ising model on Sierpiński carpet

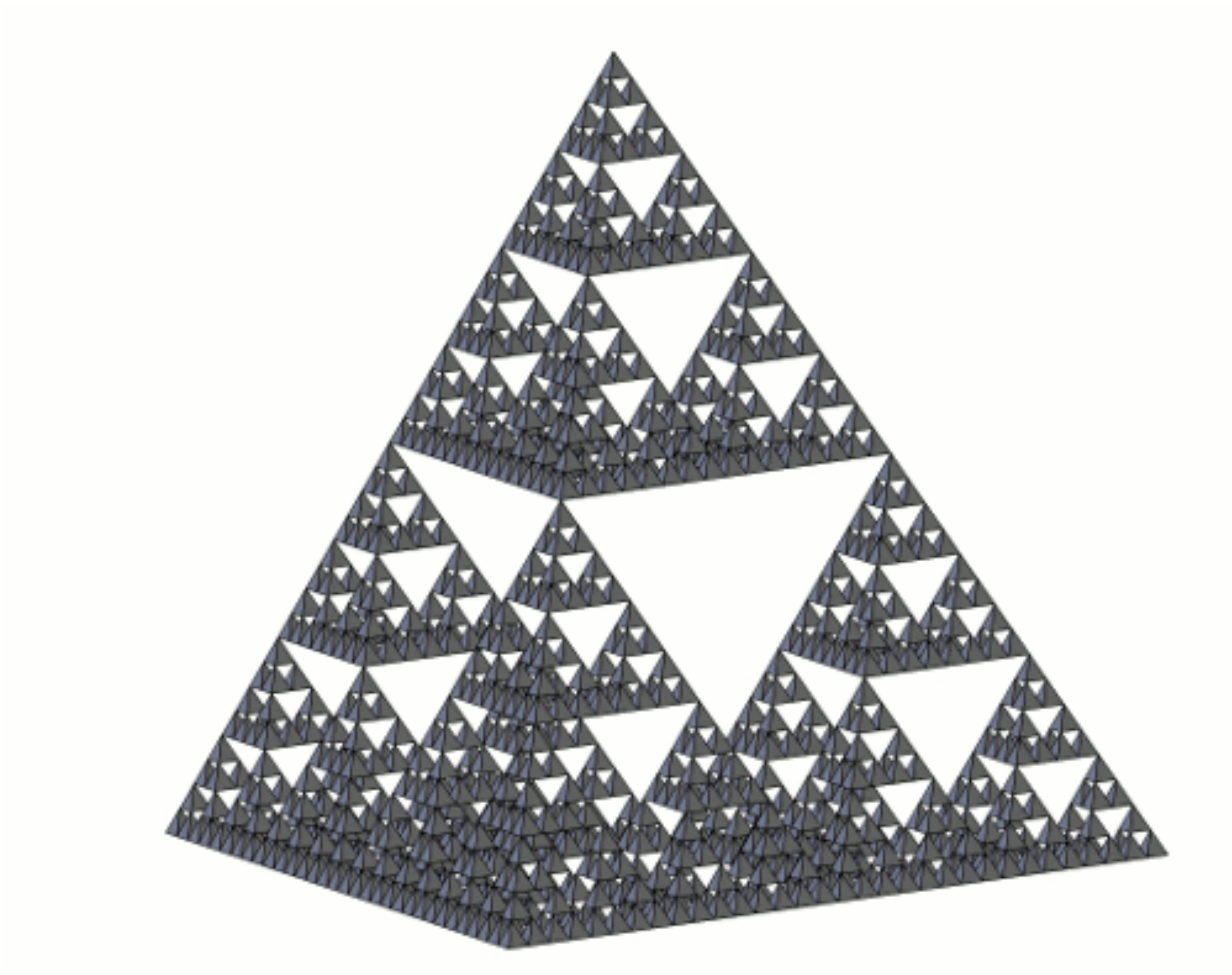
$$d_H = \log_3 8 \approx 1.893$$



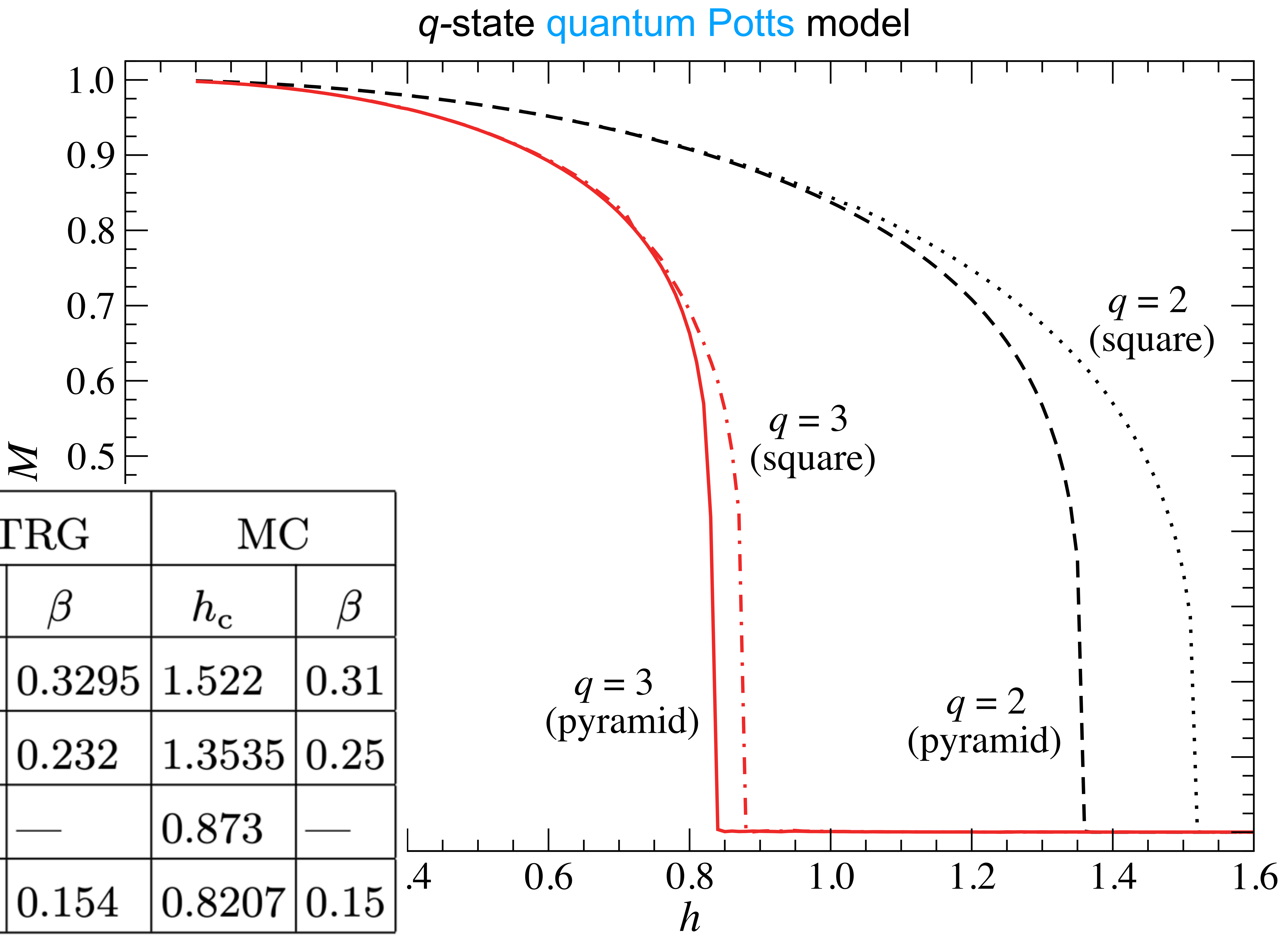




Are scaling relations also valid for non-integer (fractal) dimension?

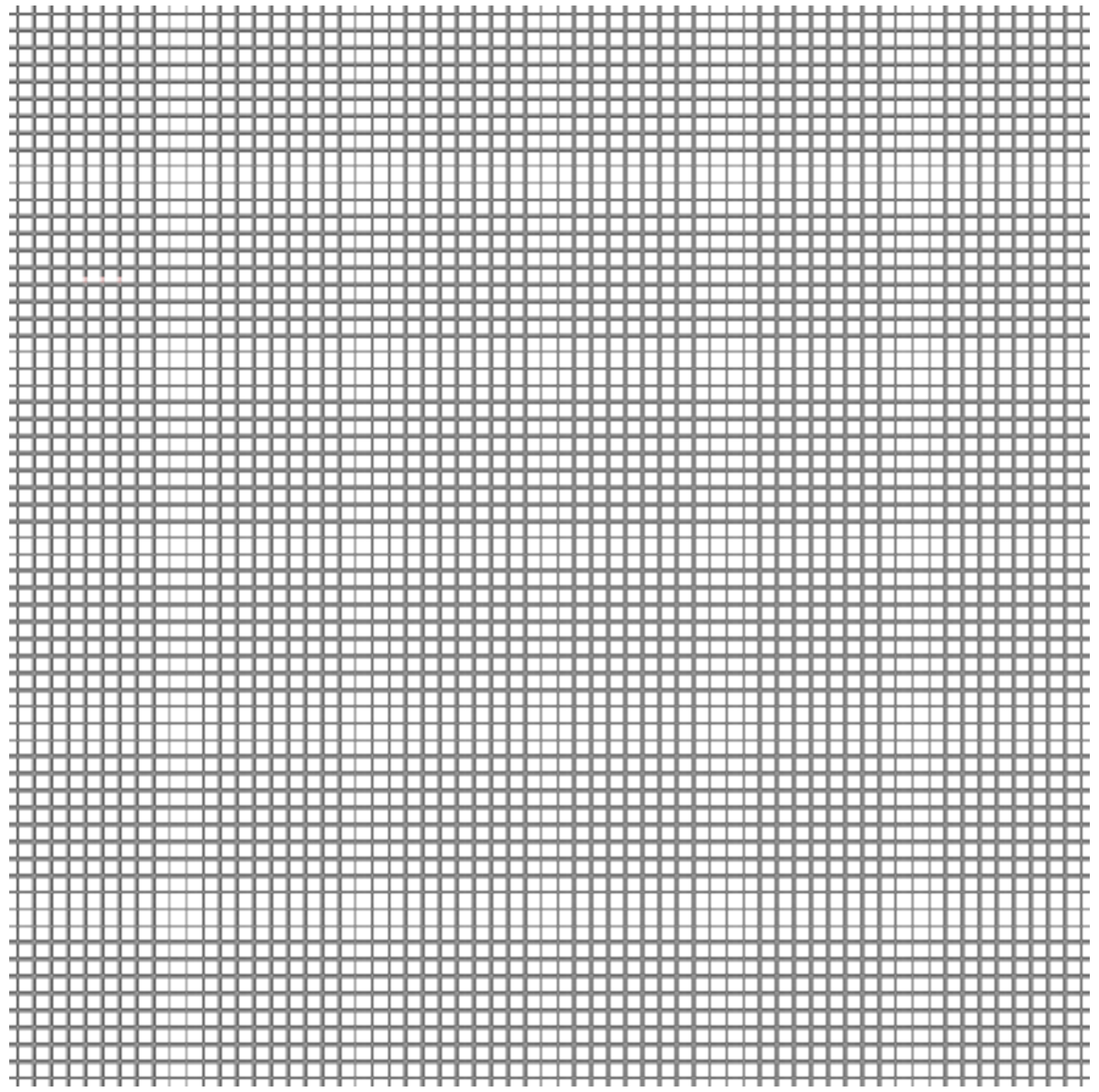


$d_H = \log_2 4 = 2$



Model (lattice)	HOTRG		MC	
	h_c	β	h_c	β
$q = 2$ (square)	1.5219	0.3295	1.522	0.31
$q = 2$ (pyramid)	1.358	0.232	1.3535	0.25
$q = 3$ (square)	0.876	—	0.873	—
$q = 3$ (pyramid)	0.832	0.154	0.8207	0.15

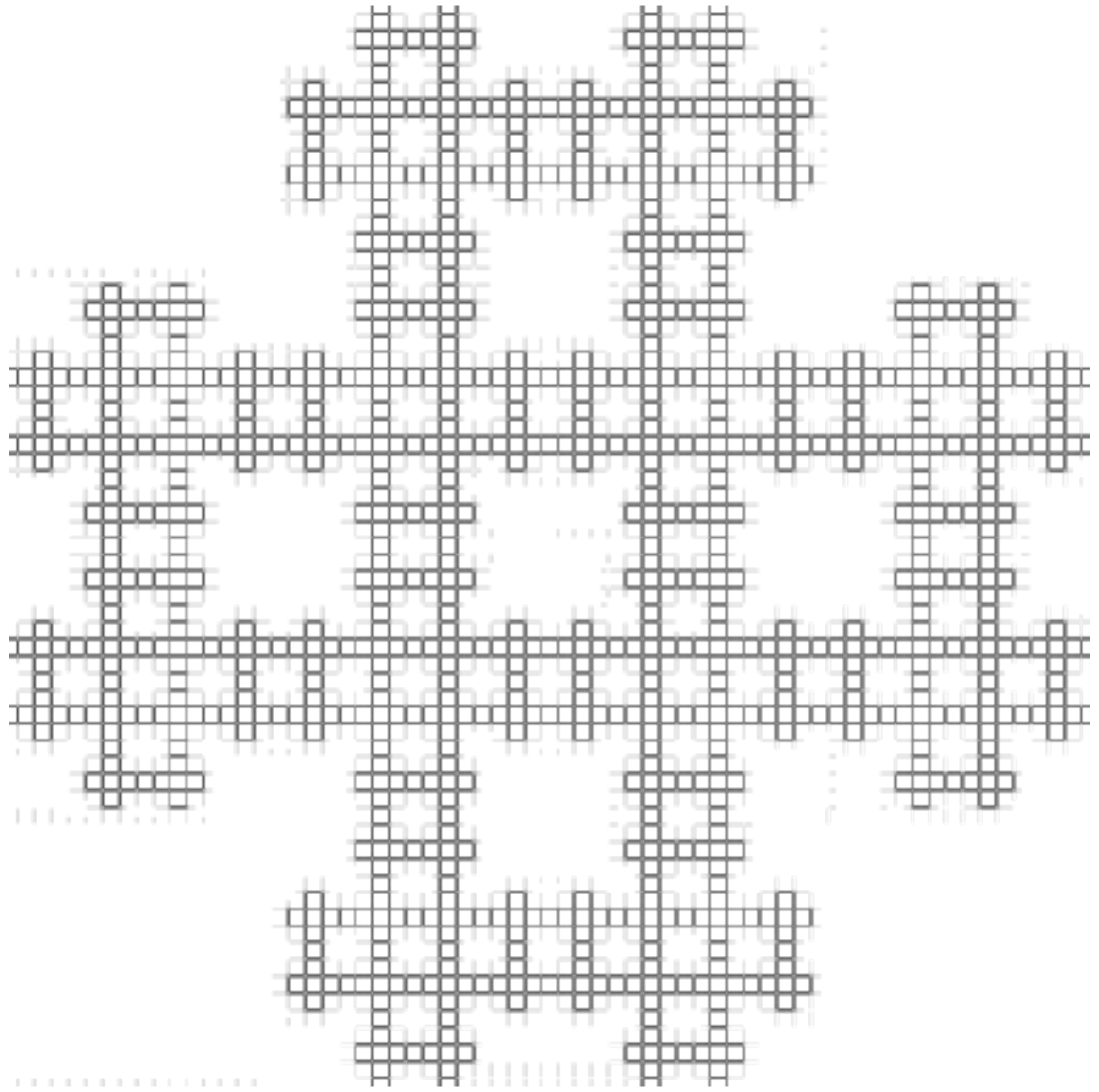
Square lattice



$d_{\mathrm{H}} = 2$

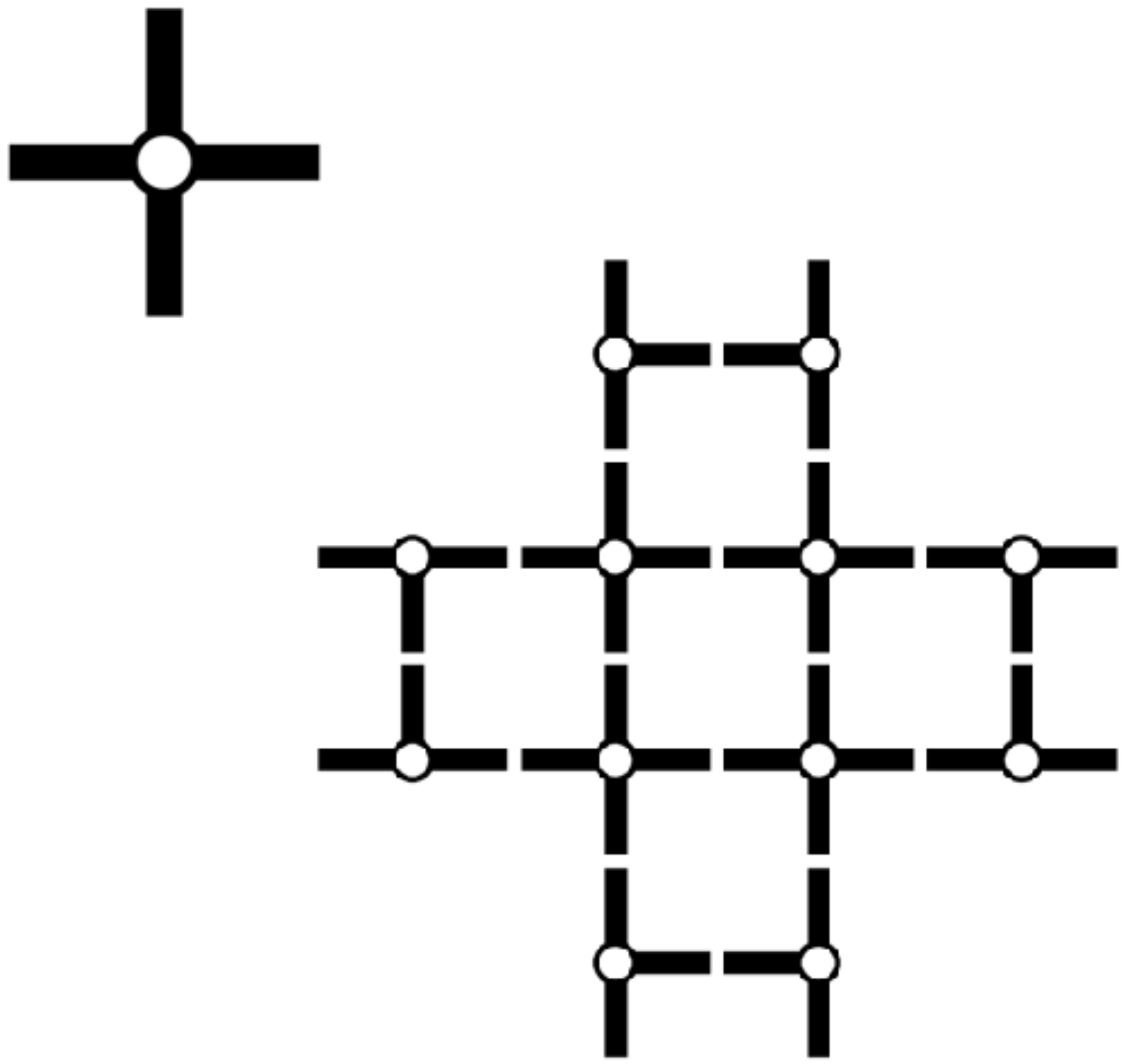
Continuous transformation?

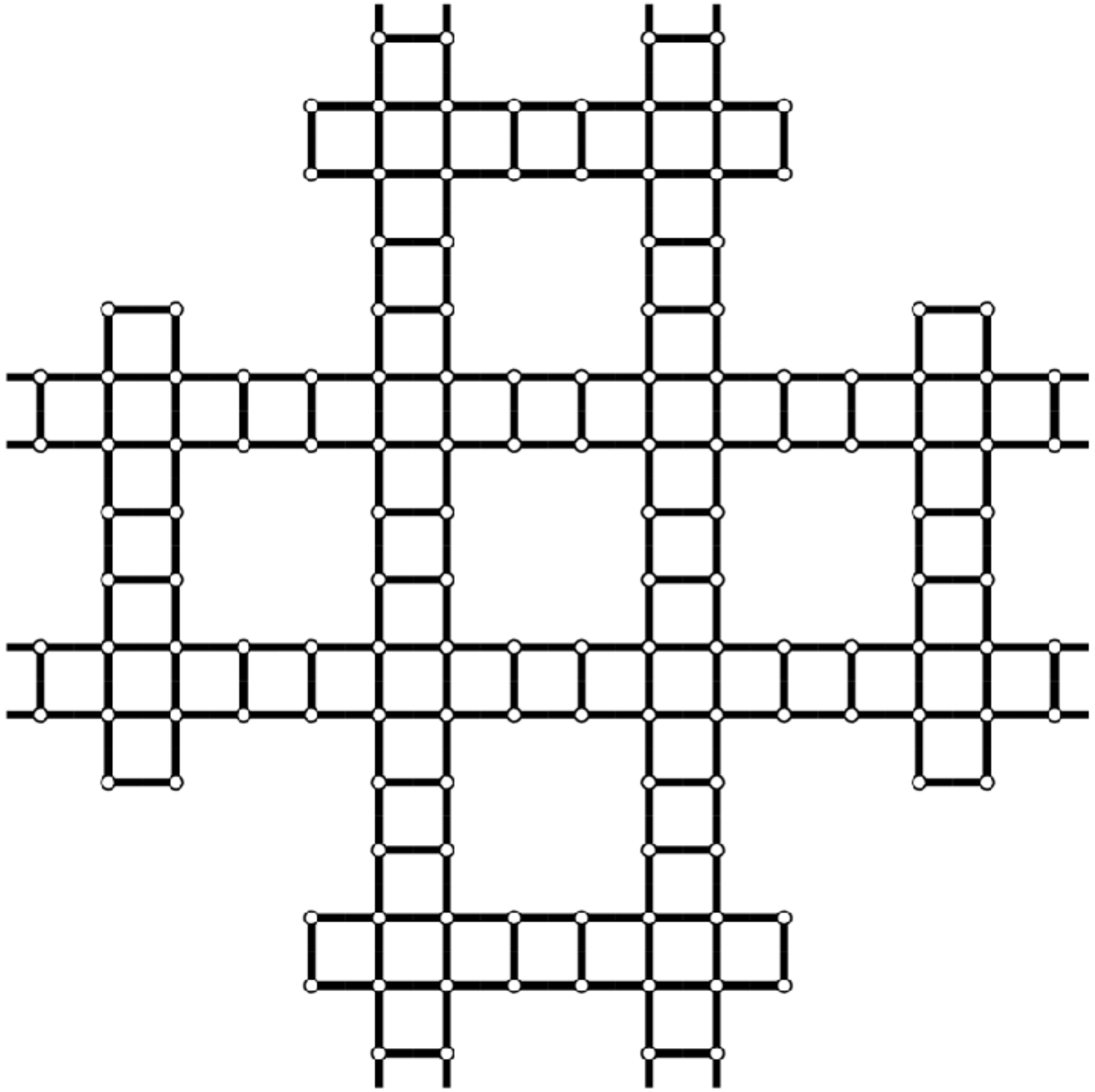
Fractal lattice



$d_{\mathrm{H}} \approx 1.792$

Fractal$_1$

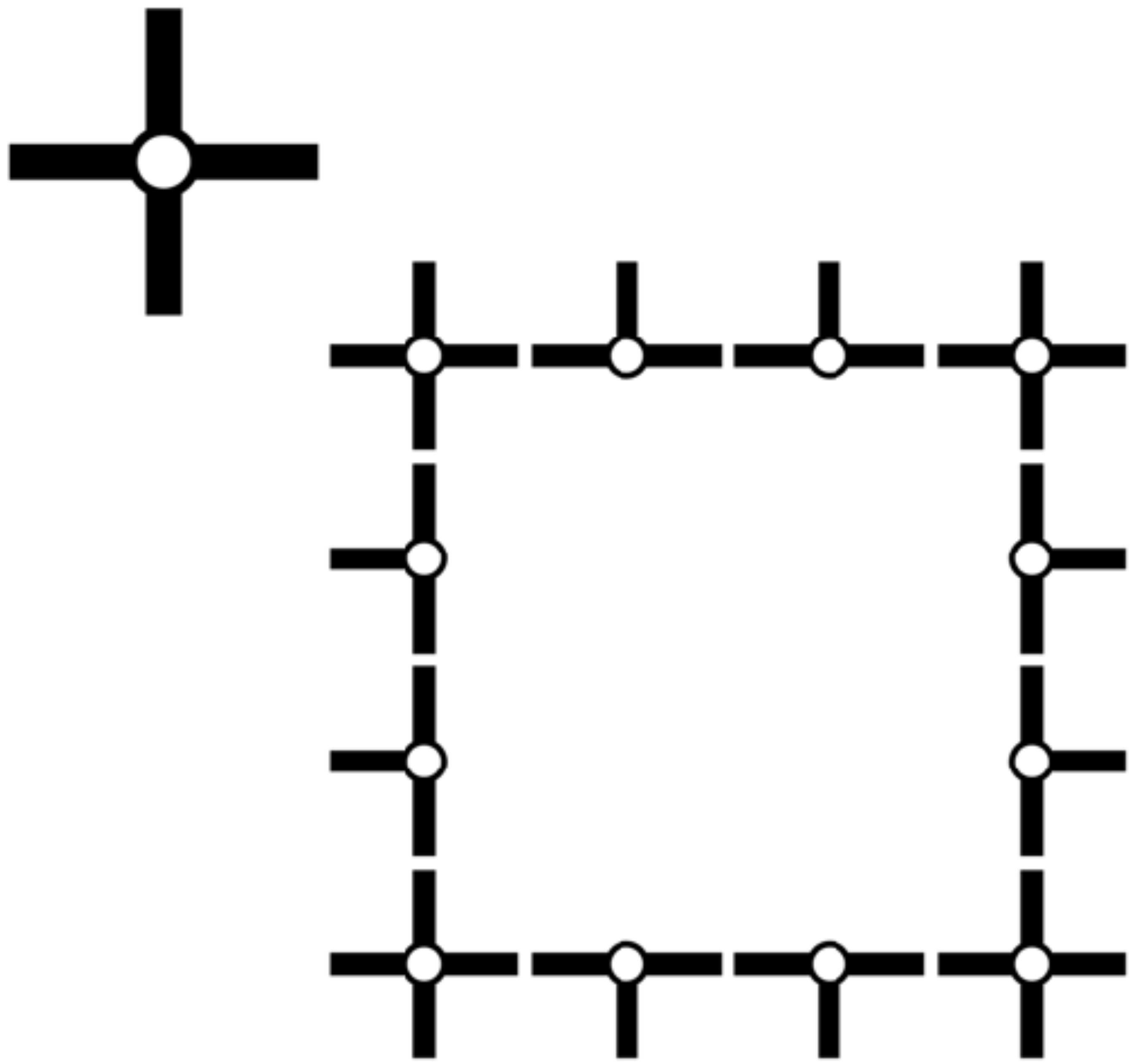




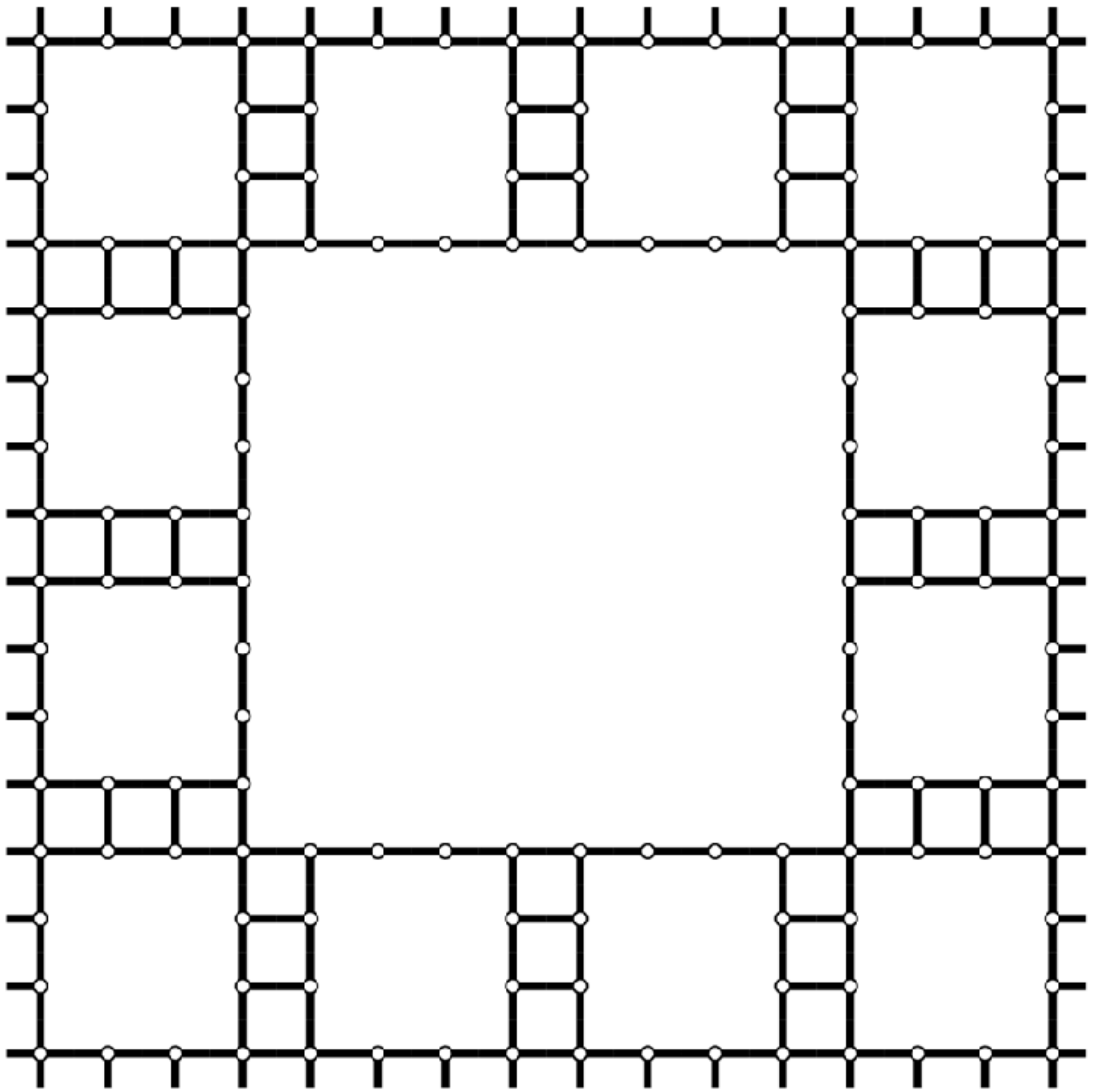
$$d_{\mathrm{H}} = \frac{\ln(12)}{\ln(4)} \approx 1.792$$

J. Genzor, A. Gendiar, T. Nishino, Phys. Rev. E **93** (2016) 012141

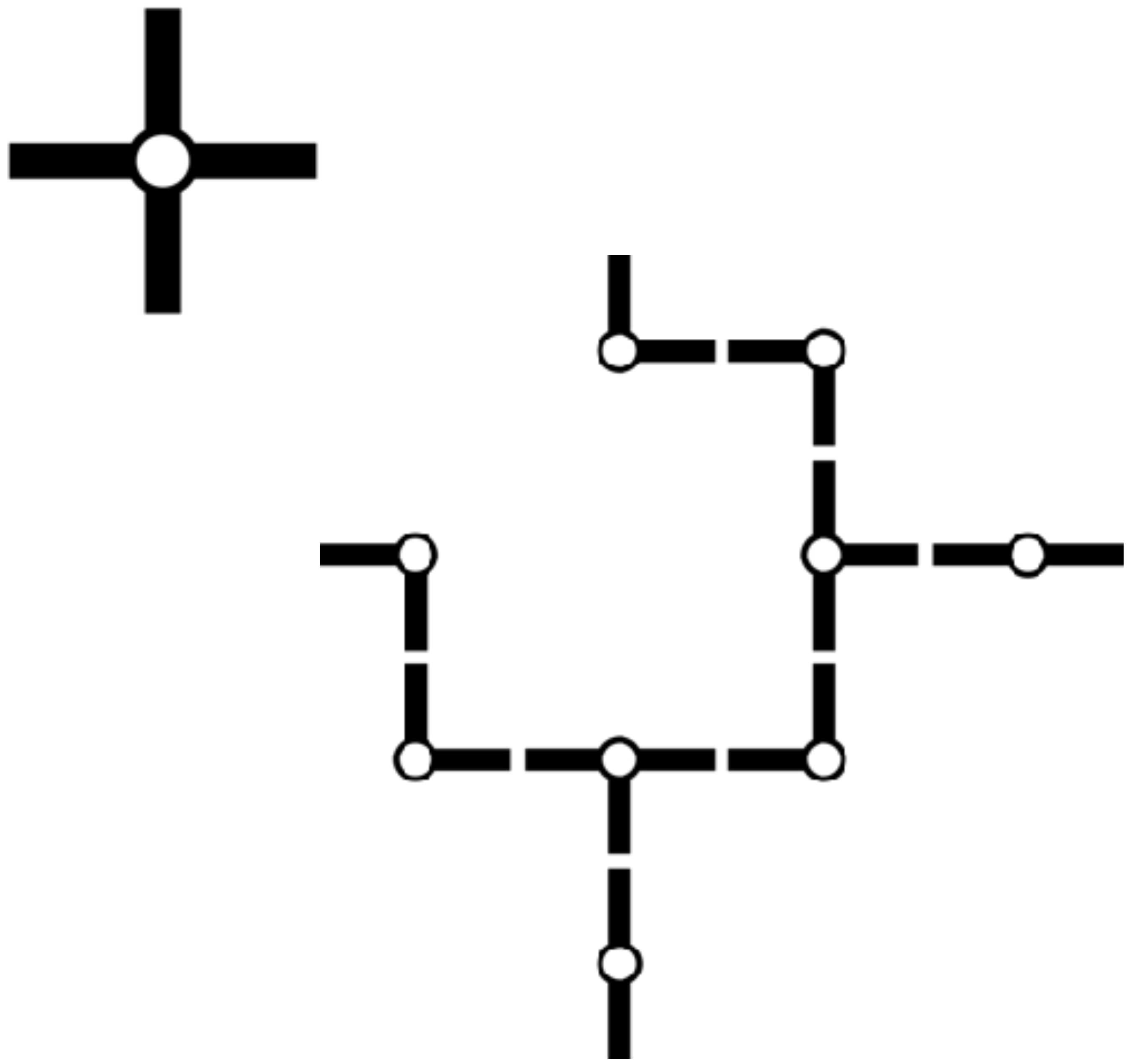
Fractal$_2$ Sierpiński Carpet

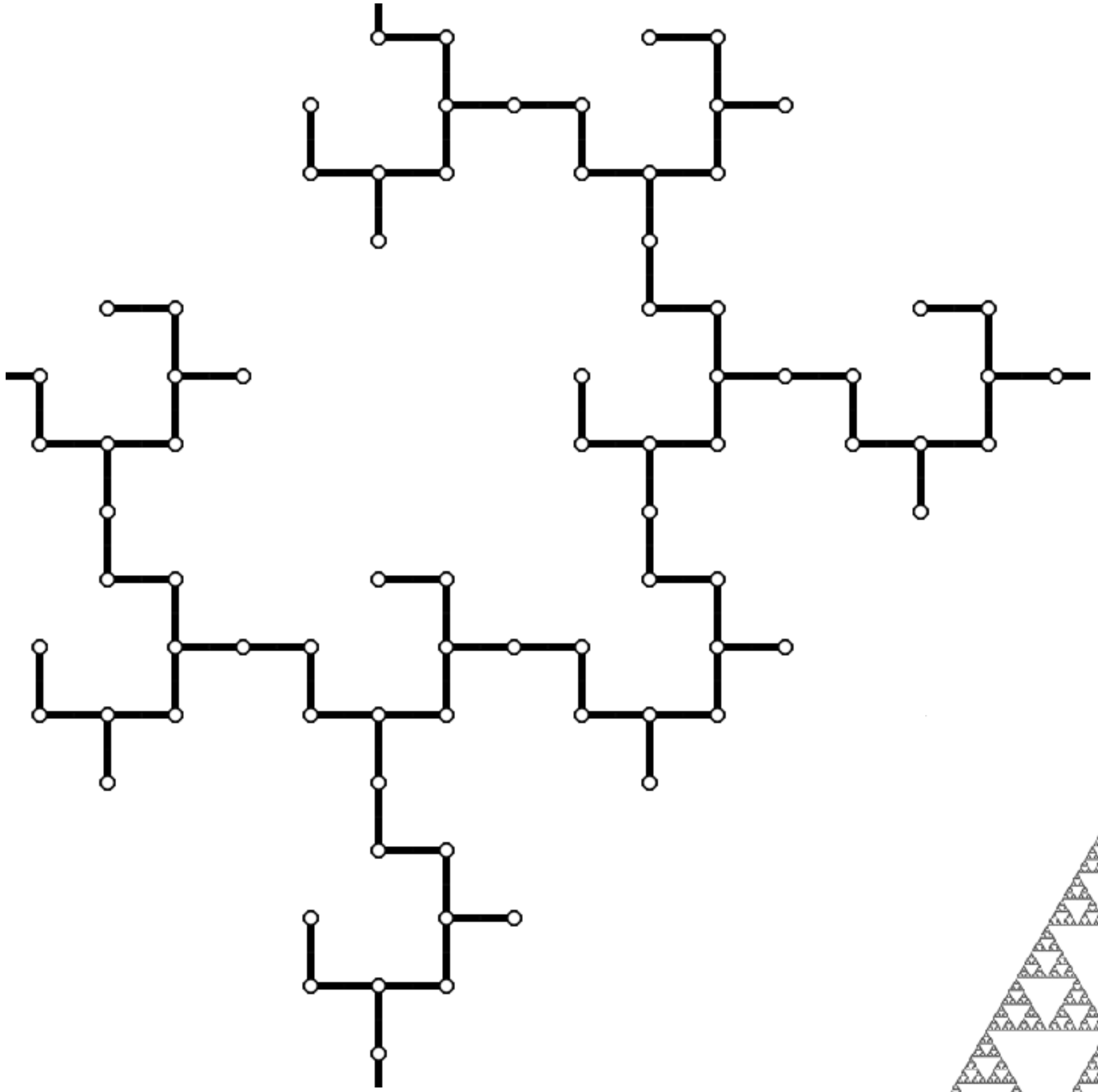


$$d_H = \frac{\ln(12)}{\ln(4)} \approx 1.792$$

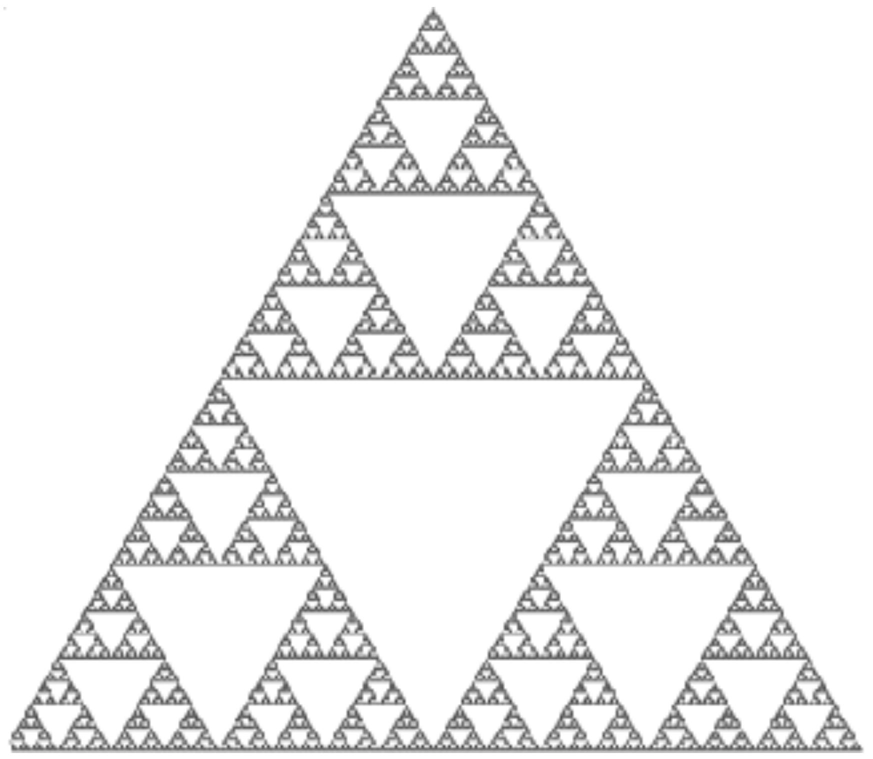


Fractal$_3$





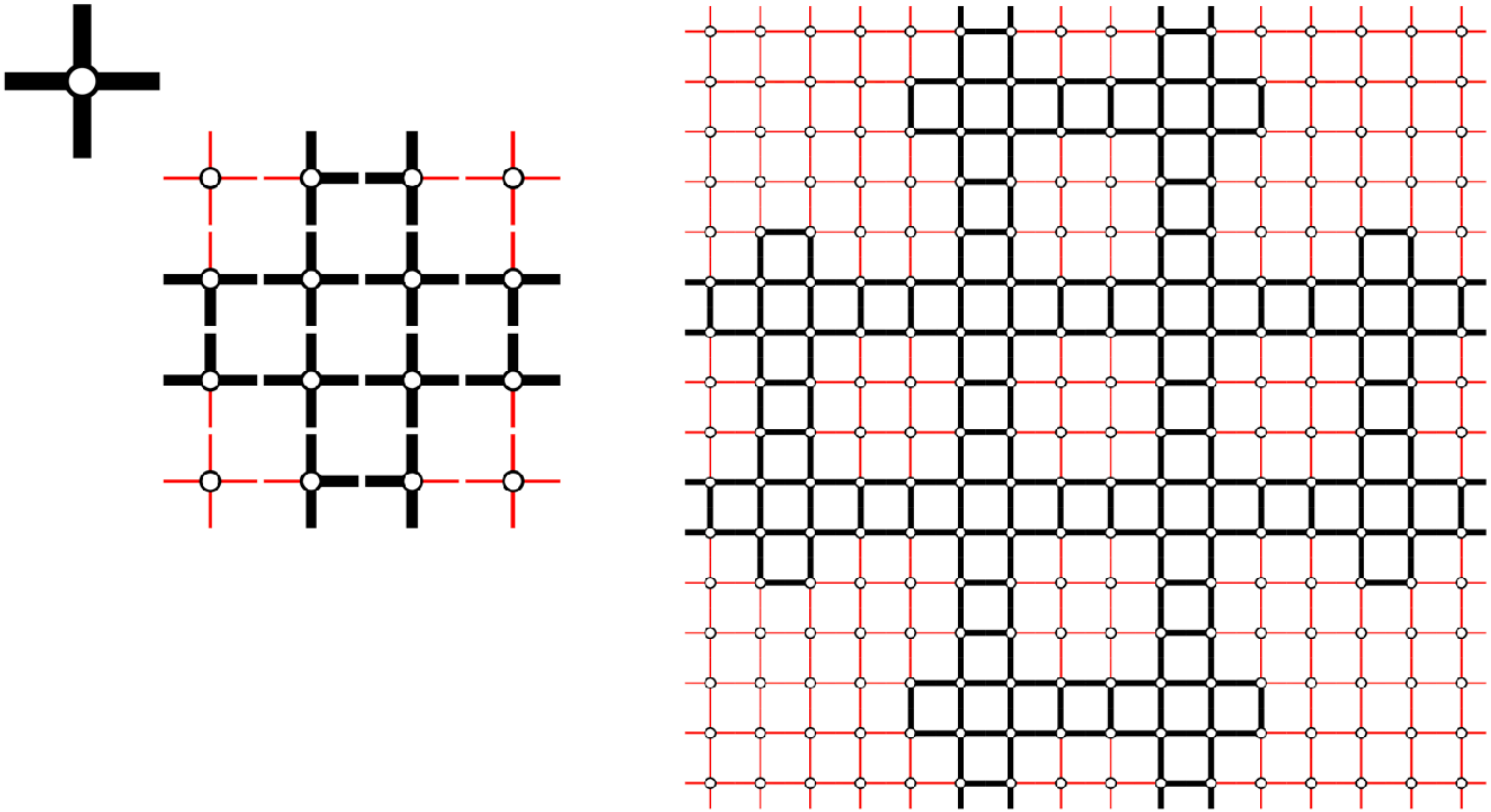
$$d_H = \frac{\ln(9)}{\ln(4)} \approx 1.585$$



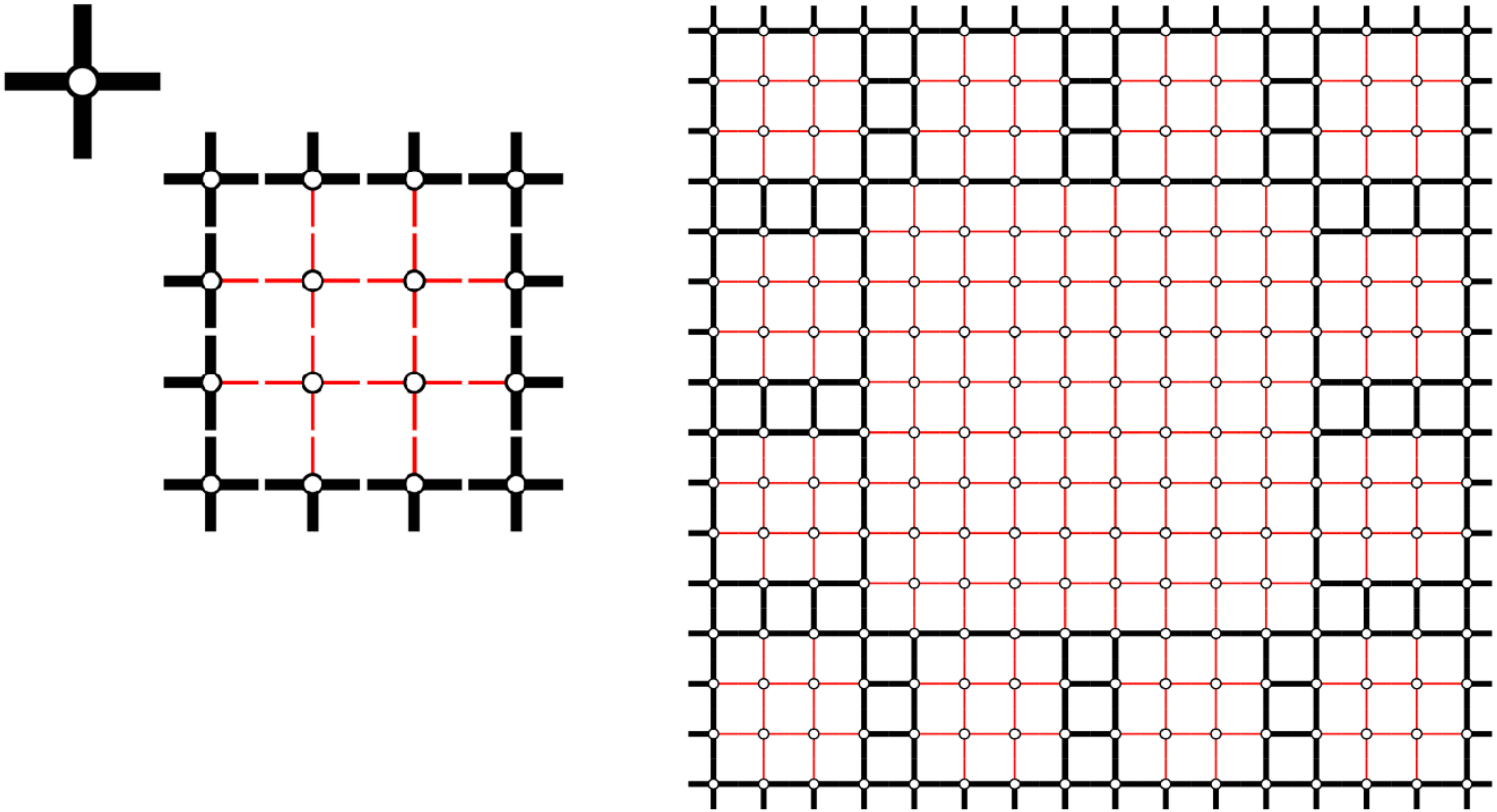
$$d_H = \frac{\ln 3}{\ln 2} \approx 1.585$$

$$\mathcal{H} = -J_1 \sum_{\langle ij \rangle_1} \sigma_i \sigma_j - J_2 \sum_{\langle ij \rangle_2} \sigma_i \sigma_j$$

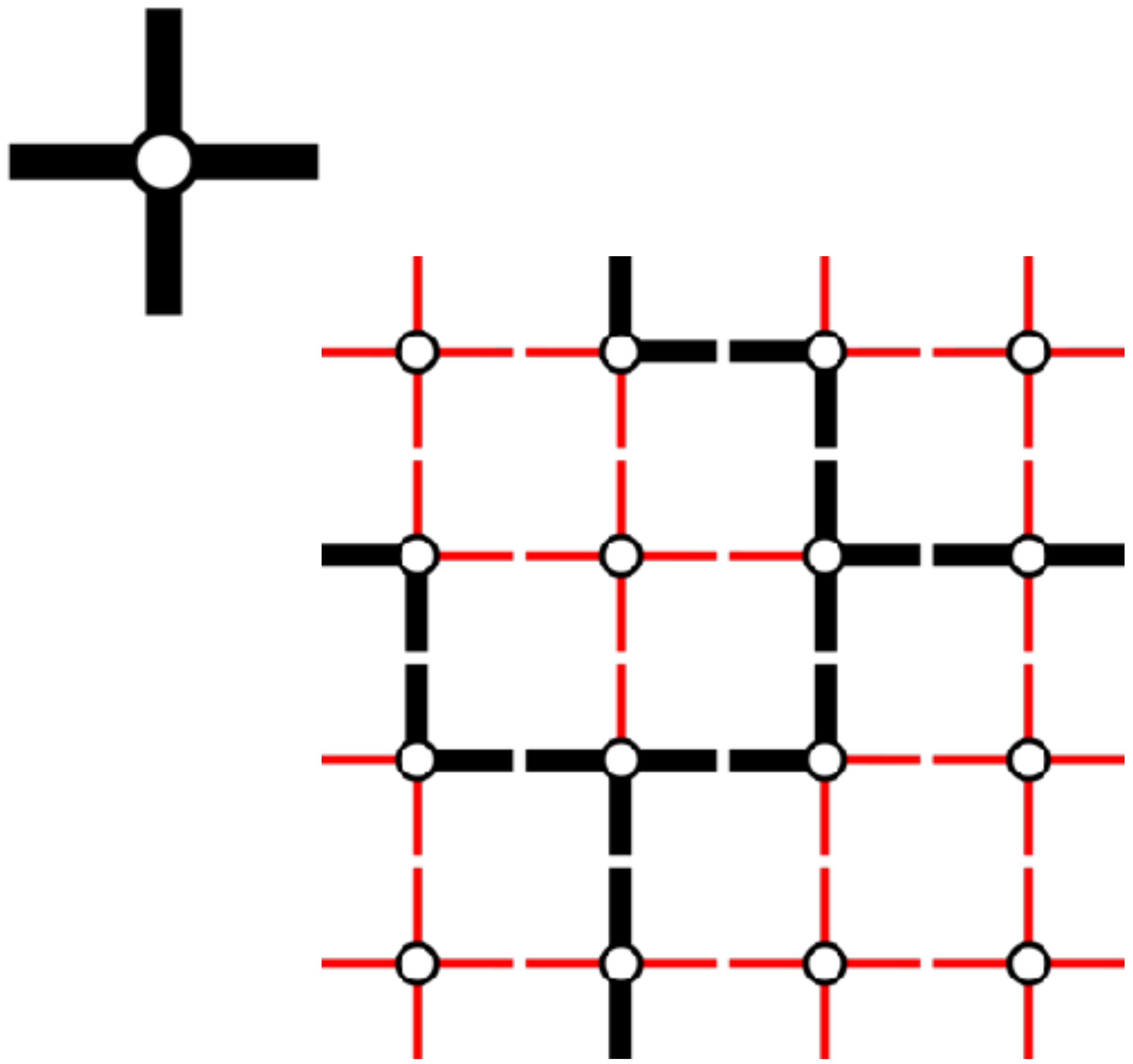
Fractal$_1$ J_1-J_2

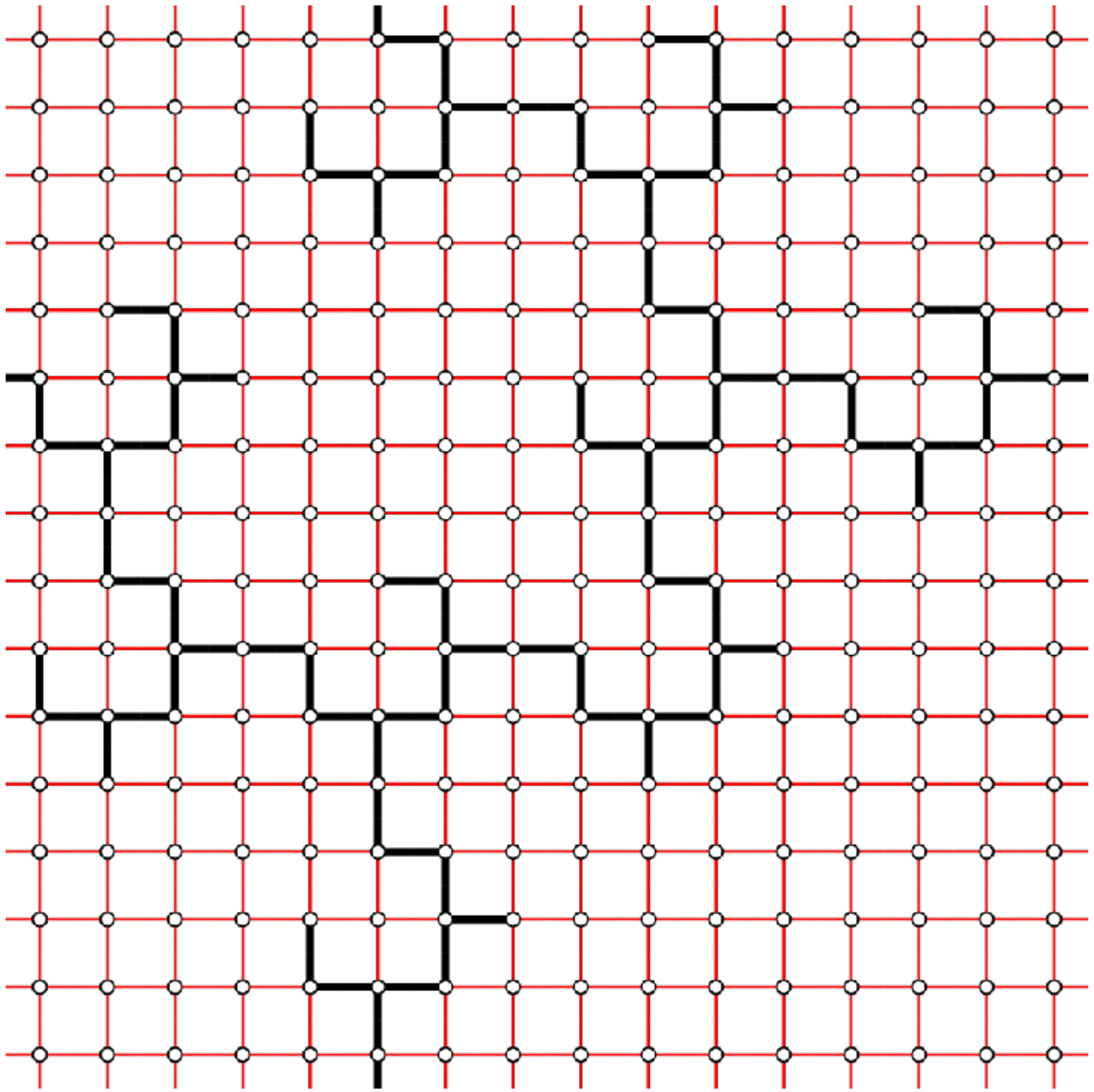


Fractal$_2$ J_1-J_2



Fractal$_3$ J_1-J_2





Numerical Results

Critical temperature T_C (for $J_1 = 1$)

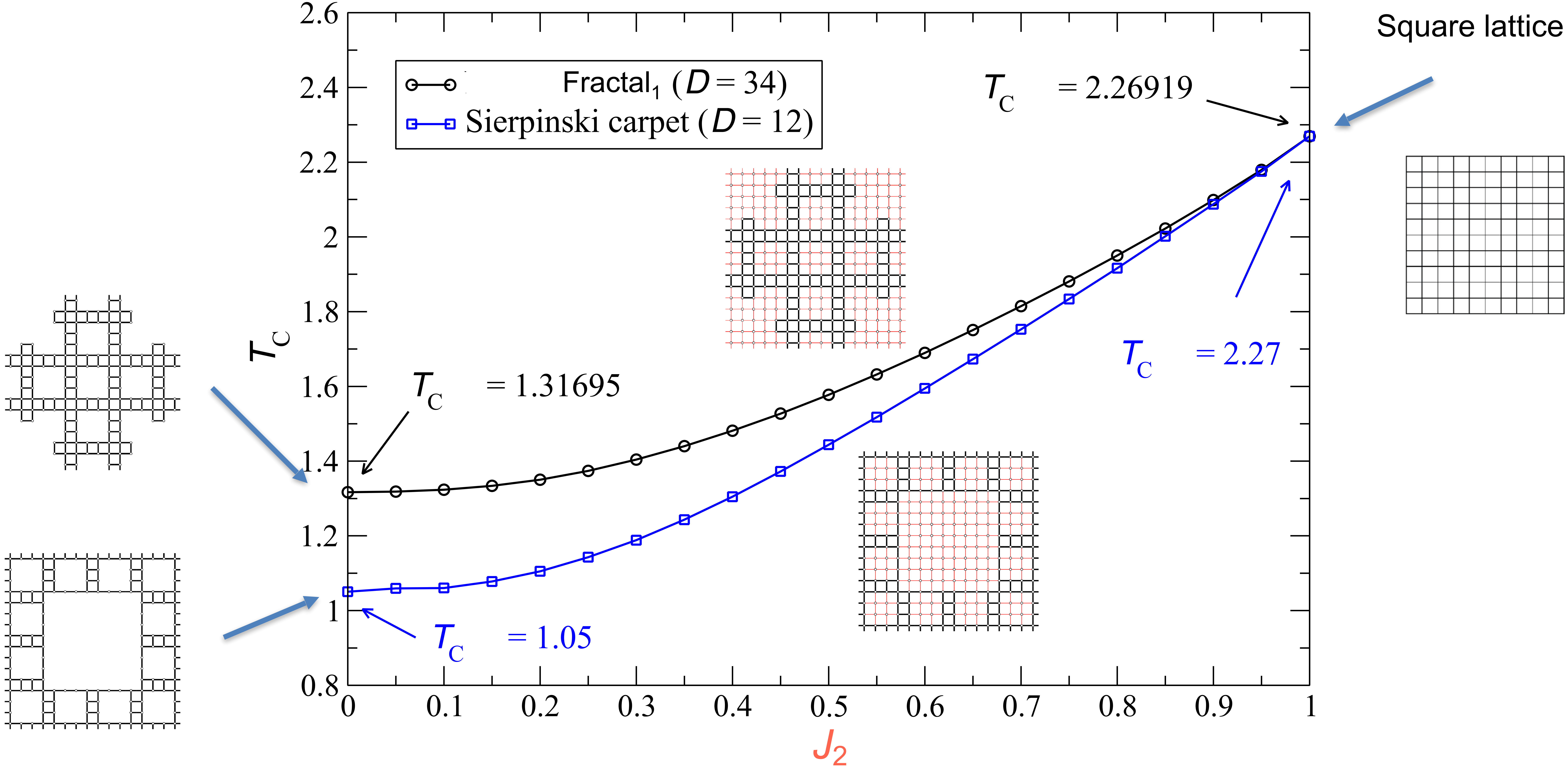
	J_2=0 (Fractal)	J_2=1 (Square lattice)
Comparison	1.31717*	
J_1-J_2	1.31695 (*D*=16-34)	2.26919 (*D*=34)
Relative error/difference	~0.02%	~0.0002%

Critical exponent β

	J_2=0 (Fractal)	J_2=1 (Square lattice)
Comparison	0.01388*	0.125
J_1-J_2	0.0153970 (*D*=34)	0.128 (*D*=34)
Relative error/difference	~10%	~2.4%

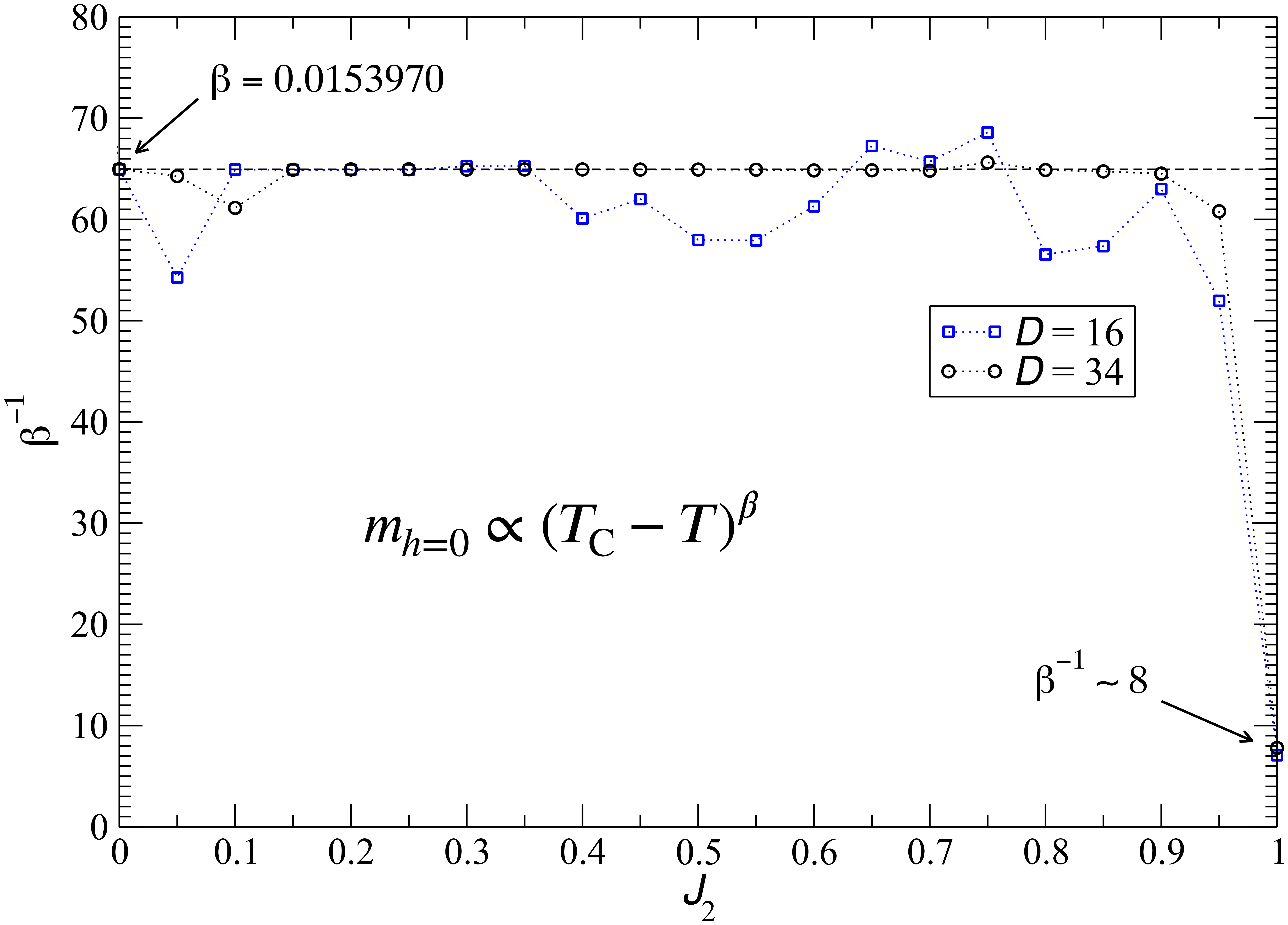
(*D*=32) J. Genzor, T. Nishino, A. Gendiar, Acta Phys. Slov. **67** (2017) 85

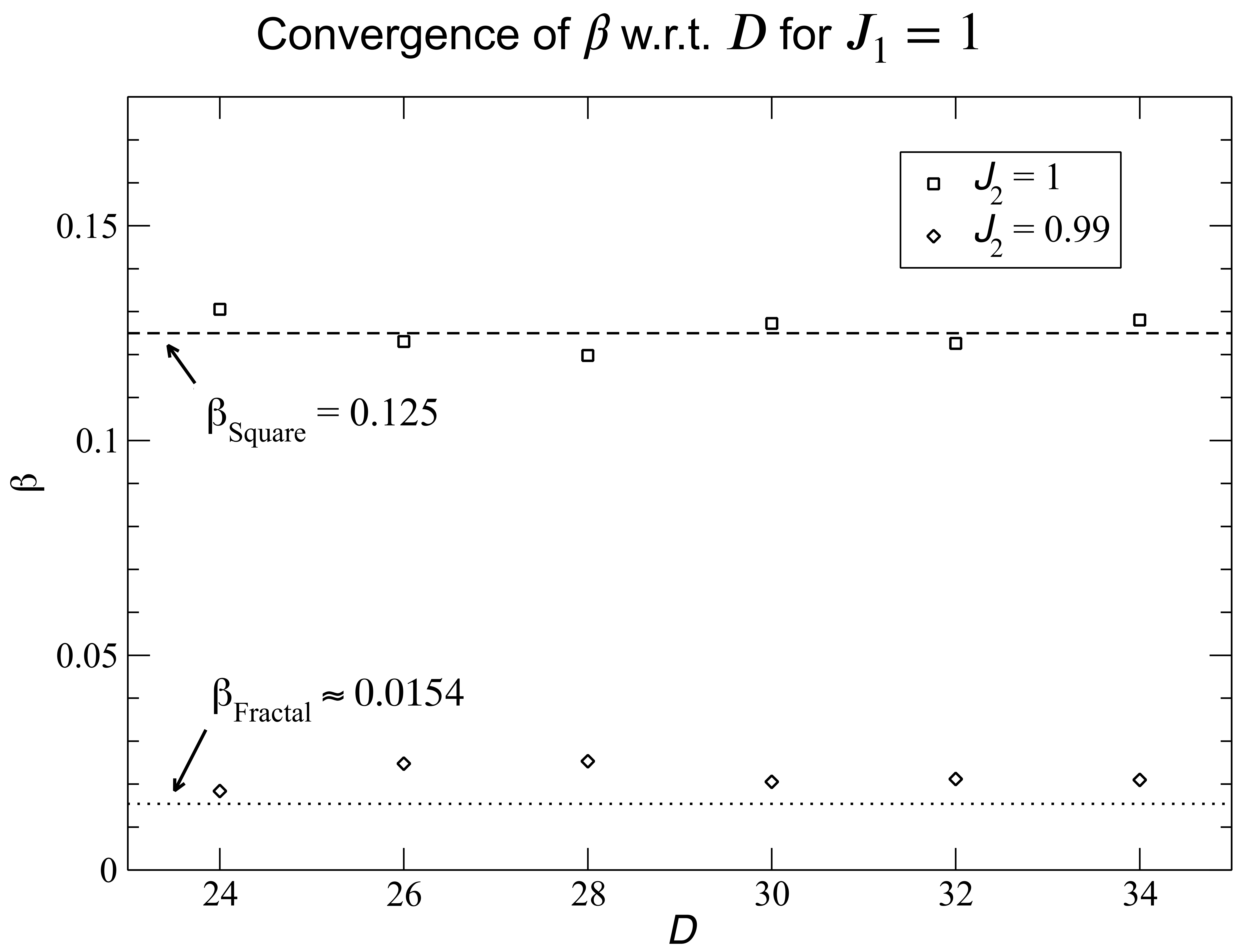
Critical temperature T_C as a function of $0 \leq J_2 \leq 1$ (if $J_1=1$)



J. Genzor, A. Gendiar, T. Nishino, Phys. Rev. E **93** (2016) 012141

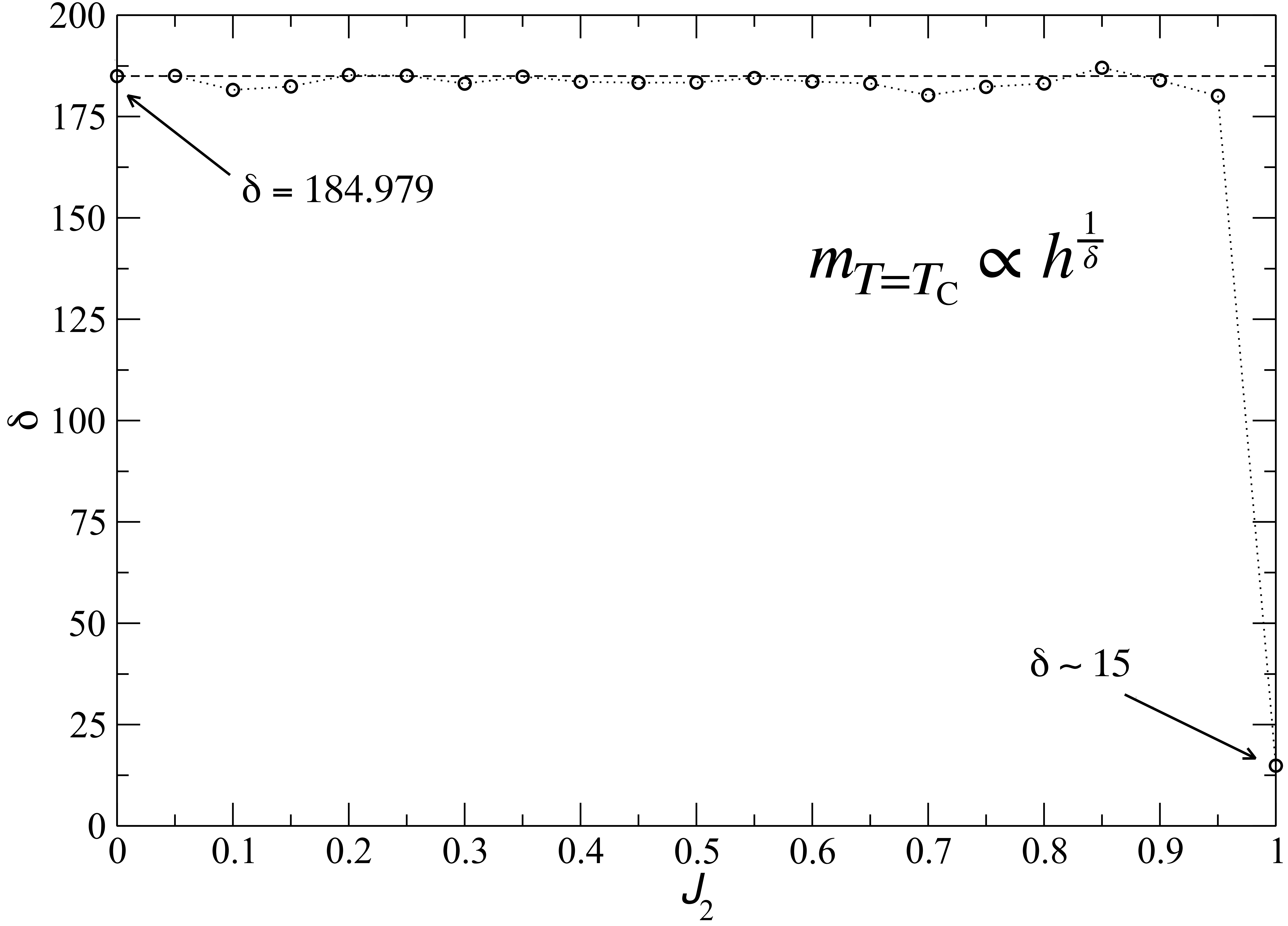
Critical exponent β for $J_1 = 1$



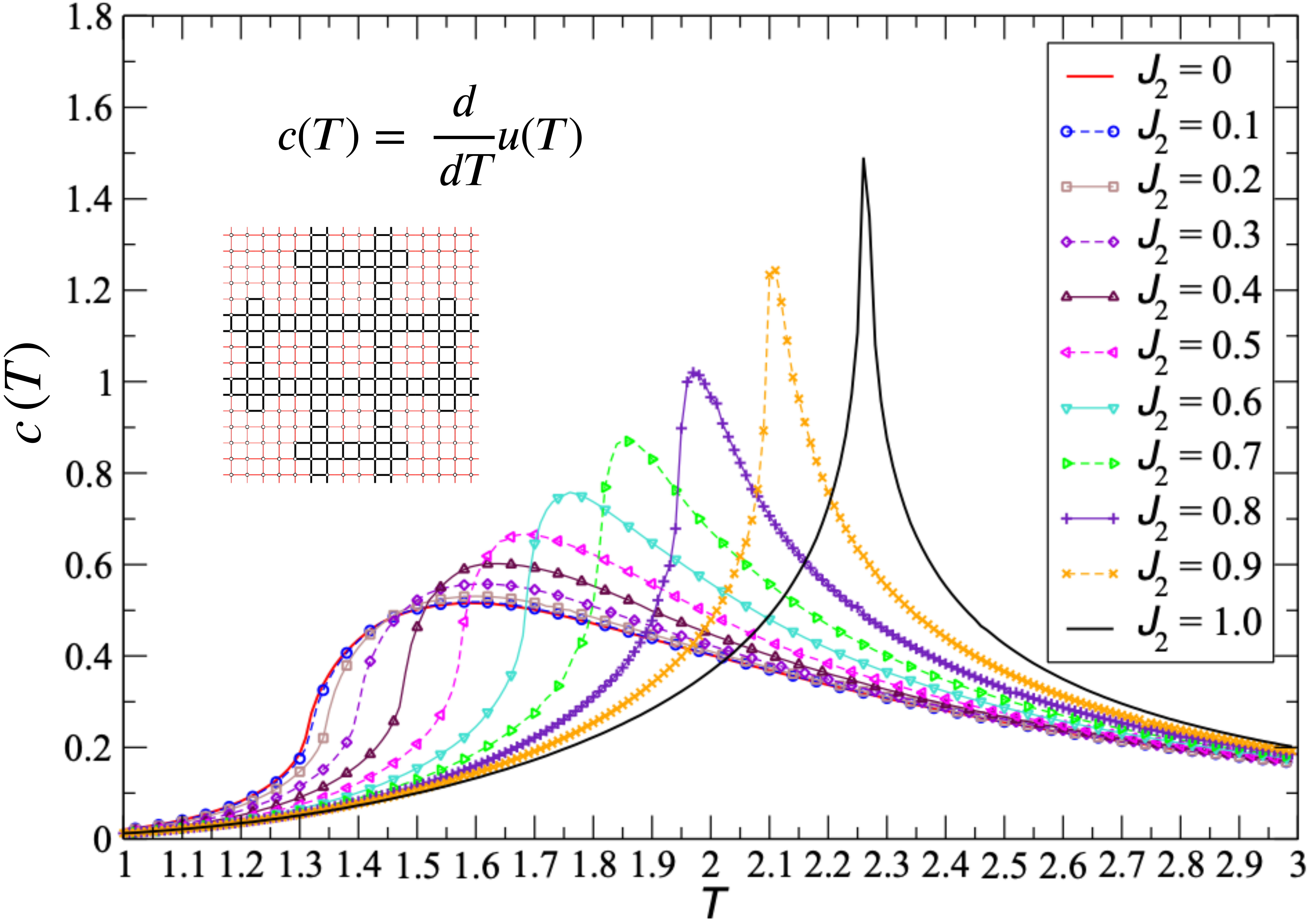


Convergence of β w.r.t. D for J1 = 1
J2 = 1
J2 = 0.99
βSquare = 0.125
βFractal ≈ 0.0154
β
0.15
0.1
0.05
0
24
26
28
30
32
34
D

Critical exponent δ for $J_1 = 1$ ($D = 34$)

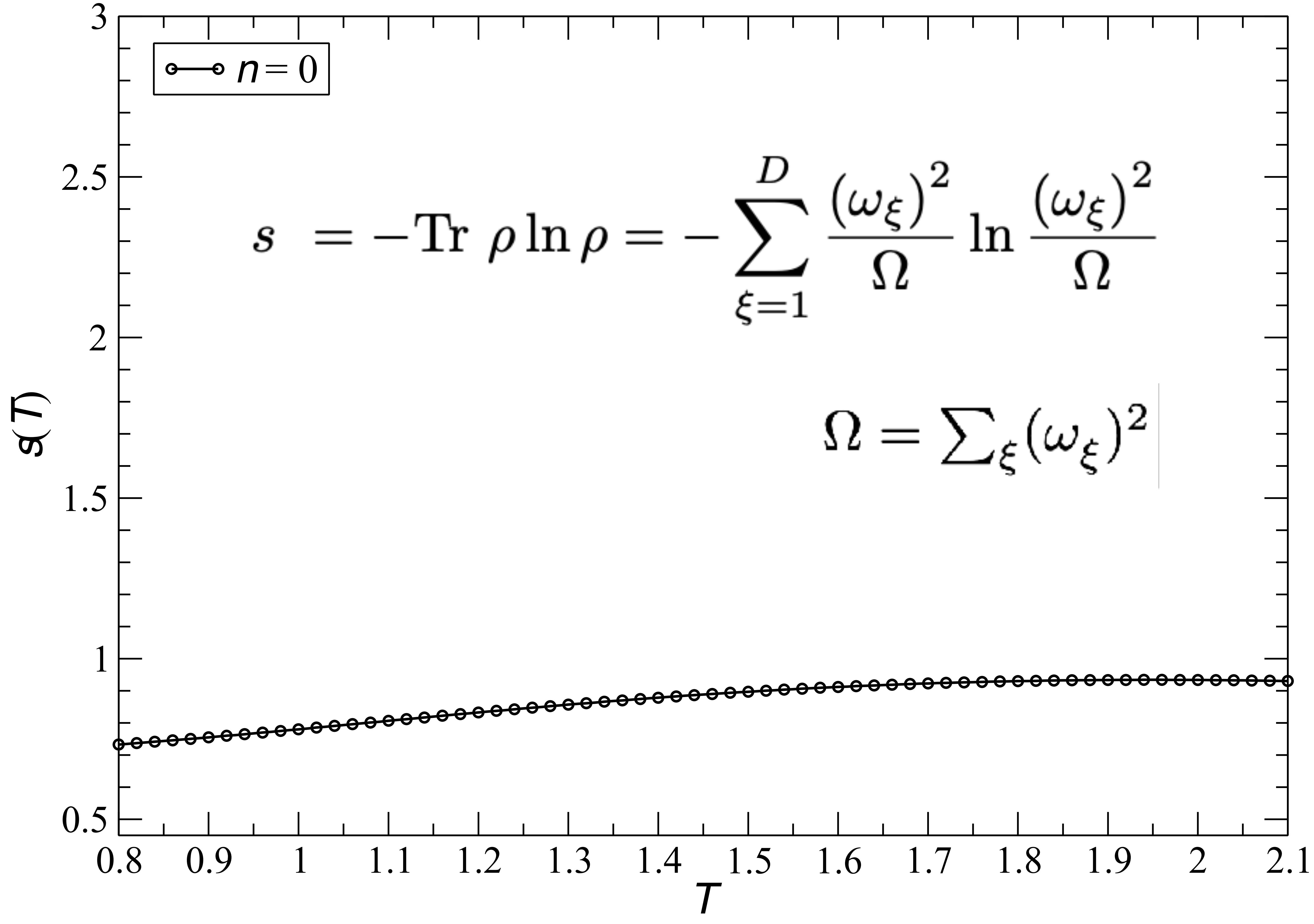


"Anomalous" specific heat for $J_1 = 1$, $0 \leq J_2 \leq 1$



Case study: Entropy flow for $J_2 = 0.5$

Entanglement entropy



$$s = -\mathrm{Tr}\, \rho \ln \rho = -\sum_{\xi=1}^{D} \frac{(\omega_\xi)^2}{\Omega} \ln \frac{(\omega_\xi)^2}{\Omega}$$

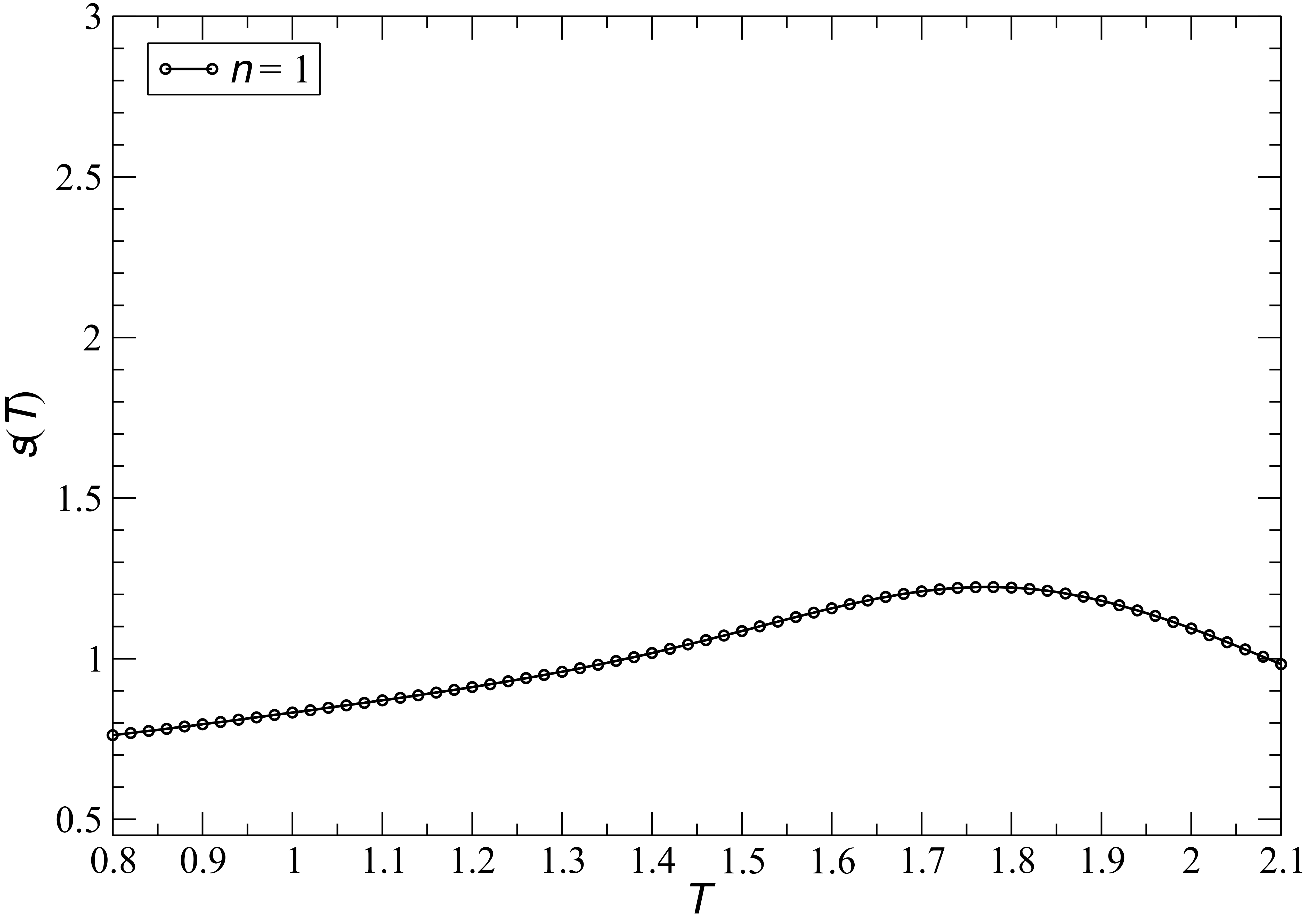
$$\Omega = \sum_\xi (\omega_\xi)^2$$

$$\mathcal{T}'^{[1],n}_{x(x'yy')} = \mathcal{T}^{[1],n}_{xx'yy'}$$

$$\mathcal{T}'^{[1],n} = U\ \omega\ V^\dagger$$

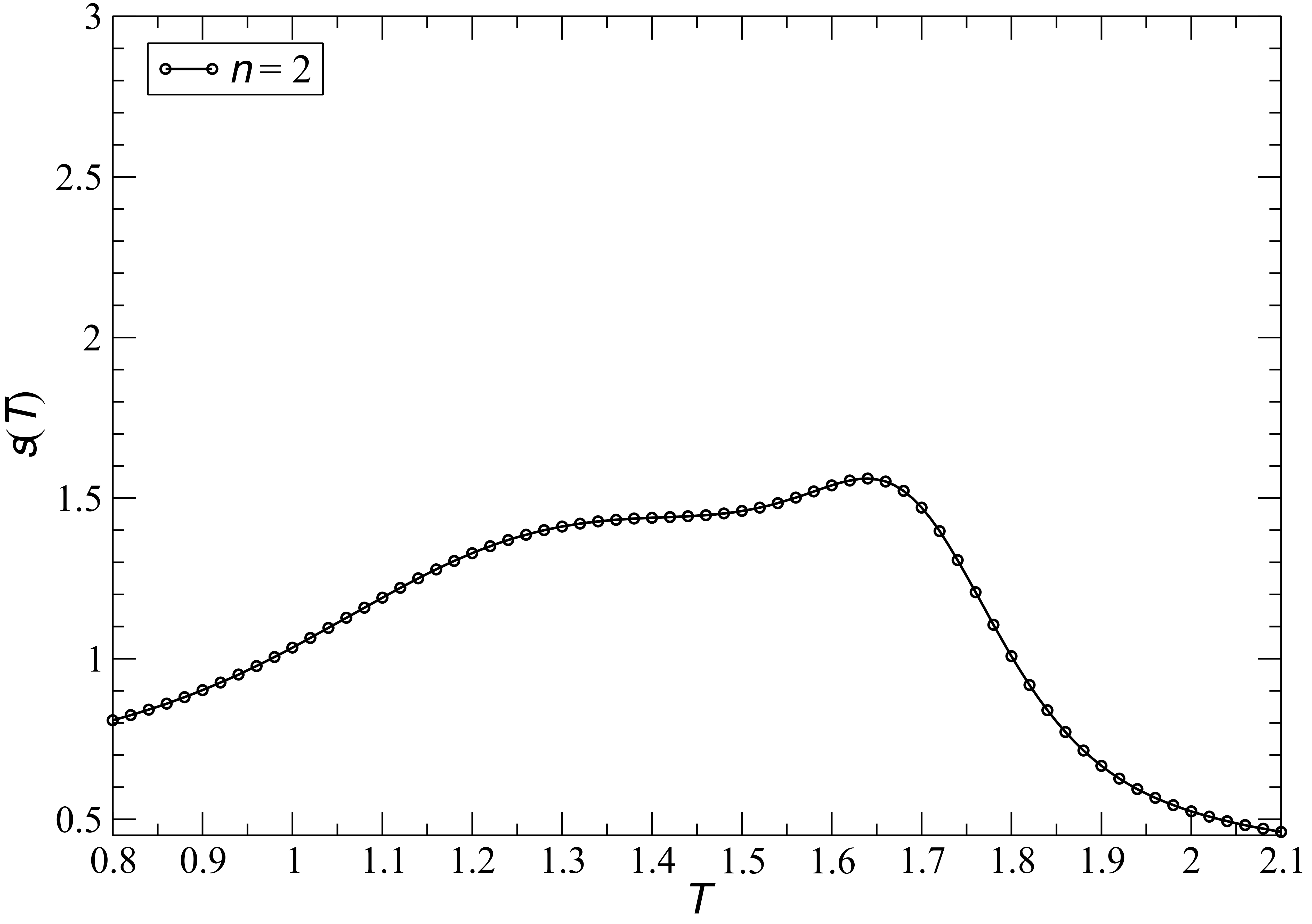
$$T_C(J_2 = 0.5) = 1.5777$$

Case study: Entropy flow for $J_2 = 0.5$



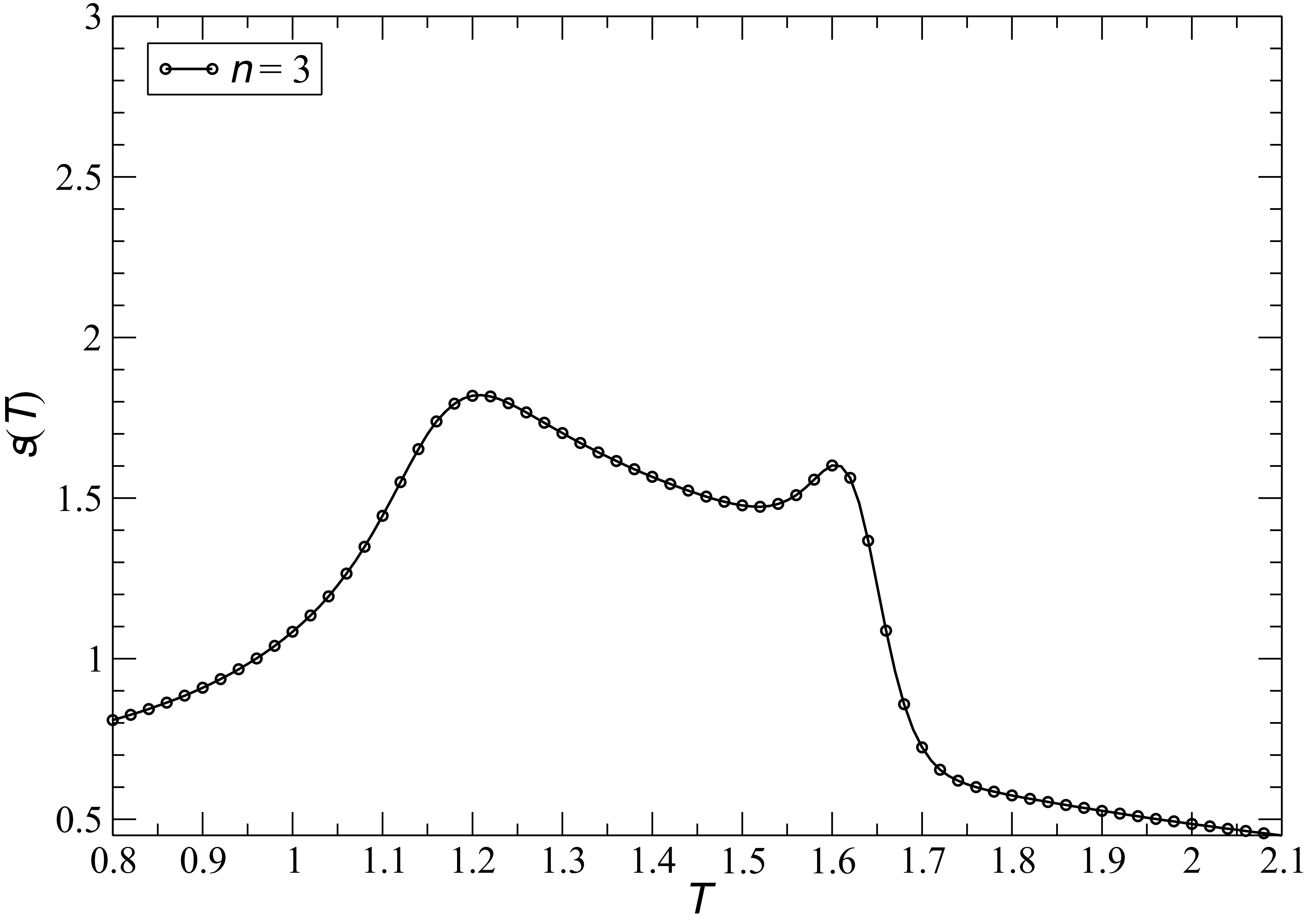
$T_C(J_2 = 0.5) = 1.5777$

Case study: Entropy flow for $J_2 = 0.5$



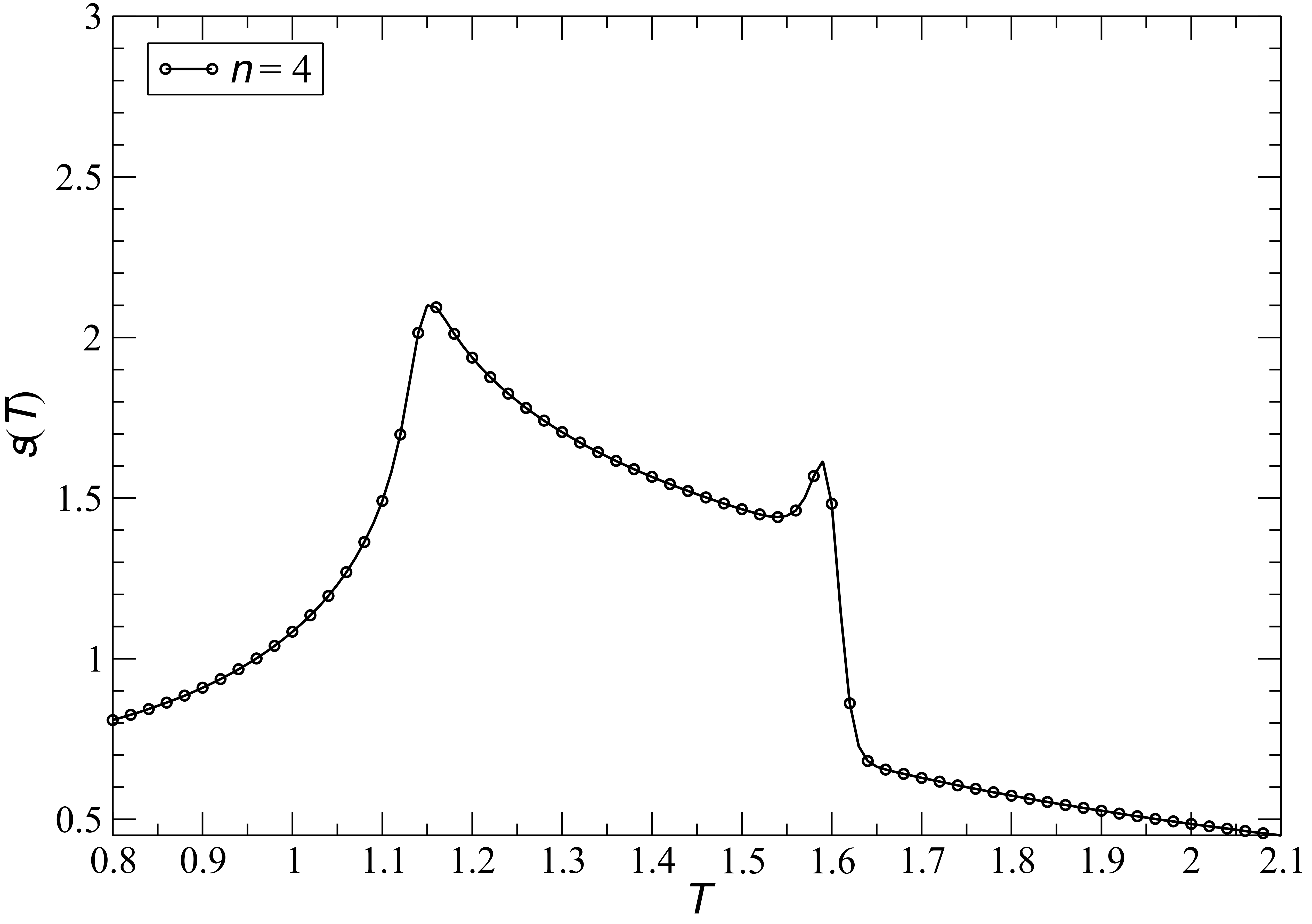
$$T_C(J_2 = 0.5) = 1.5777$$

Case study: Entropy flow for $J_2 = 0.5$



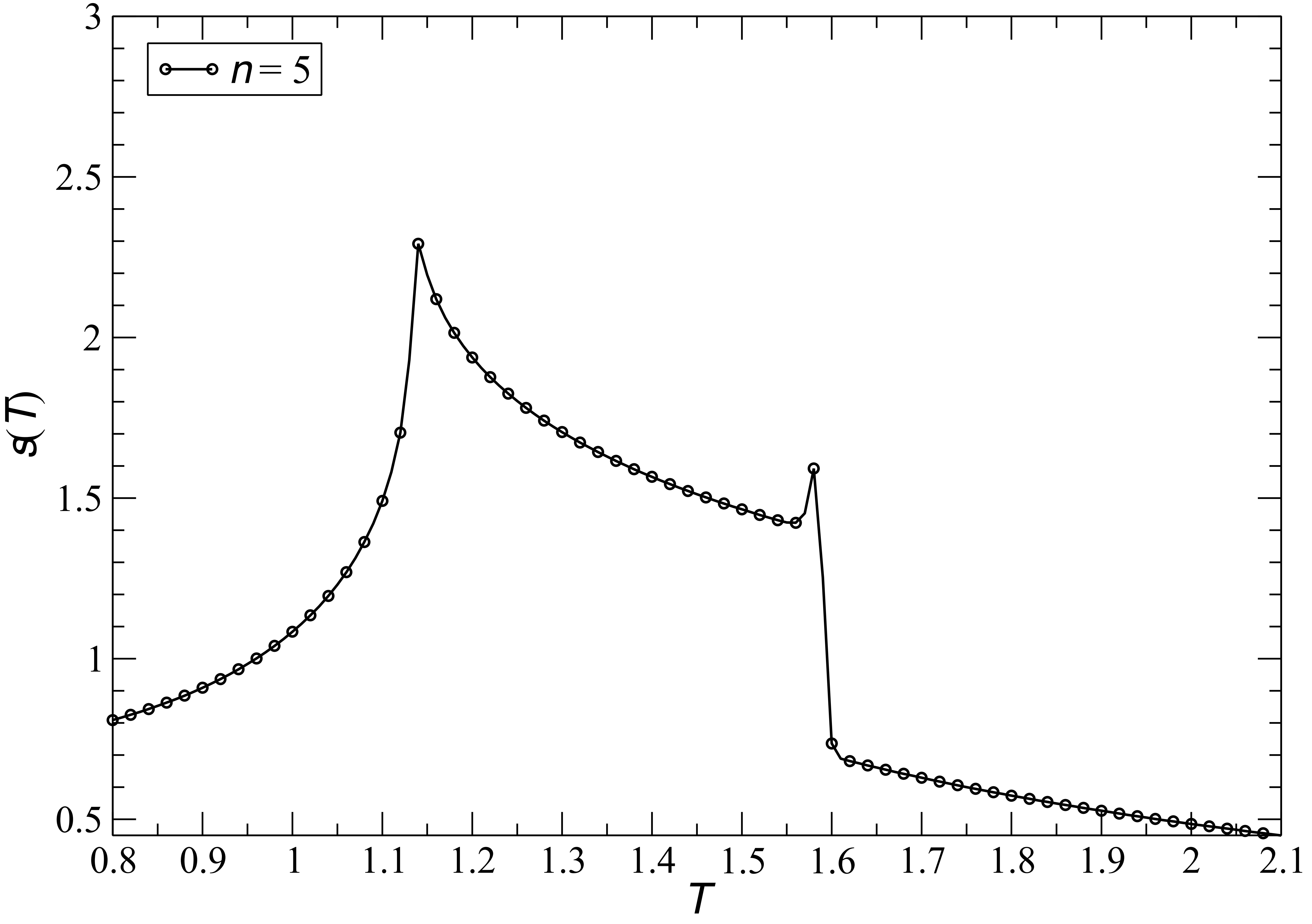
$$T_C(J_2 = 0.5) = 1.5777$$

Case study: Entropy flow for $J_2 = 0.5$



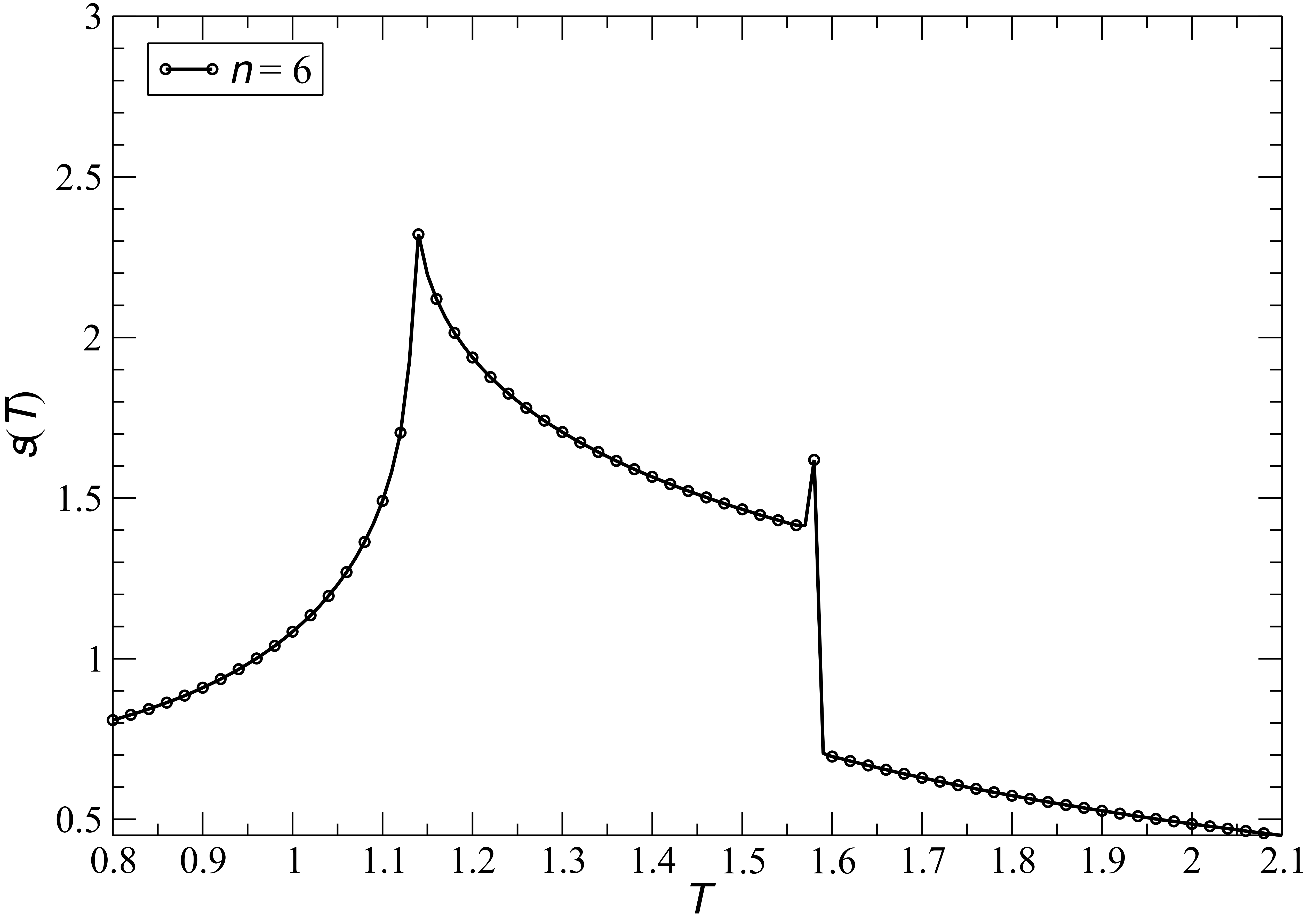
$$T_C(J_2 = 0.5) = 1.5777$$

Case study: Entropy flow for $J_2 = 0.5$



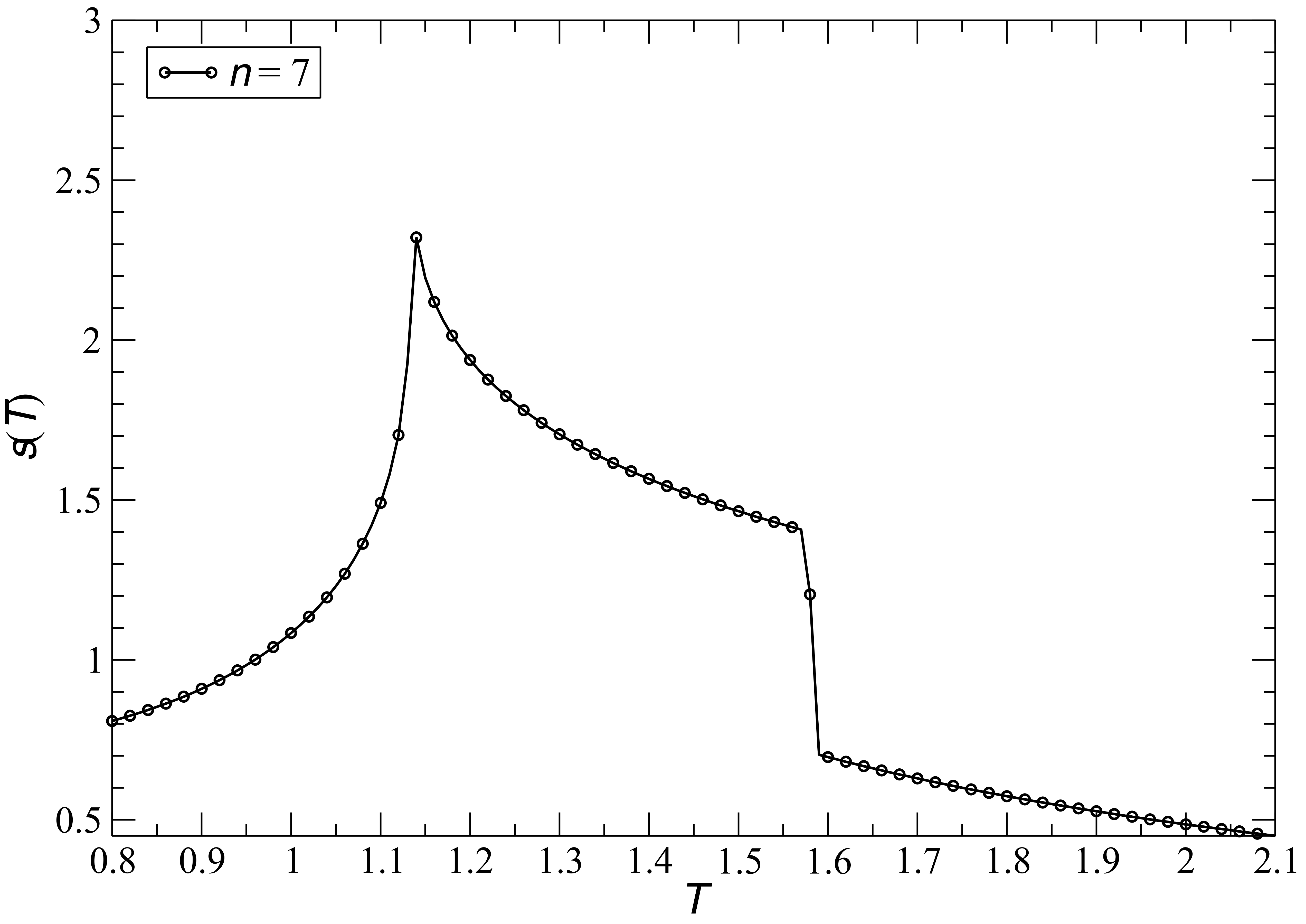
$$T_C(J_2 = 0.5) = 1.5777$$

Case study: Entropy flow for $J_2 = 0.5$



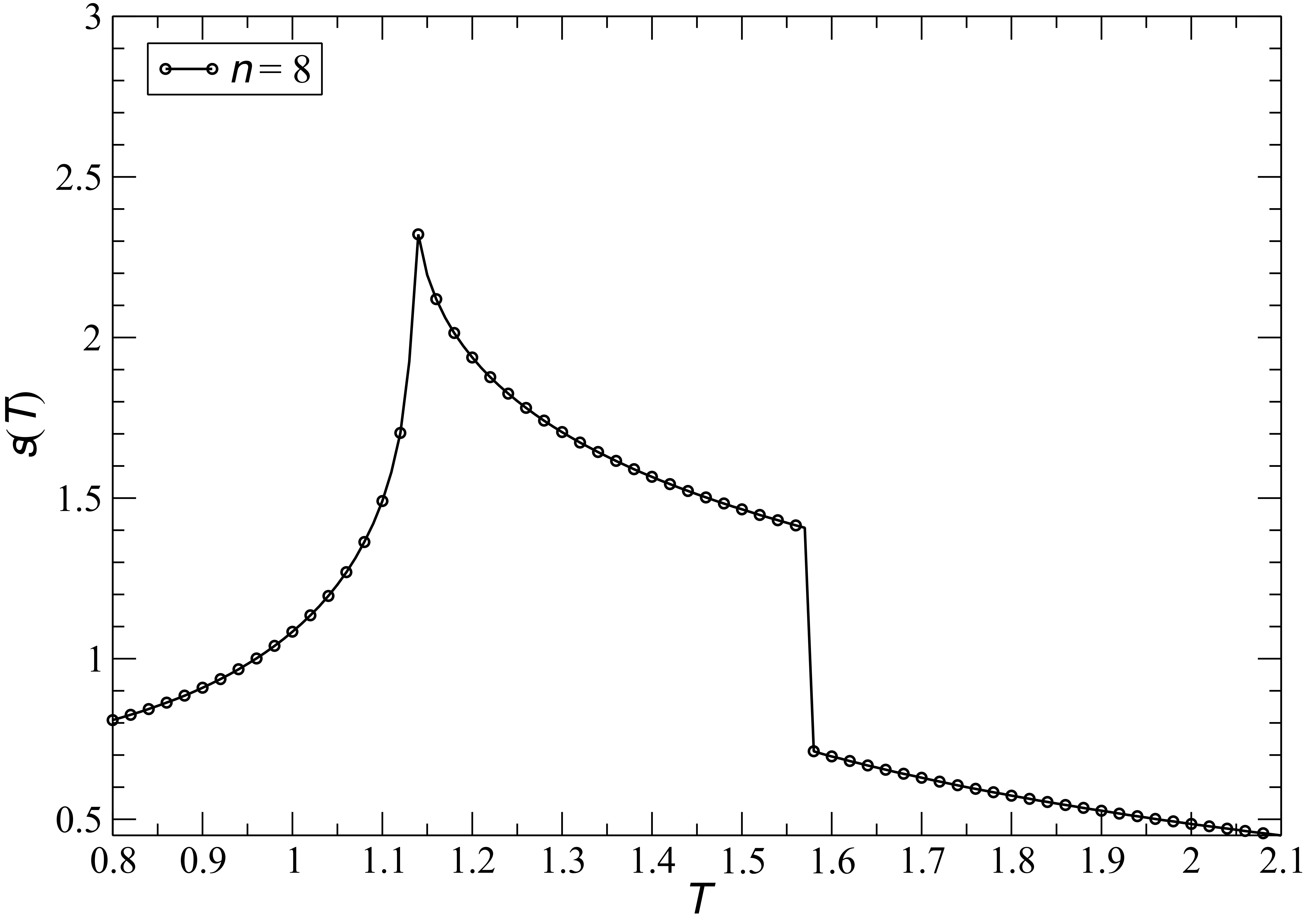
$$T_C(J_2 = 0.5) = 1.5777$$

Case study: Entropy flow for $J_2 = 0.5$



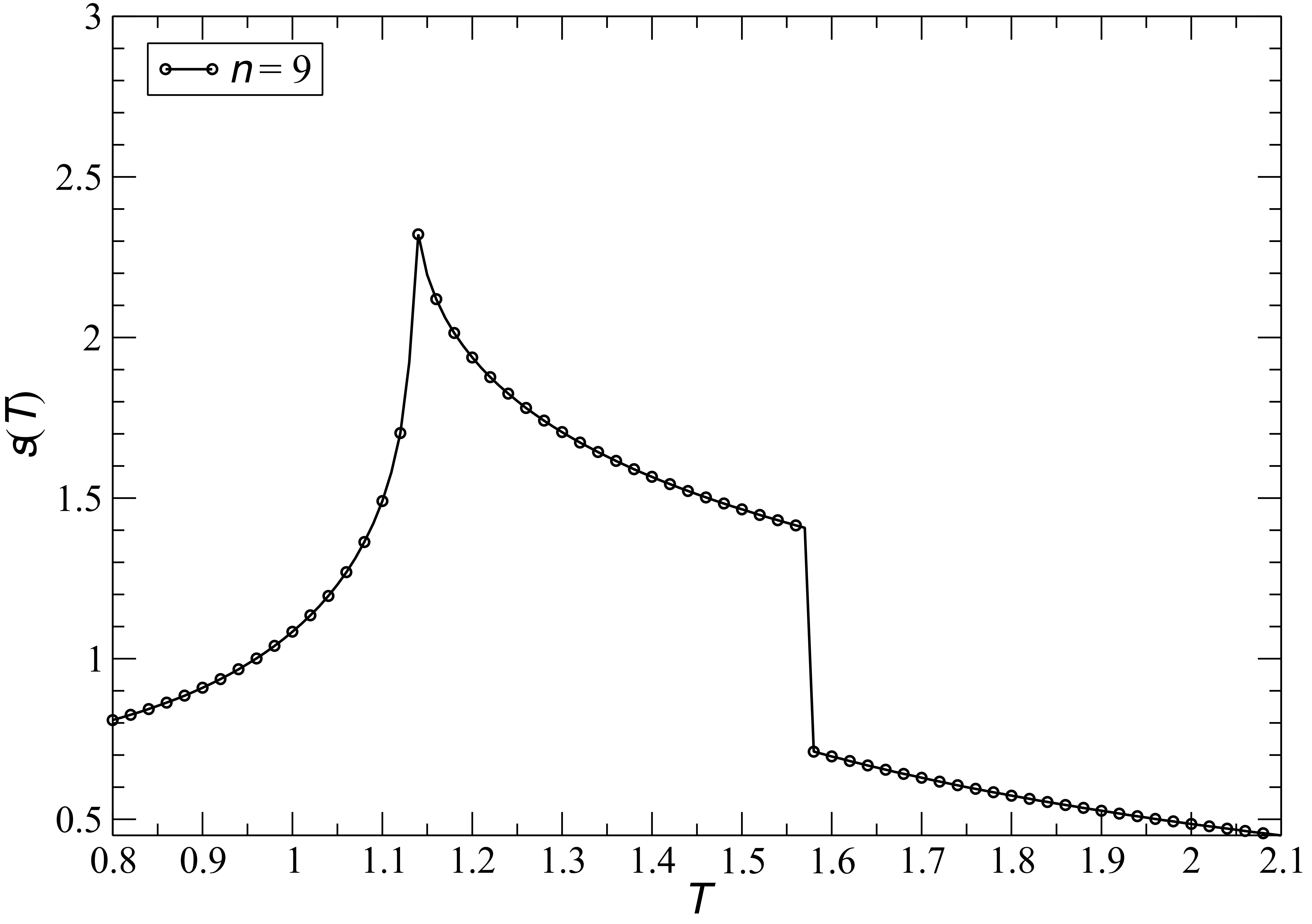
$$T_C(J_2 = 0.5) = 1.5777$$

Case study: Entropy flow for $J_2 = 0.5$



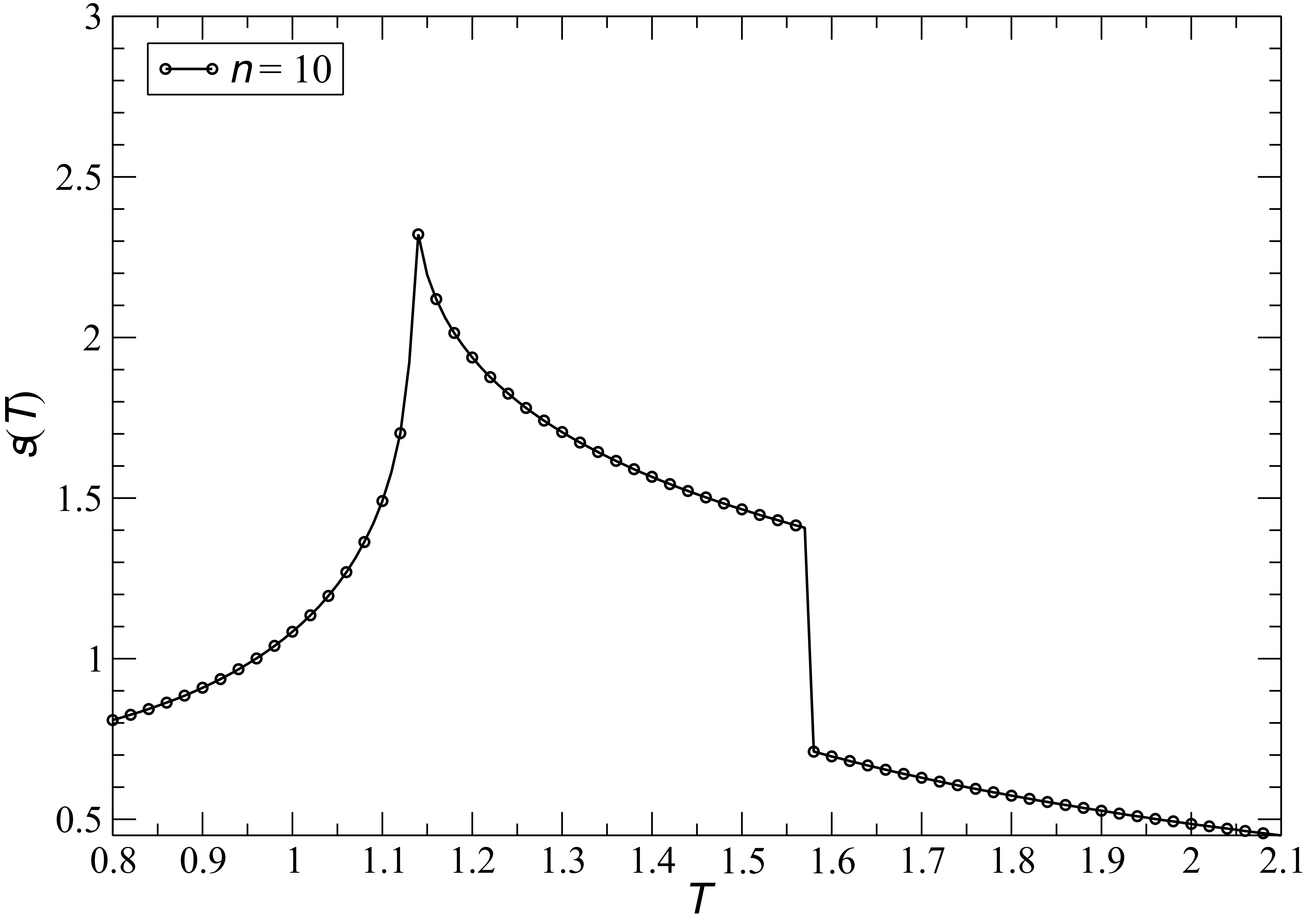
$$T_C(J_2 = 0.5) = 1.5777$$

Case study: Entropy flow for $J_2 = 0.5$



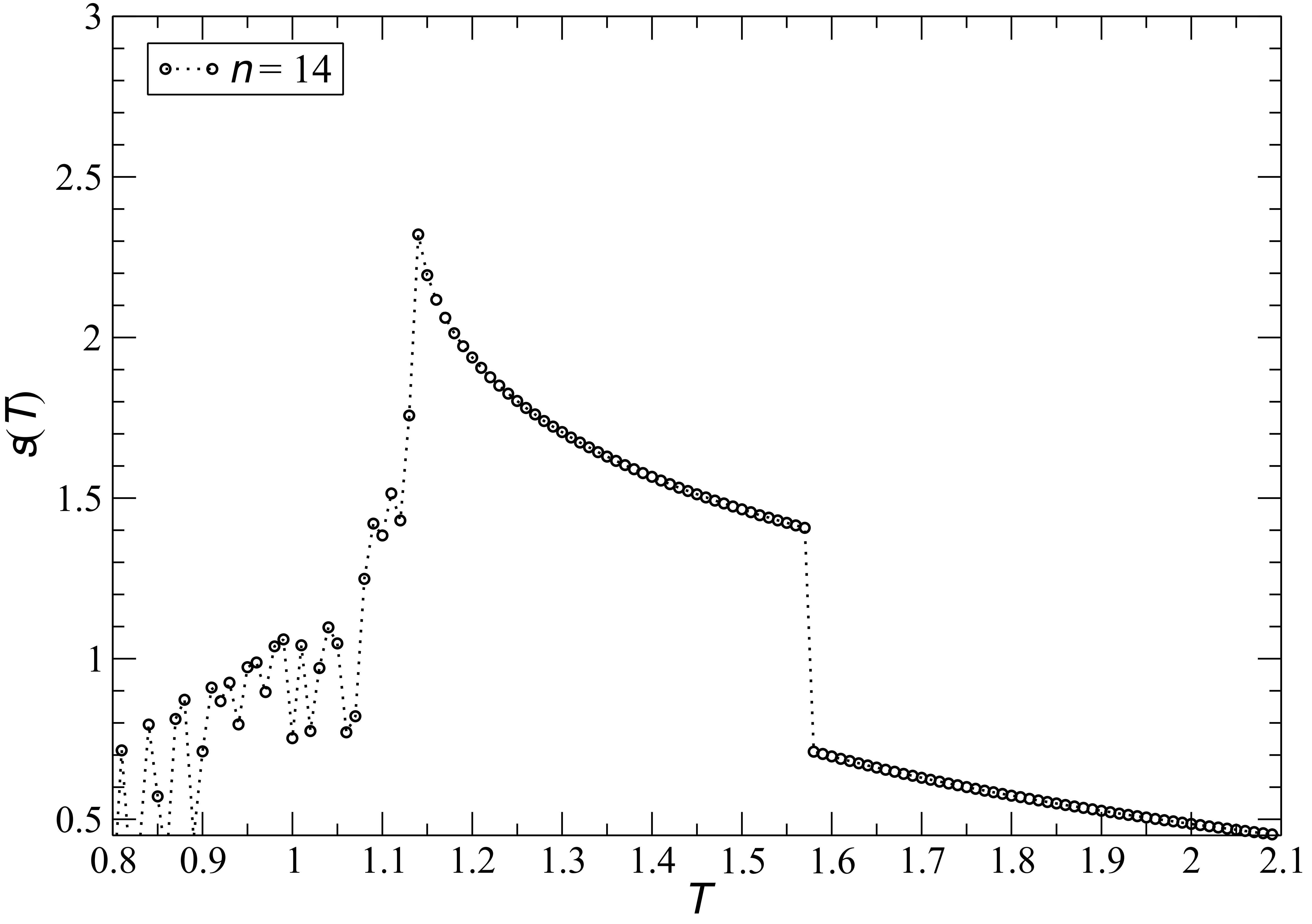
$$T_C(J_2 = 0.5) = 1.5777$$

Case study: Entropy flow for $J_2 = 0.5$



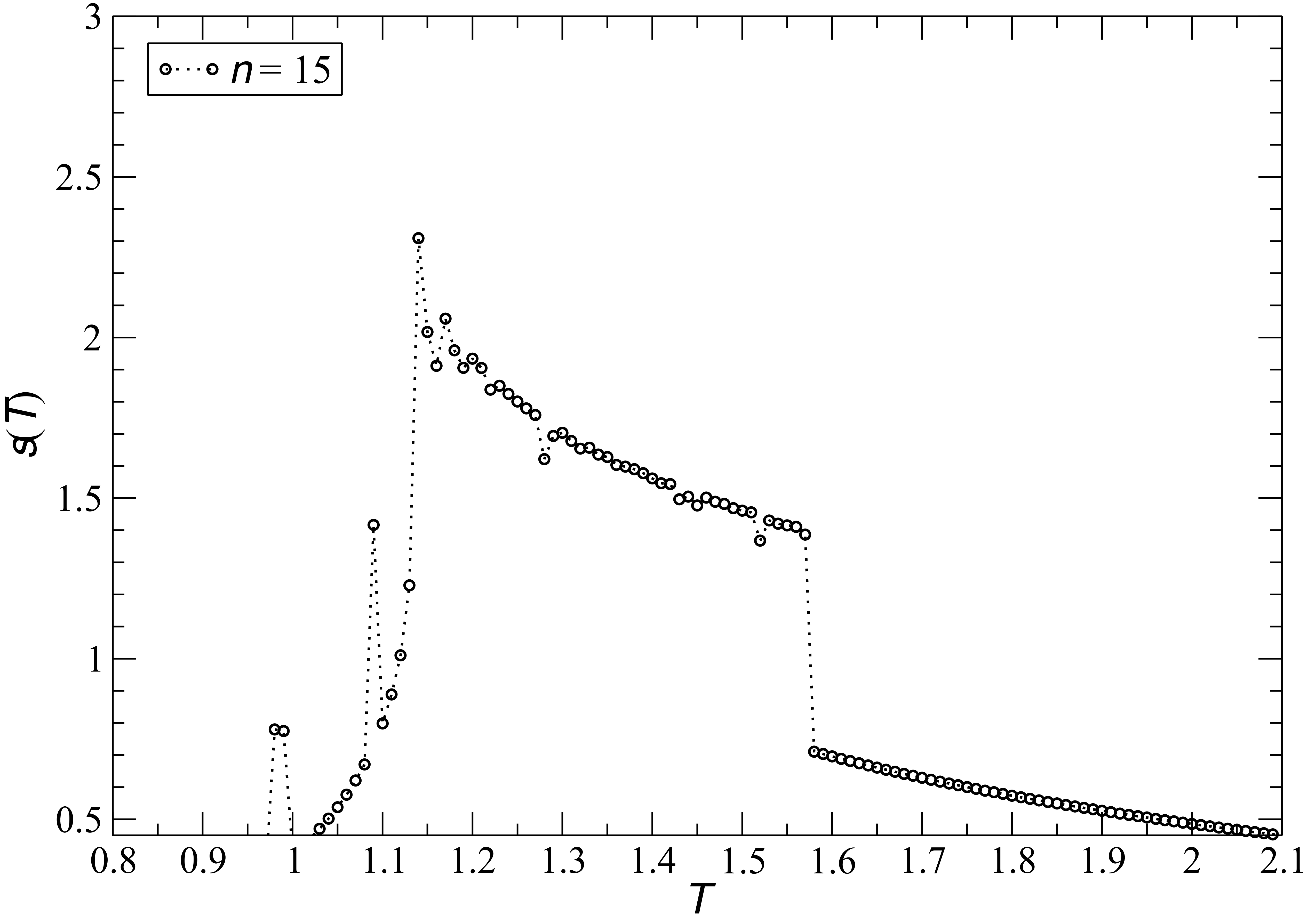
$$T_C(J_2 = 0.5) = 1.5777$$

Case study: Entropy flow for $J_2 = 0.5$



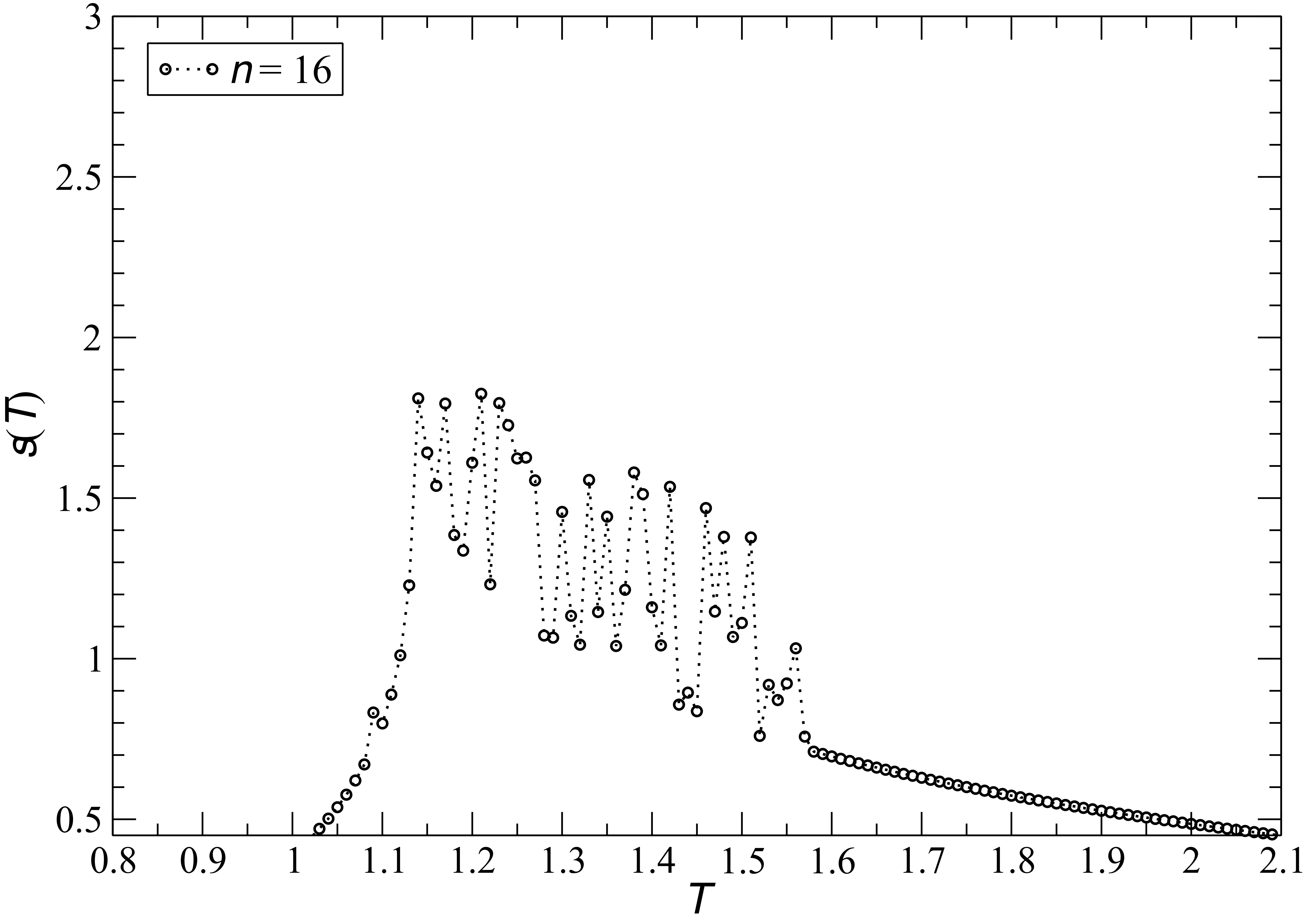
$$T_C(J_2 = 0.5) = 1.5777$$

Case study: Entropy flow for $J_2 = 0.5$



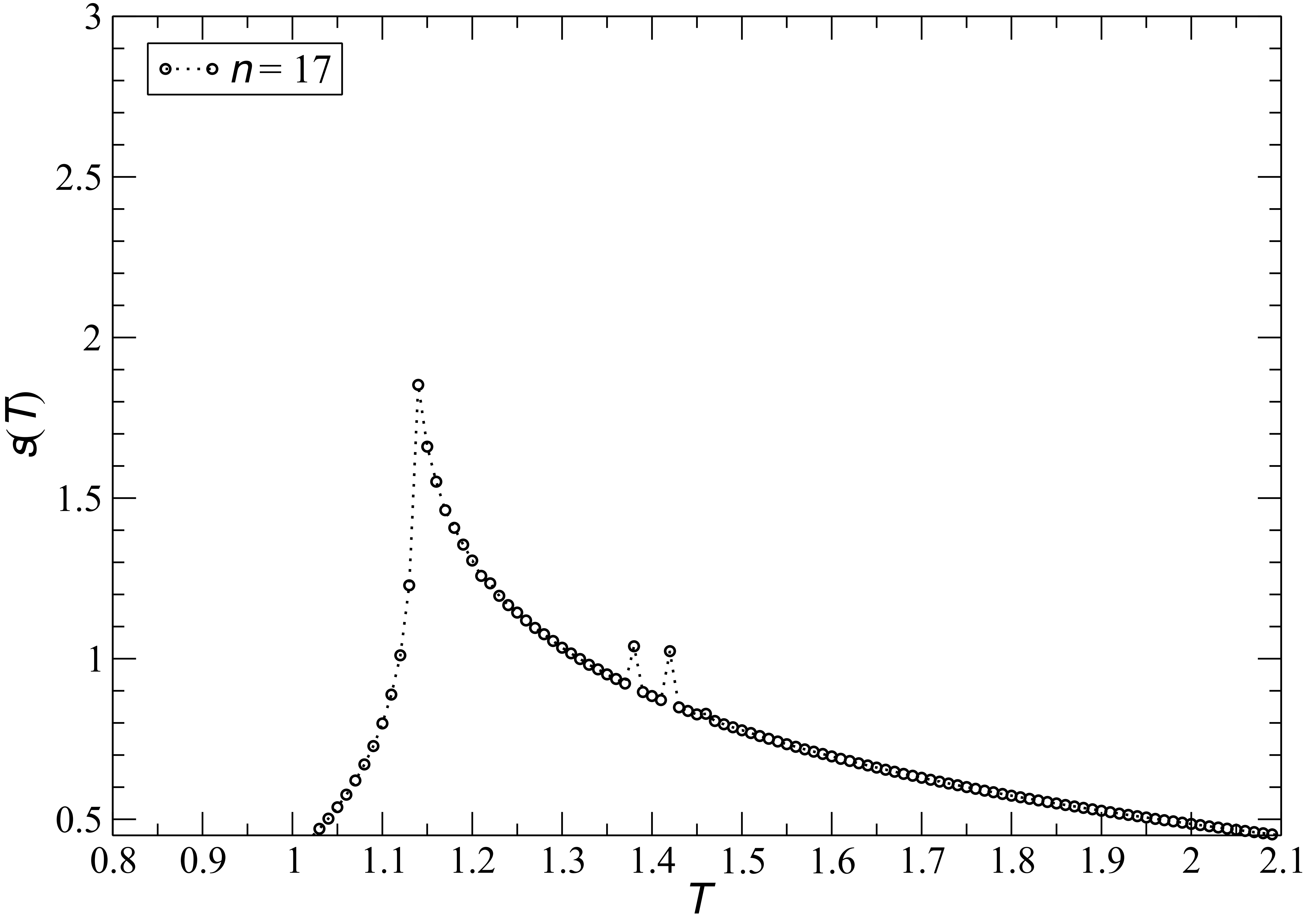
$$T_C(J_2 = 0.5) = 1.5777$$

Case study: Entropy flow for $J_2 = 0.5$



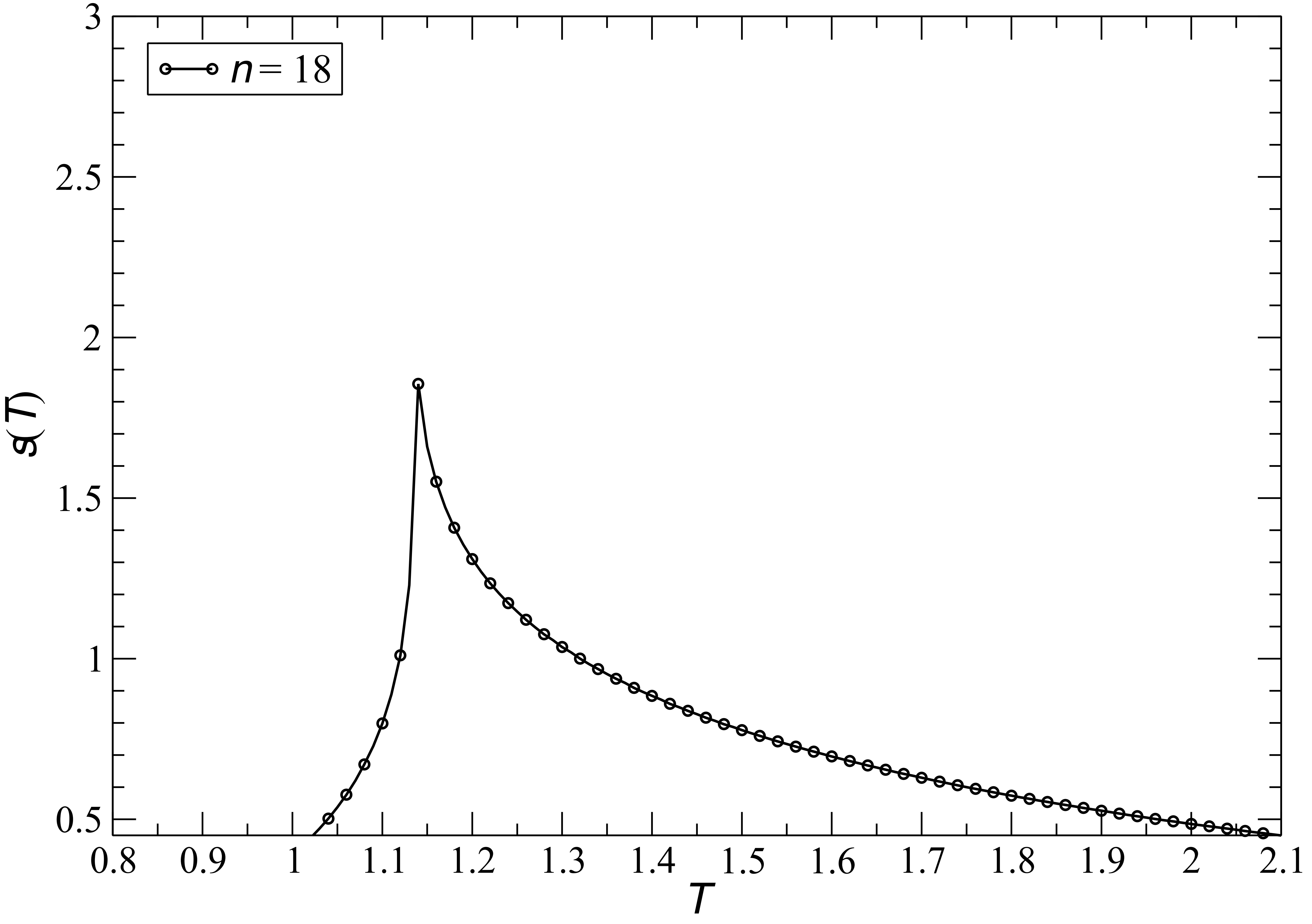
$$T_C(J_2 = 0.5) = 1.5777$$

Case study: Entropy flow for $J_2 = 0.5$



$$T_C(J_2 = 0.5) = 1.5777$$

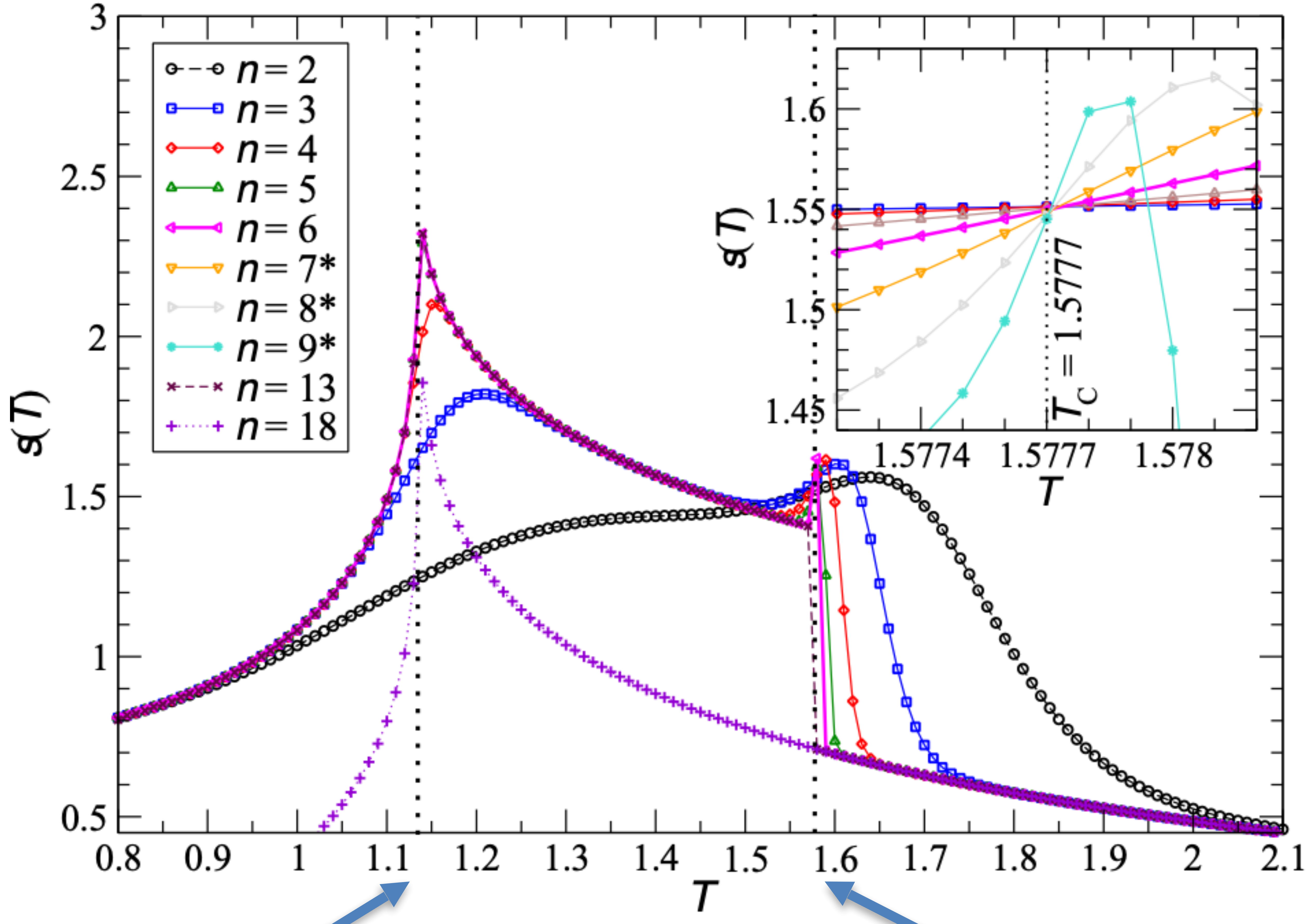
Case study: Entropy flow for $J_2 = 0.5$

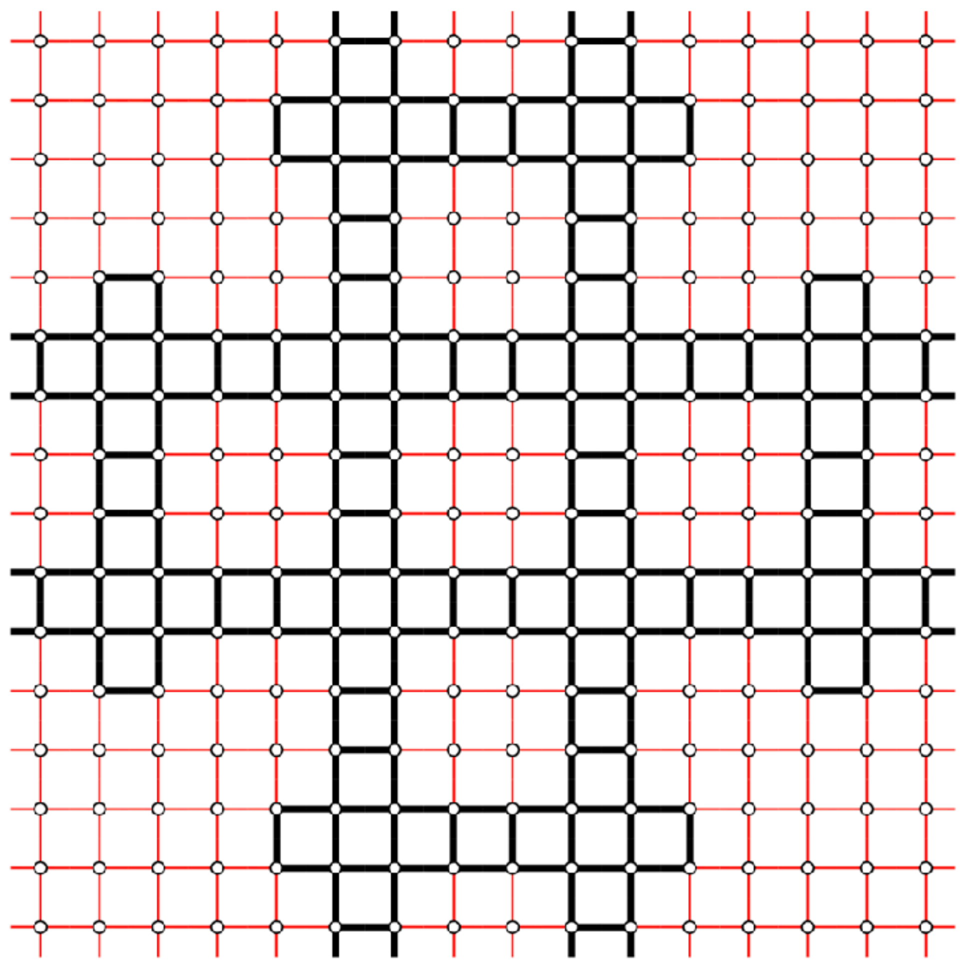


$$T_C(J_2 = 0.5) = 1.5777$$

Case study: Entropy flow for $J_2 = 0.5$

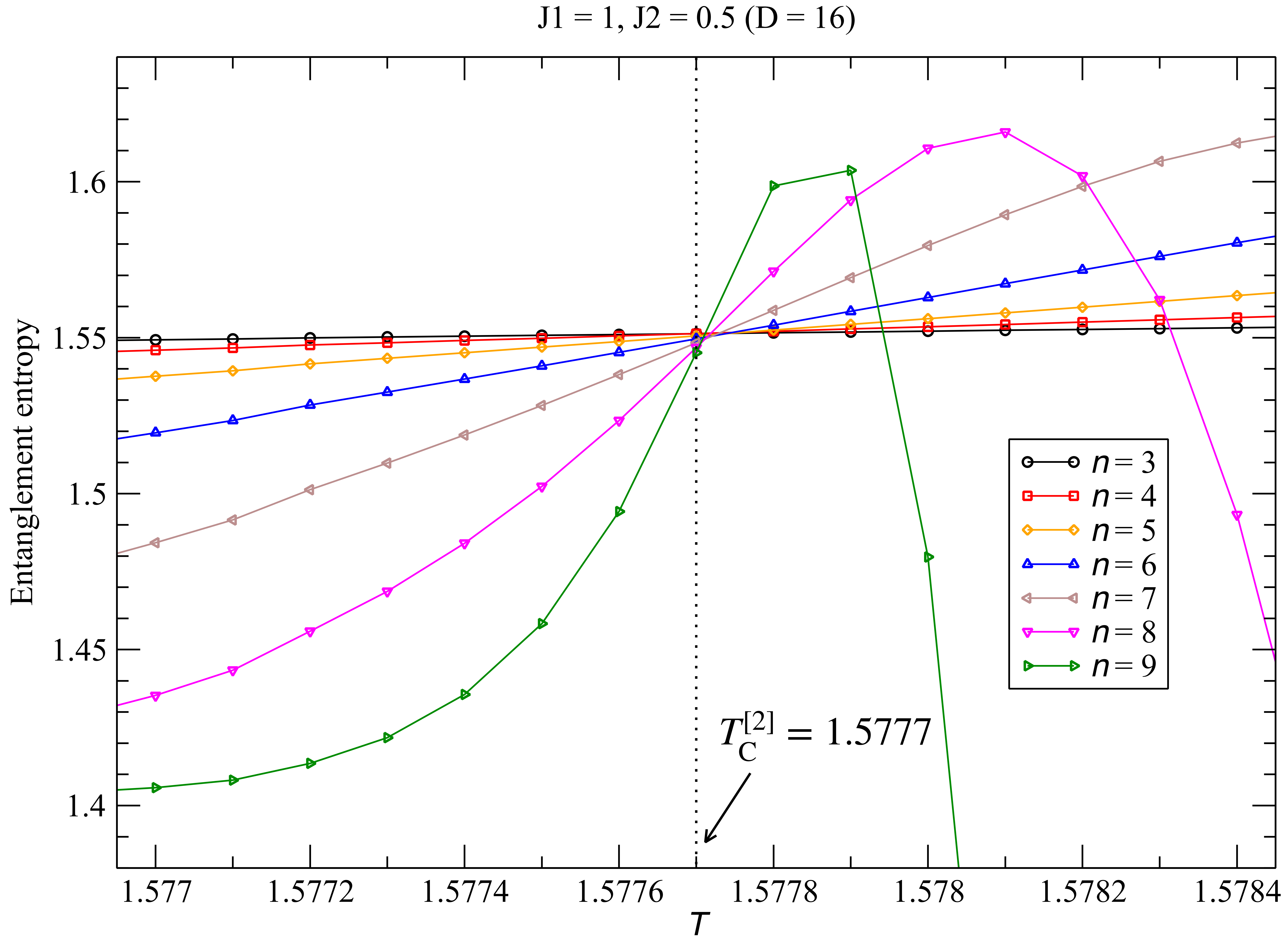
$$T_C^{[1]}(J_2) = J_2 T_C^{\text{Ising}} = J_2 \frac{2}{\ln\left(1+\sqrt{2}\right)}$$



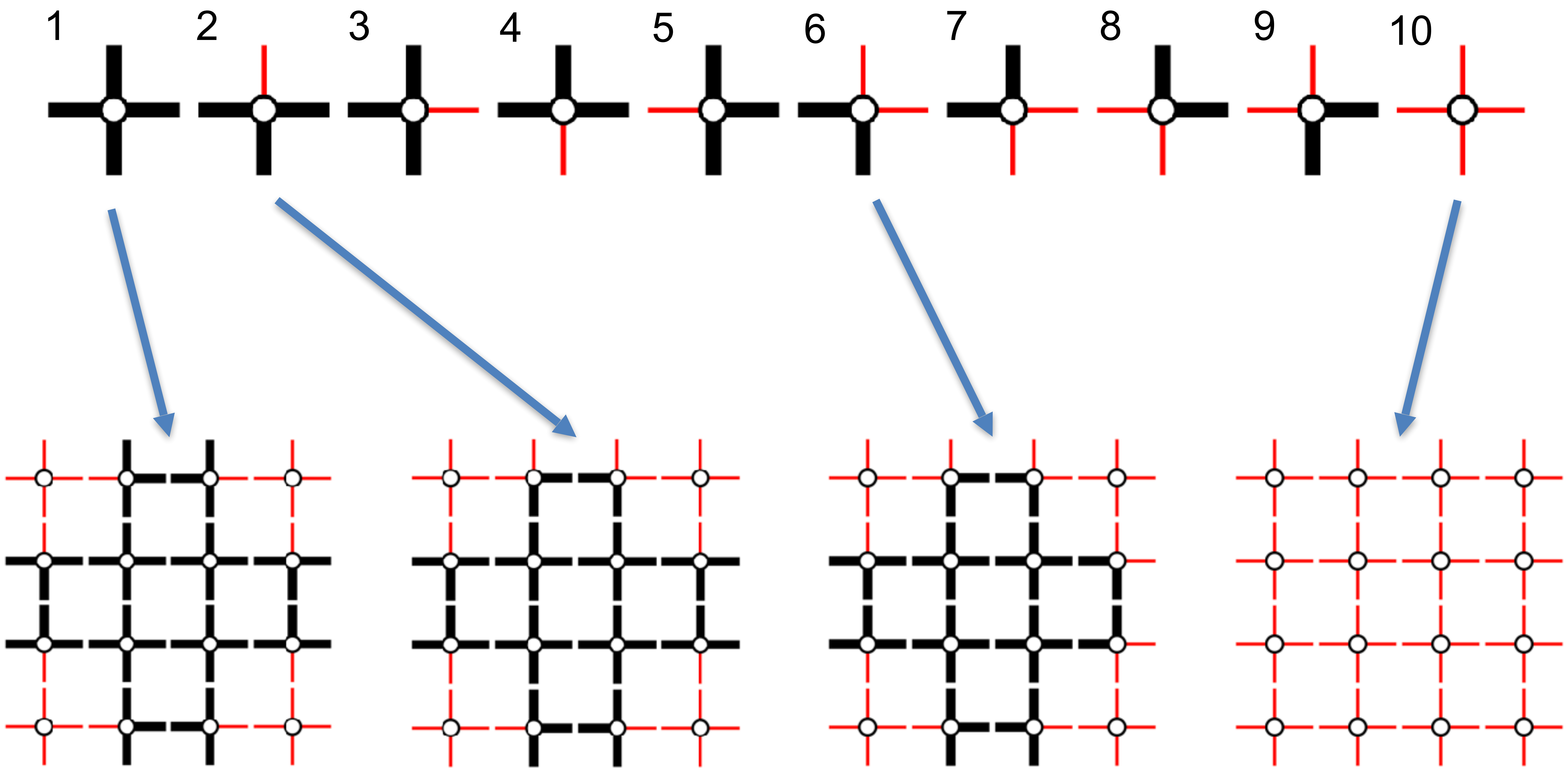


$$T_C^{[1]}(0.5) = \frac{2\times 0.5}{\ln\left(1+\sqrt{2}\right)} \approx 1.13$$

$$T_C^{[2]}(0.5) = 1.5777$$

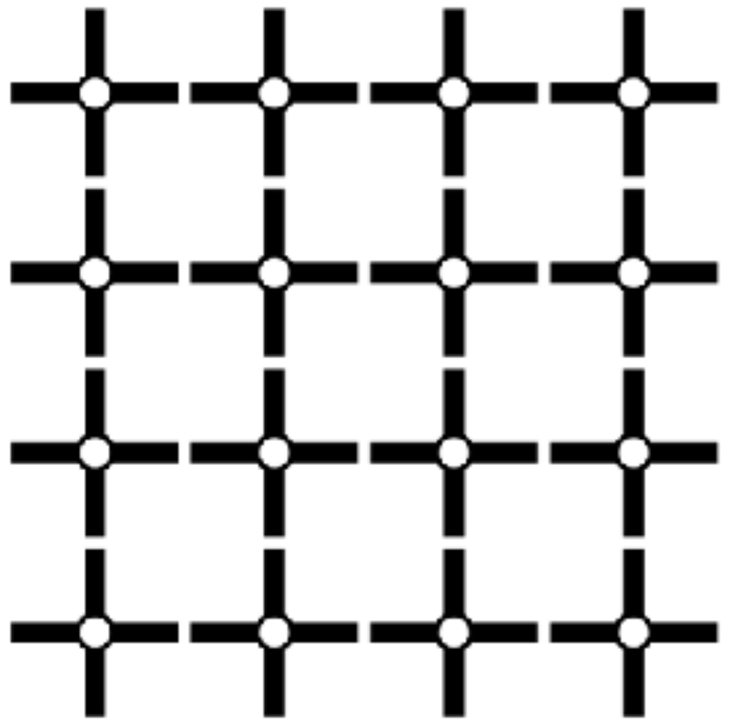

J1 = 1, J2 = 0.5 (D = 16)
Entanglement entropy
T
$T_C^{[2]} = 1.5777$
n = 3
n = 4
n = 5
n = 6
n = 7
n = 8
n = 9
1.4
1.45
1.5
1.55
1.6
1.577
1.5772
1.5774
1.5776
1.5778
1.578
1.5782
1.5784

Extension Patterns (Fractal$_1$)

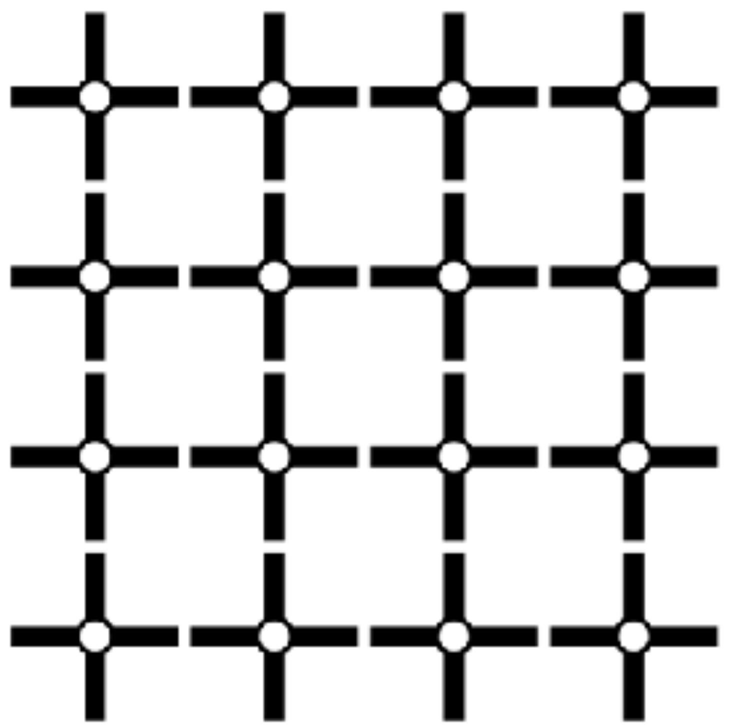


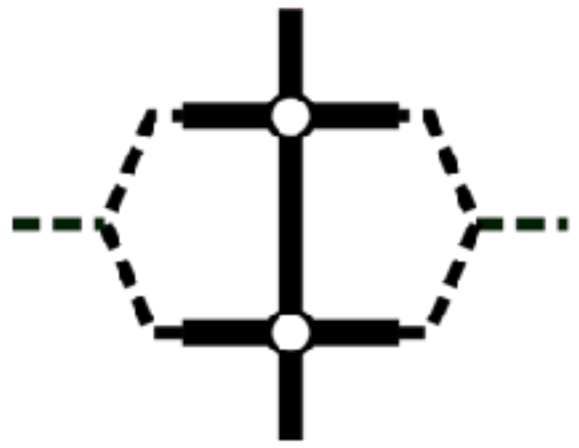
10 extension relations

HOTRG: 4-Steps “Unrolling”

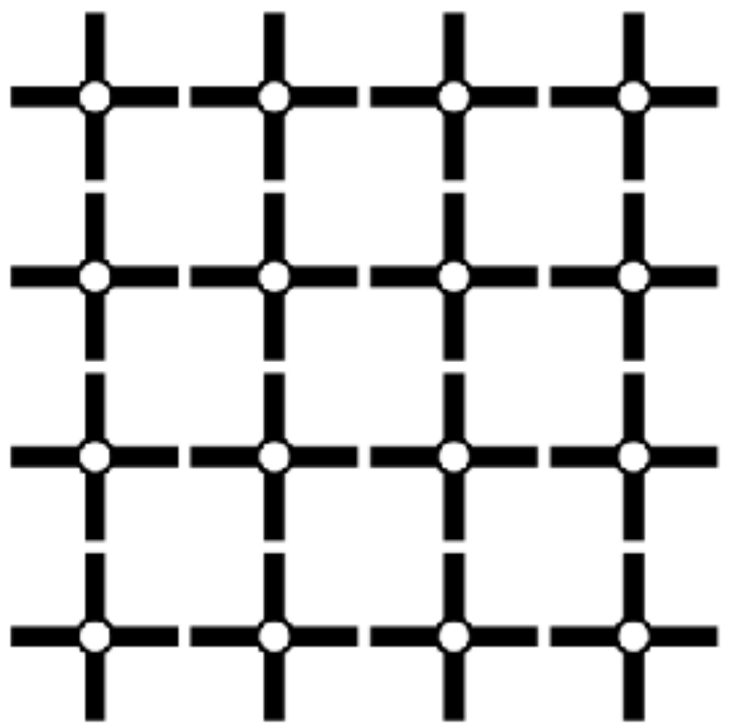


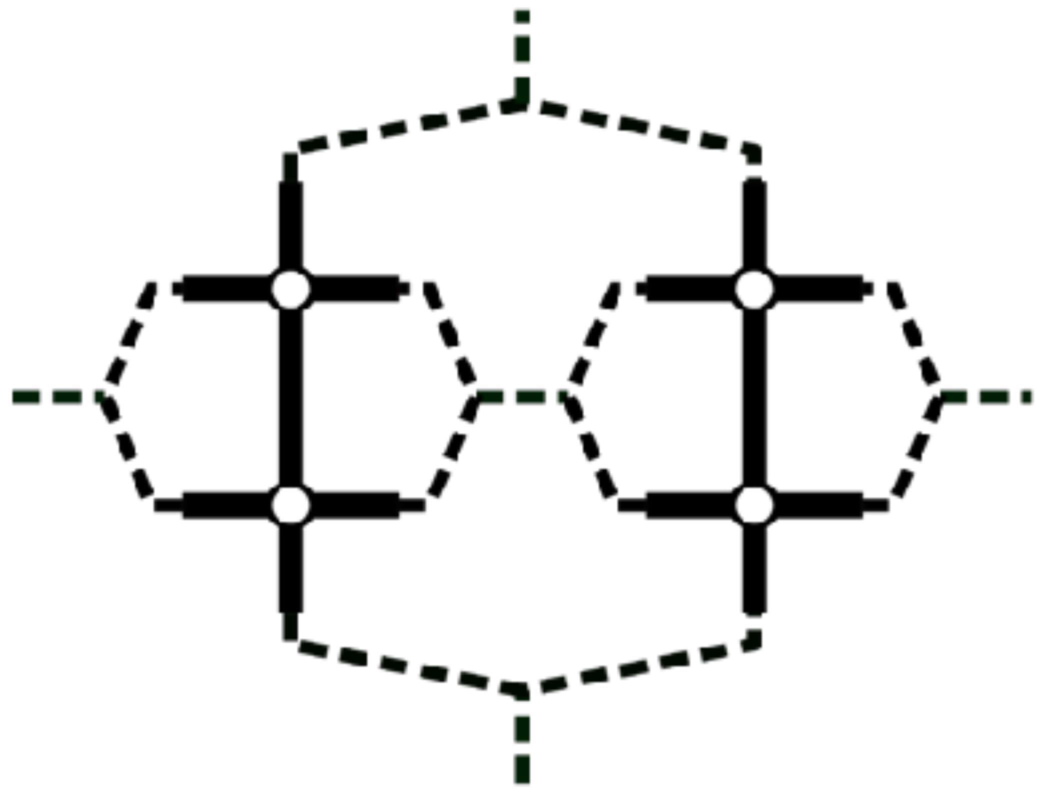
HOTRG: 4-Steps “Unrolling”



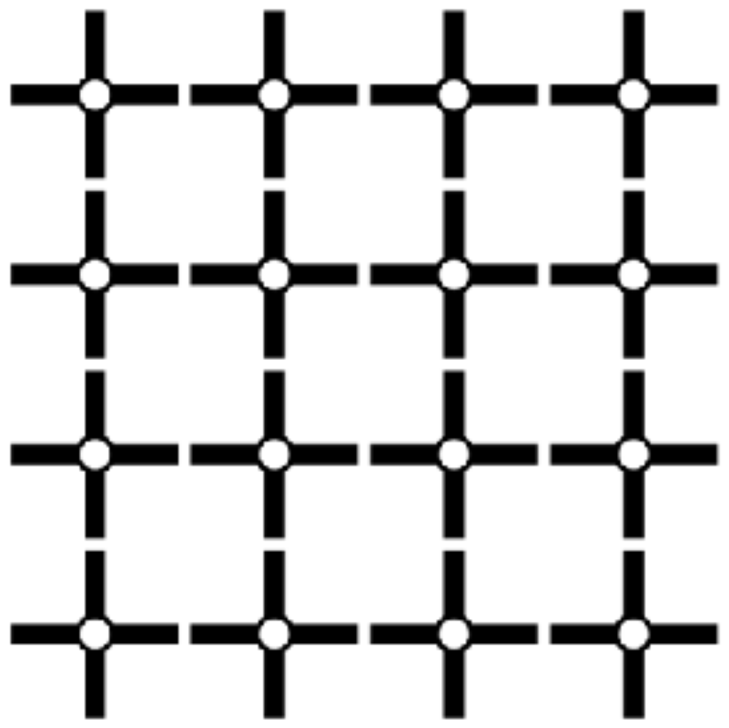


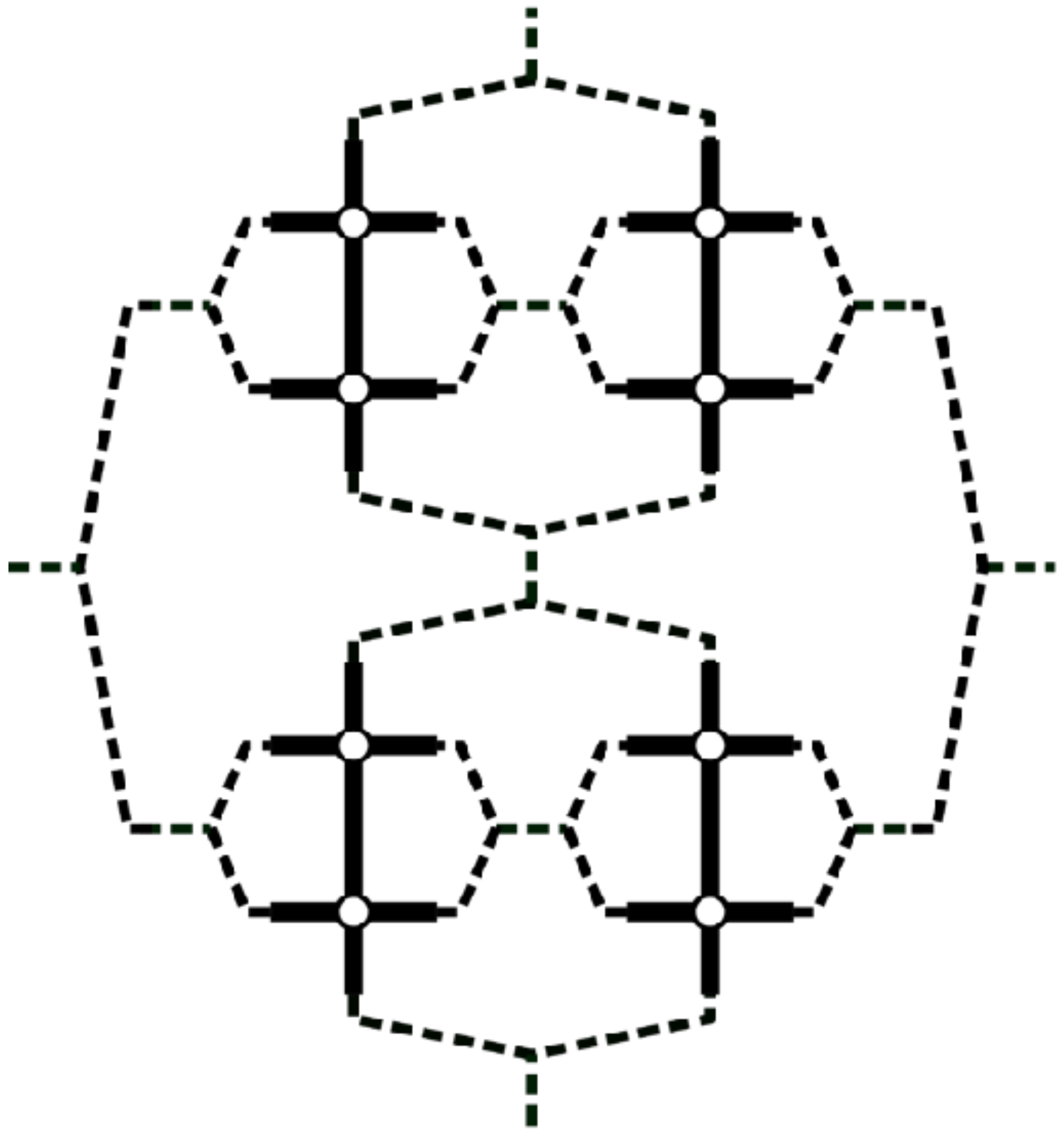
HOTRG: 4-Steps “Unrolling”



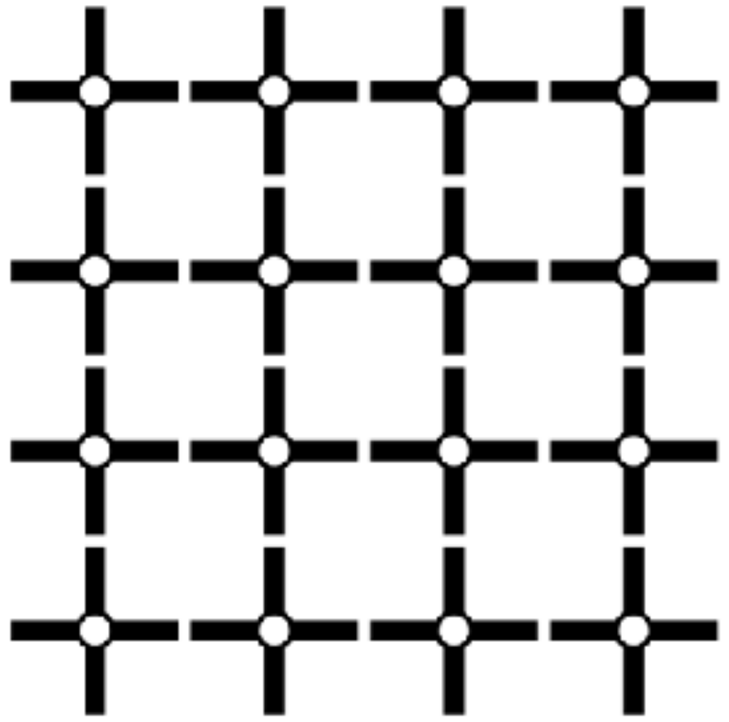


HOTRG: 4-Steps "Unrolling"

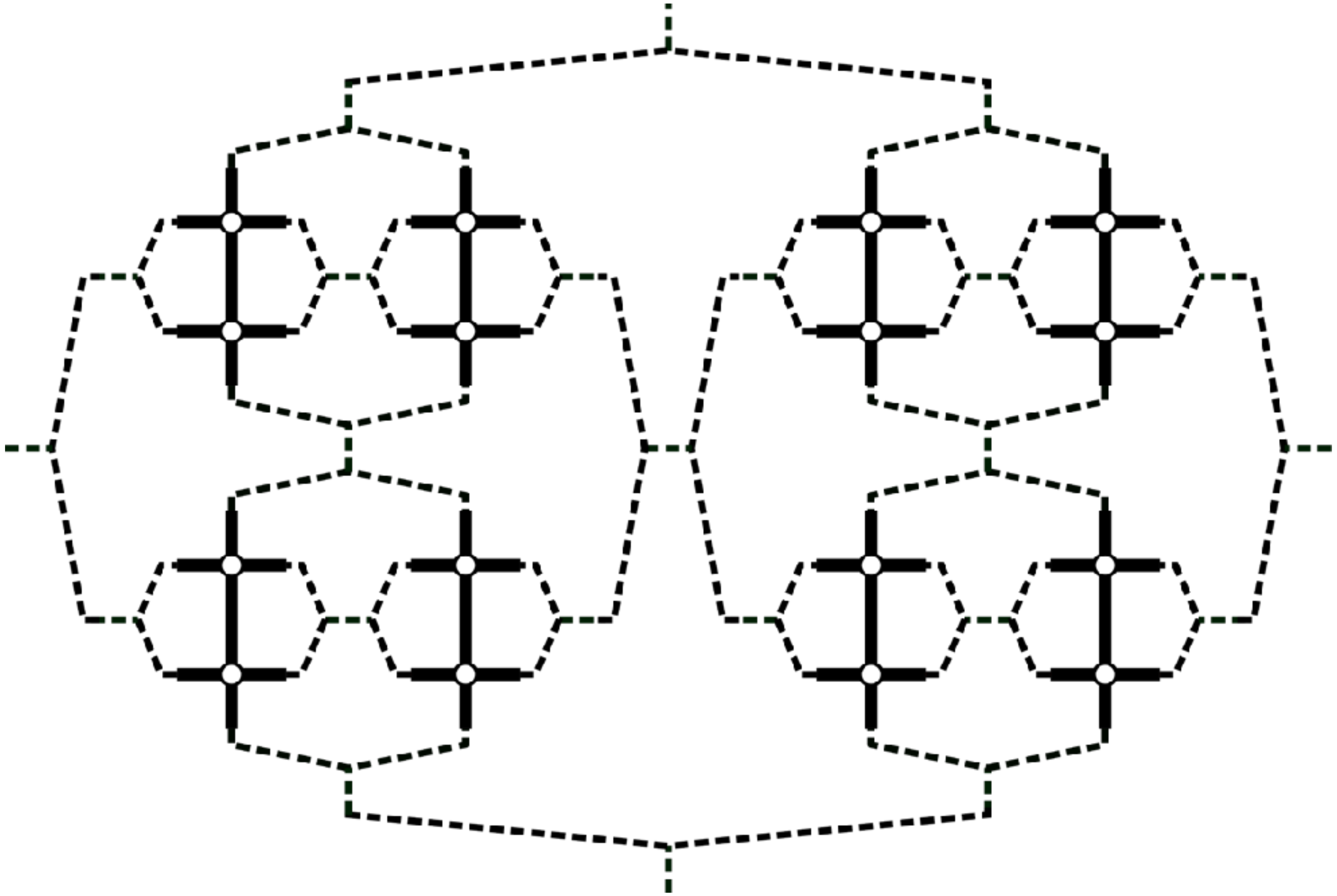




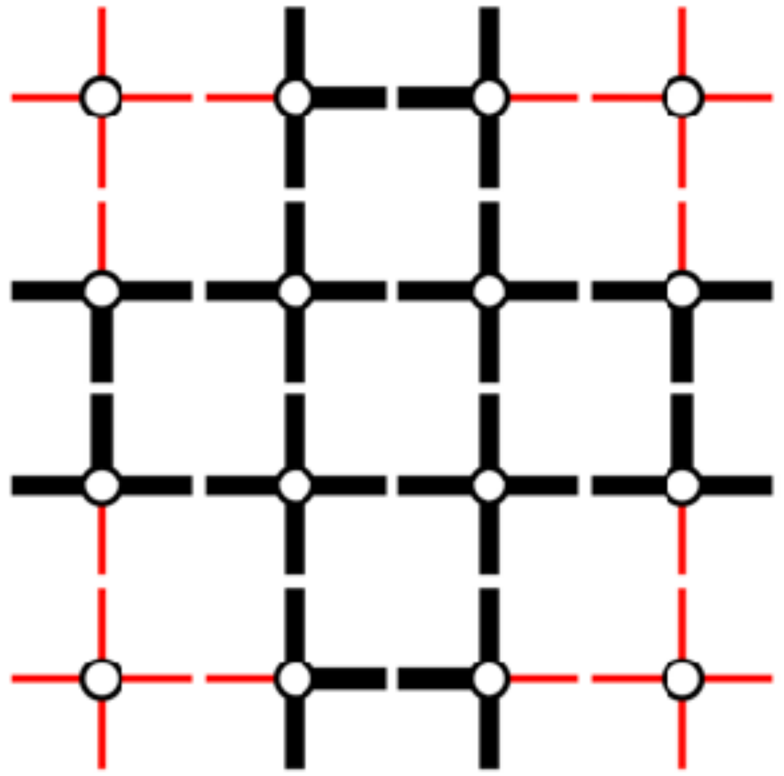
HOTRG: 4-Steps "Unrolling"

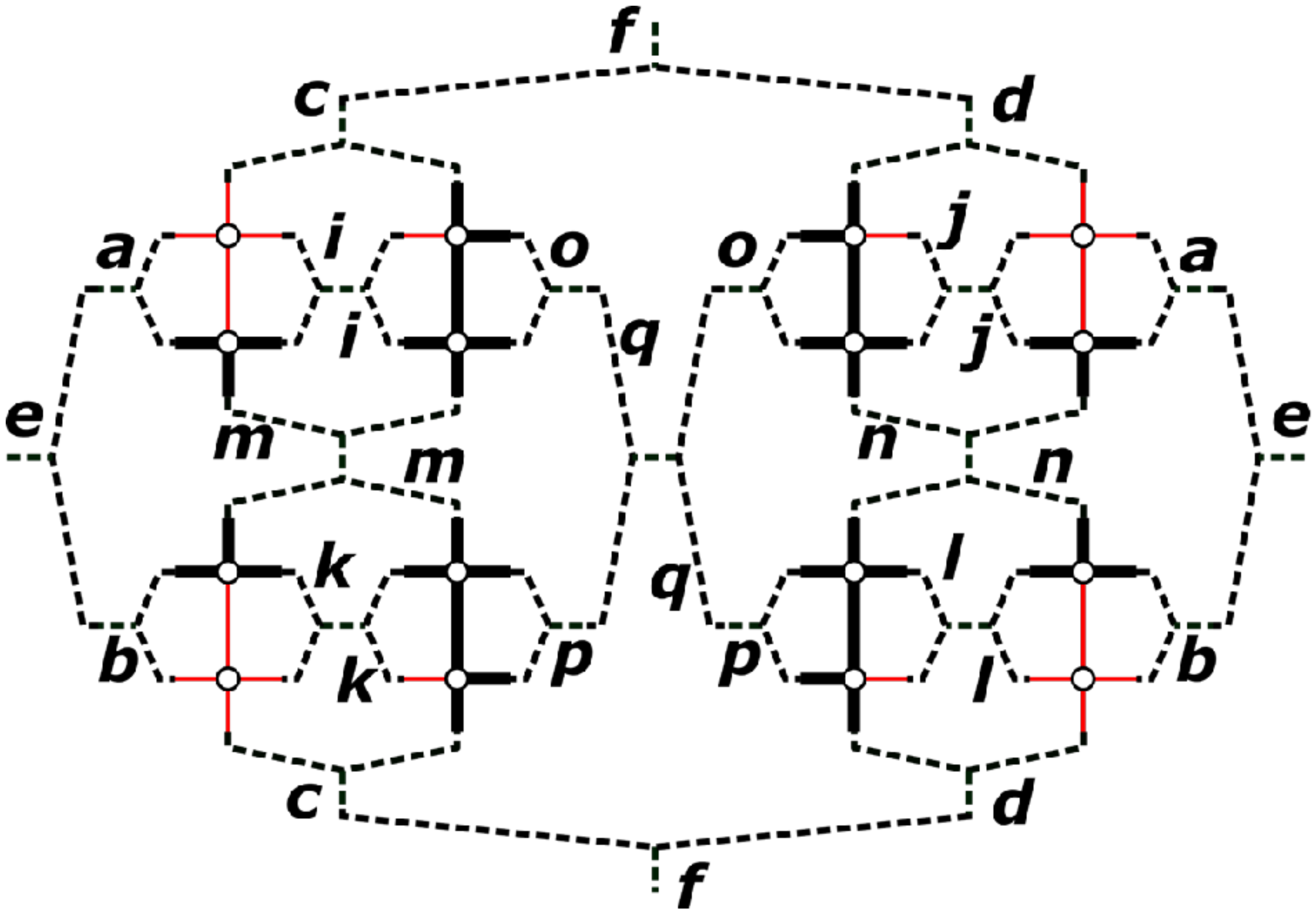


Notice which projectors are identical



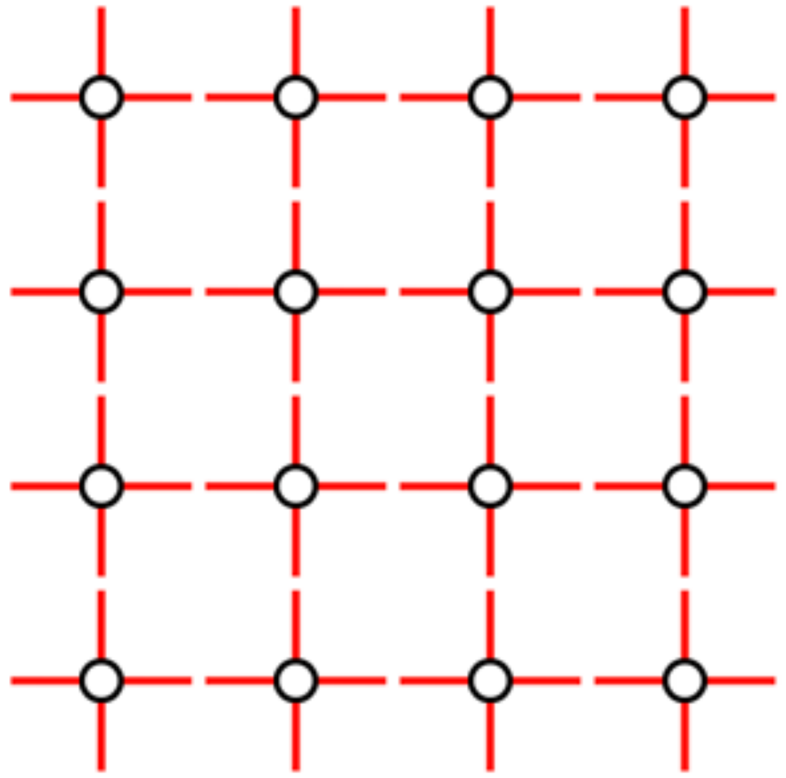
Inhomogeneous Projector Patterns

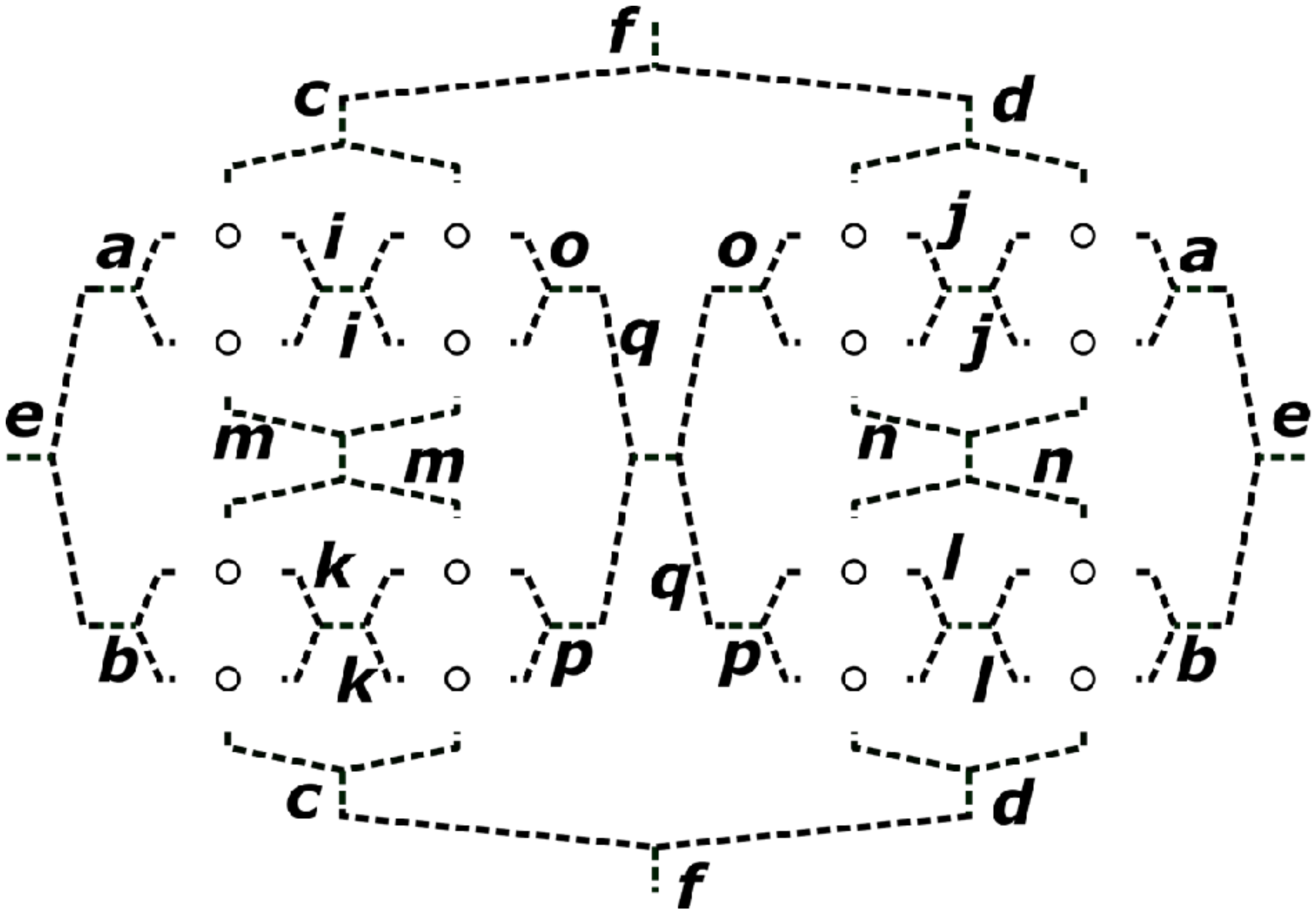




15 different projectors…

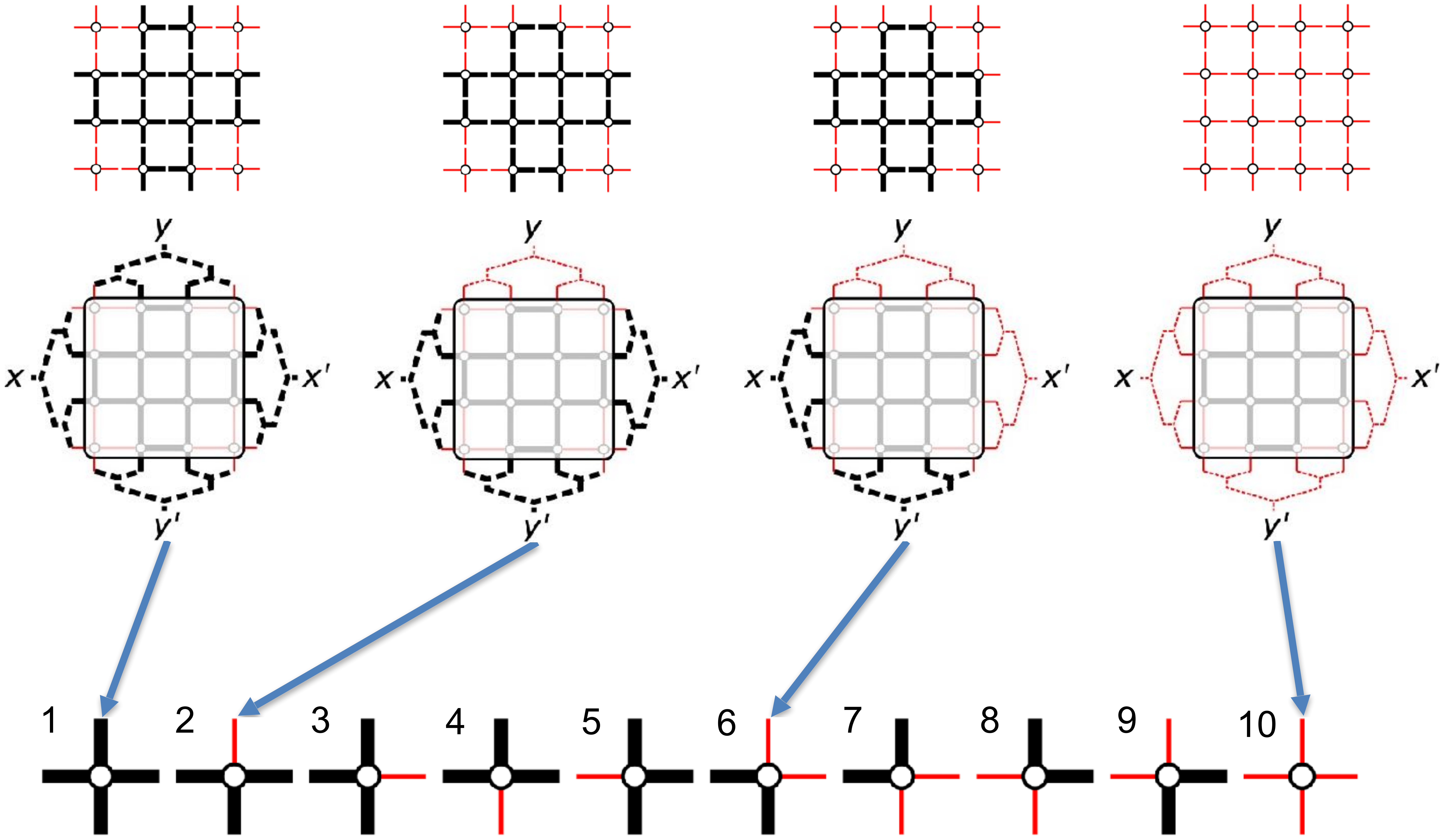
Inhomogeneous Projector Patterns





15 different projectors…

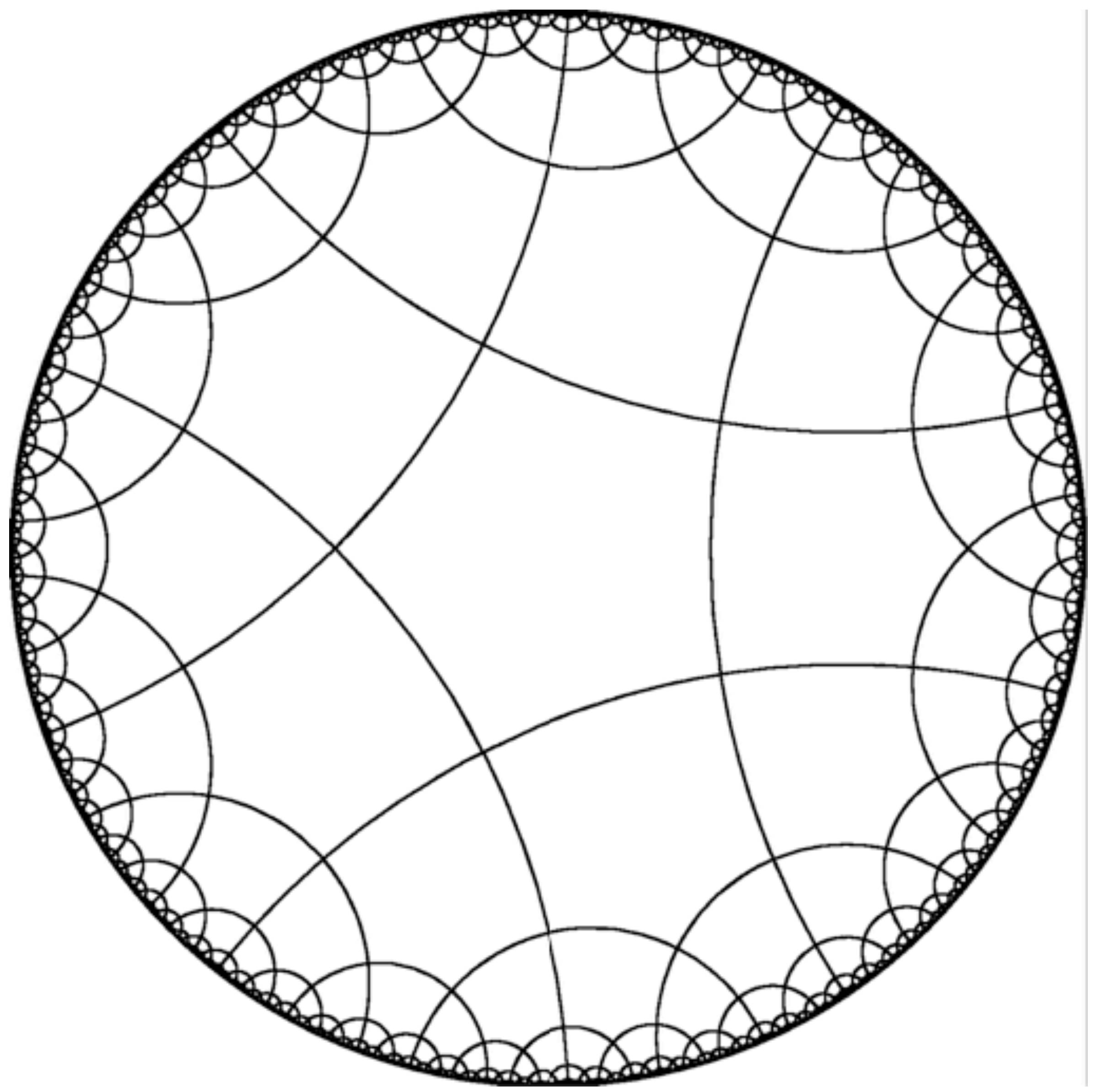
Projector Patterns: **External** Projectors



Part III

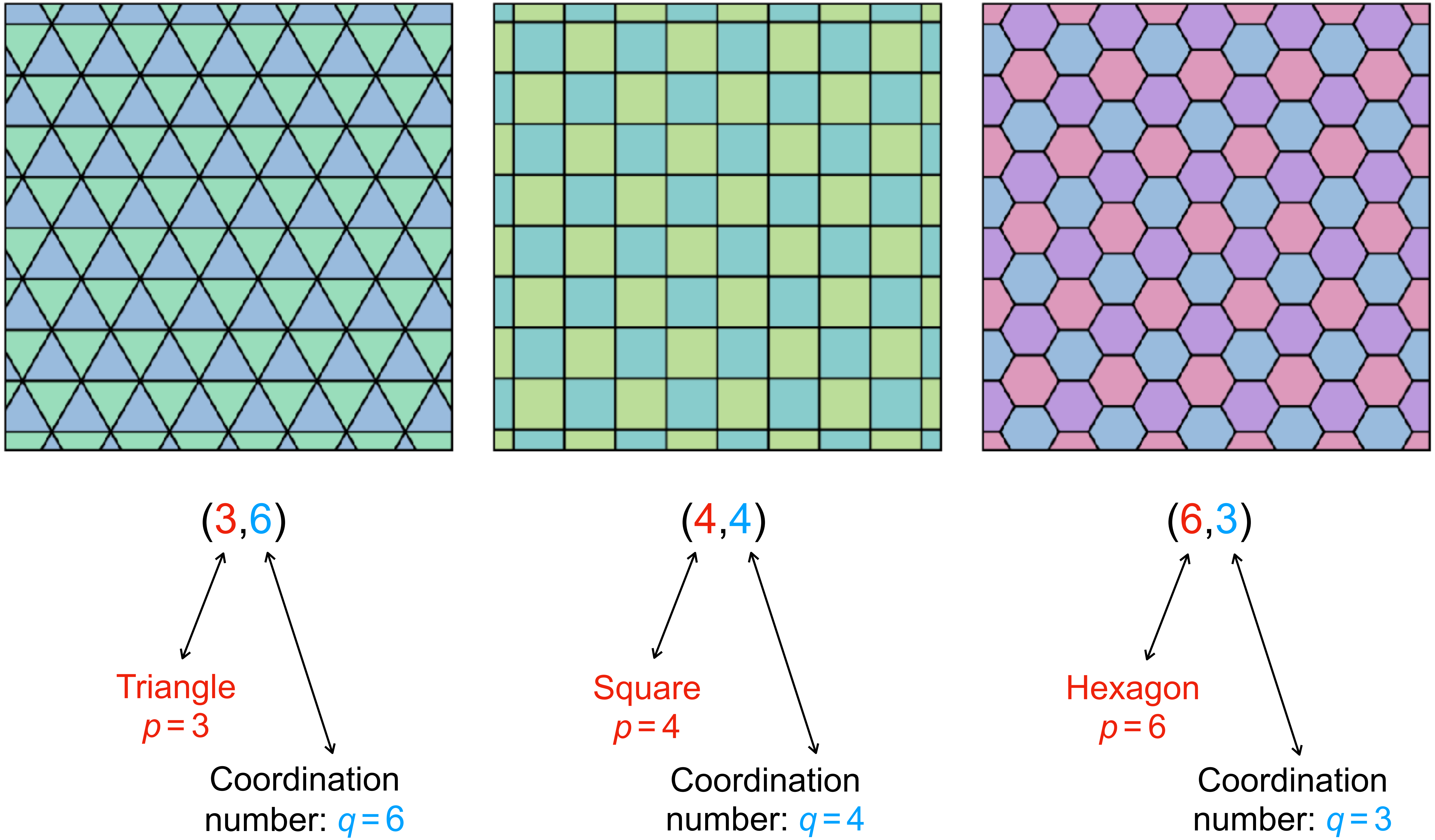
*Phase transitions on **hyperbolic** (anti-de Sitter) spaces*

1. Classification of the hyperbolic lattices
2. Analysis of classical/quantum spin models



Euclidean geometry: How to create a 2D plain using identical polygon tiling?

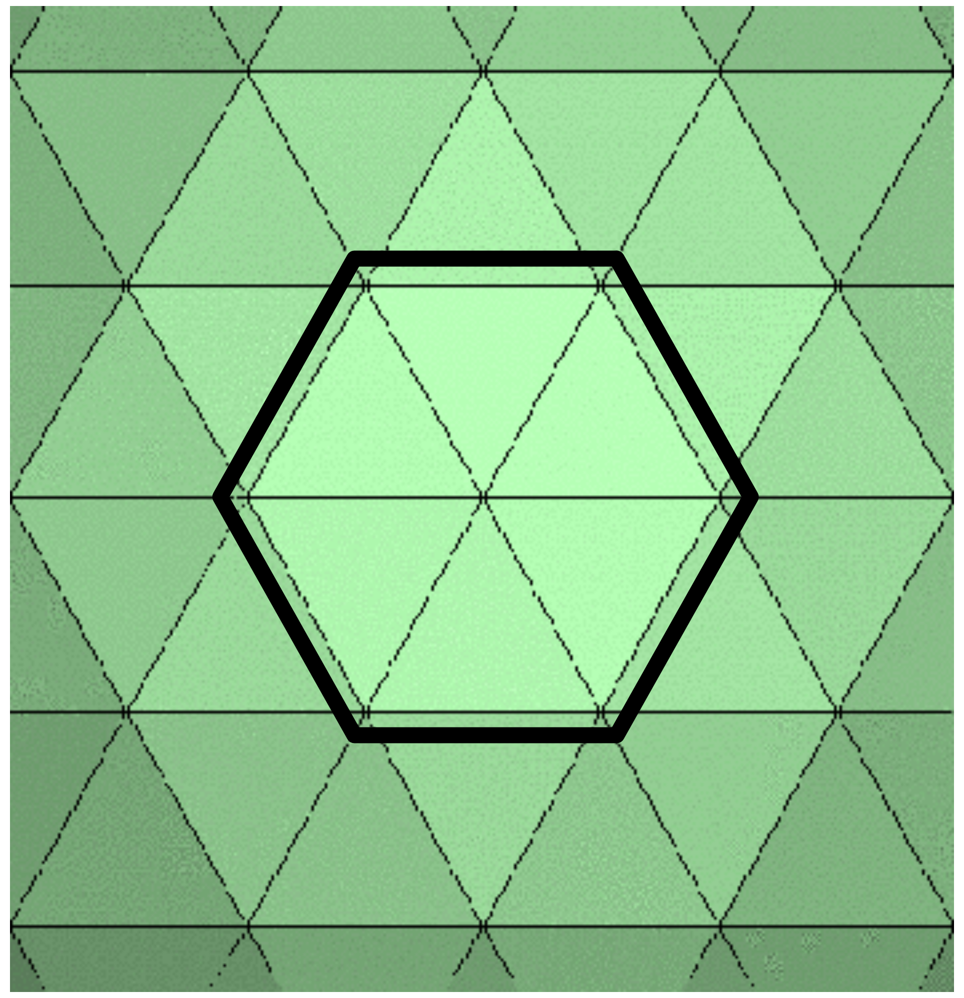
We consider a pair of two integers, known as Schläfli symbol (p, q)

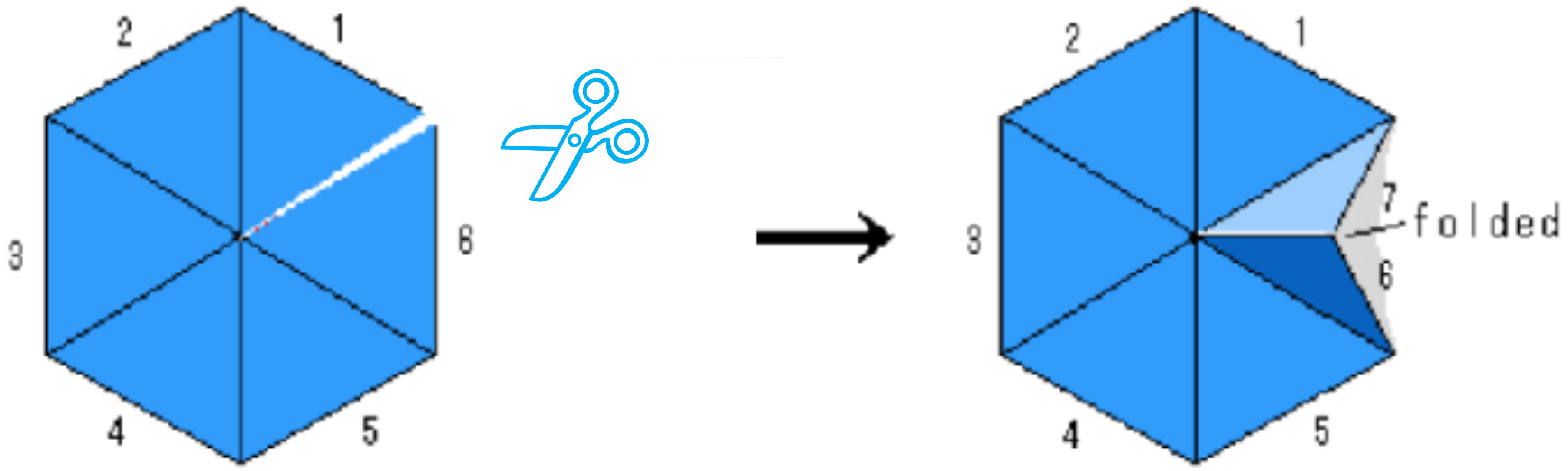


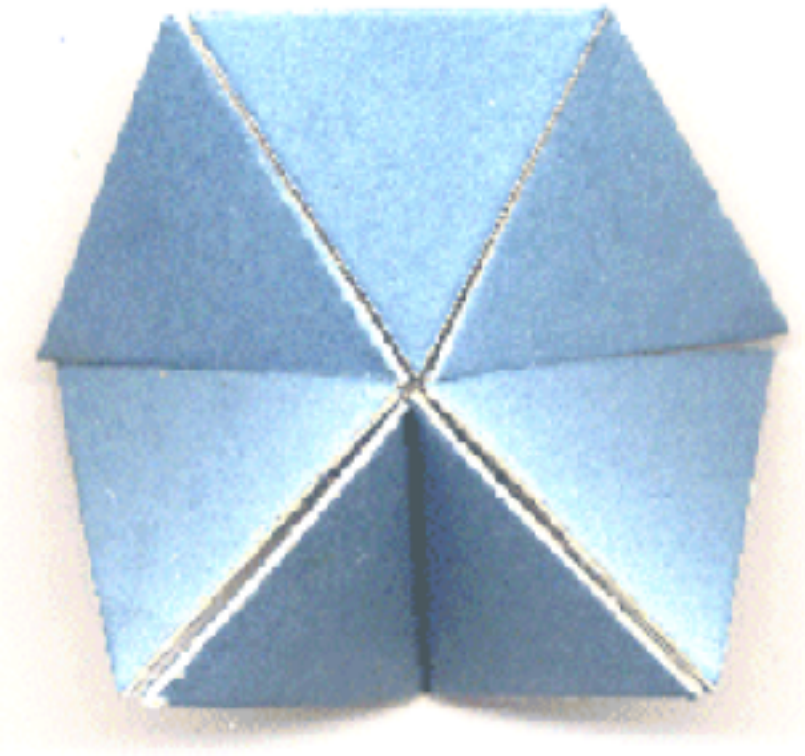
How to **imagine** and **create** a hyperbolic surface using, e.g., triangular tessellations?

Aim: To construct a curved triangular surface (3,7)?

(Triangles, coord. #) = (3,6)

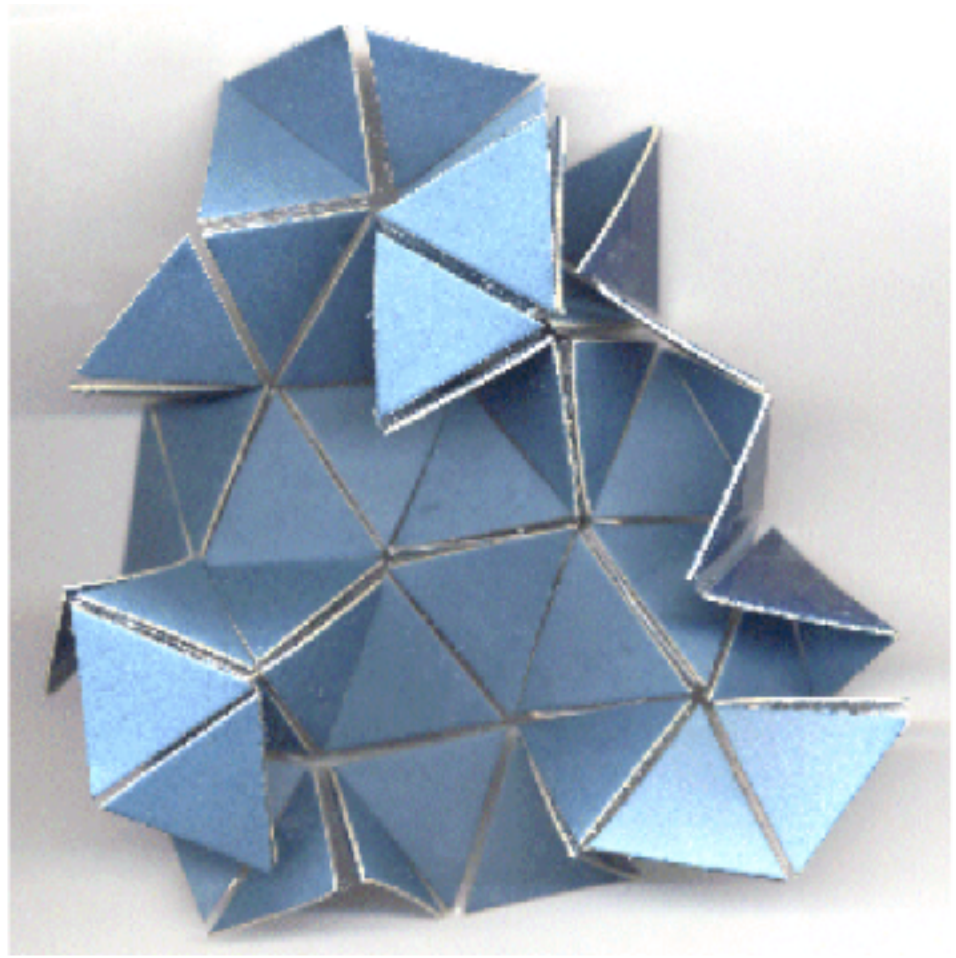


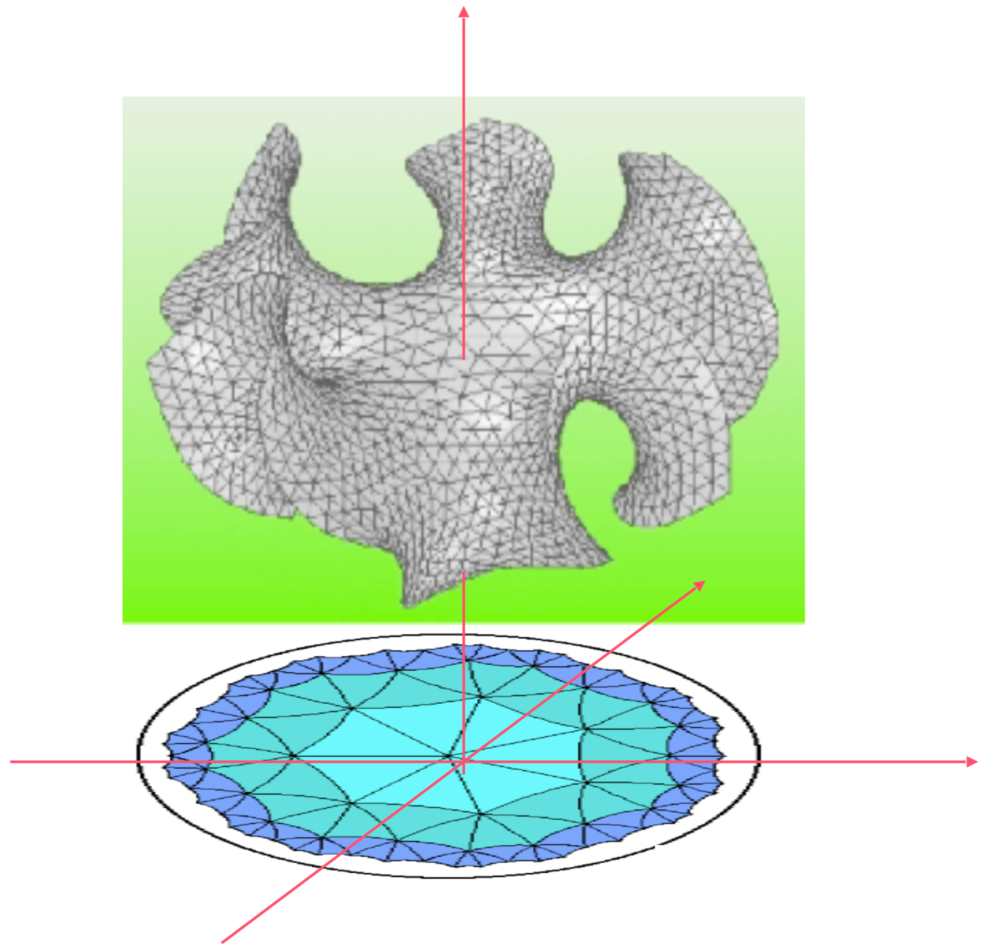


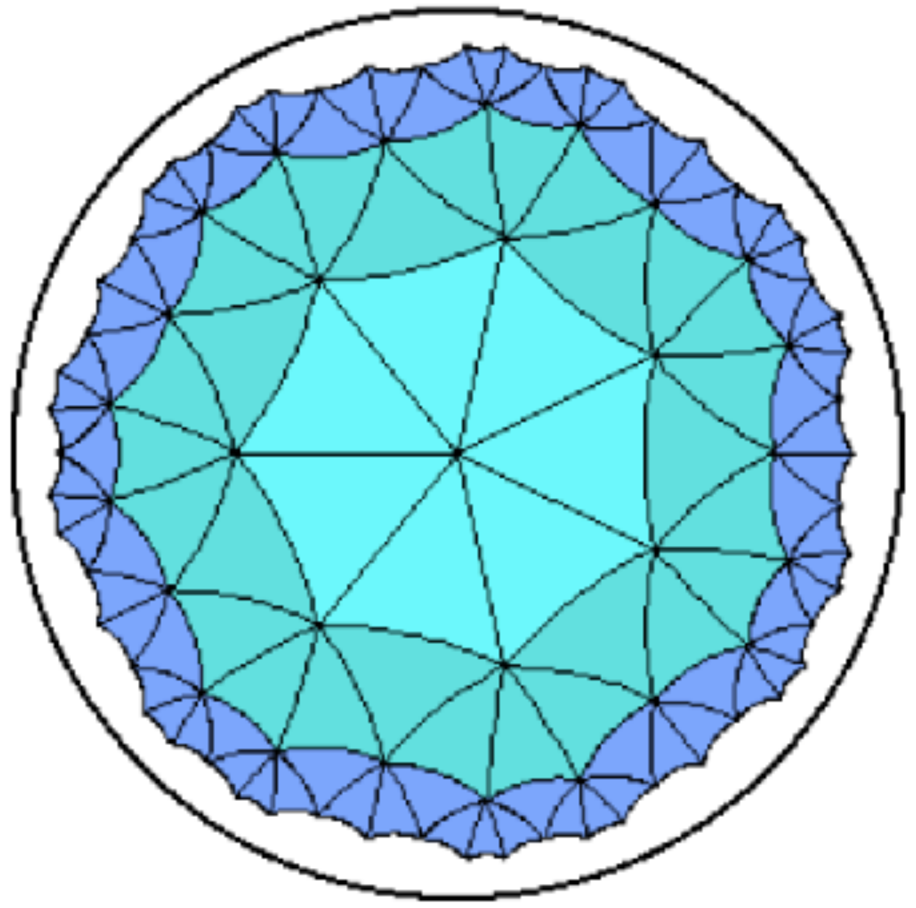


Poincaré disc
is a mapping of curved surfaces onto a unit circle

(Triangles, coord. #) = (3,7)

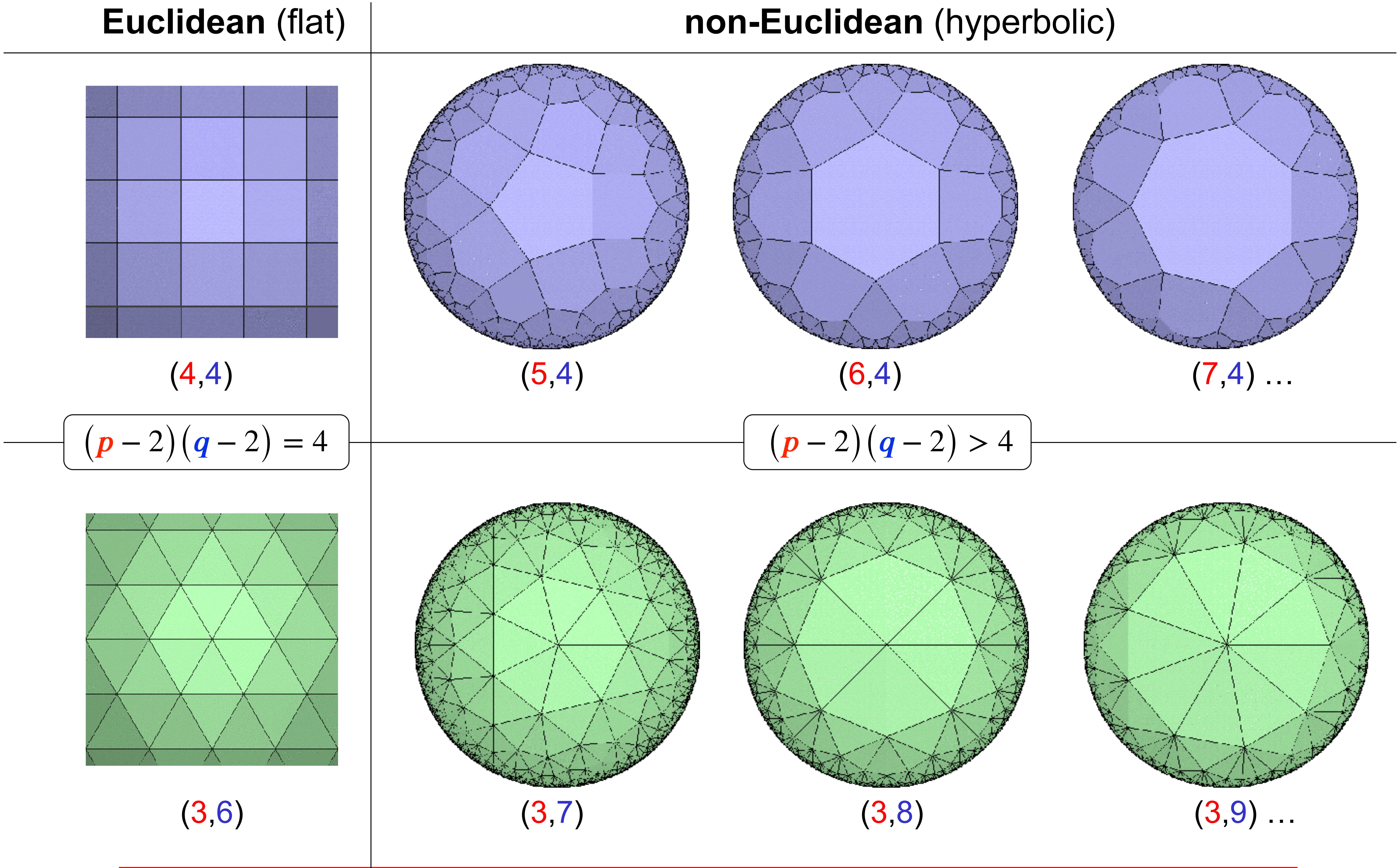






Poincaré disc
for (3,7)

Schläfli symbol (p,q) denotes a regular p-gonal tiling with the coordination number q



The Hausdorff dimension d_H of all the hyperbolic lattices is ***infinite***!

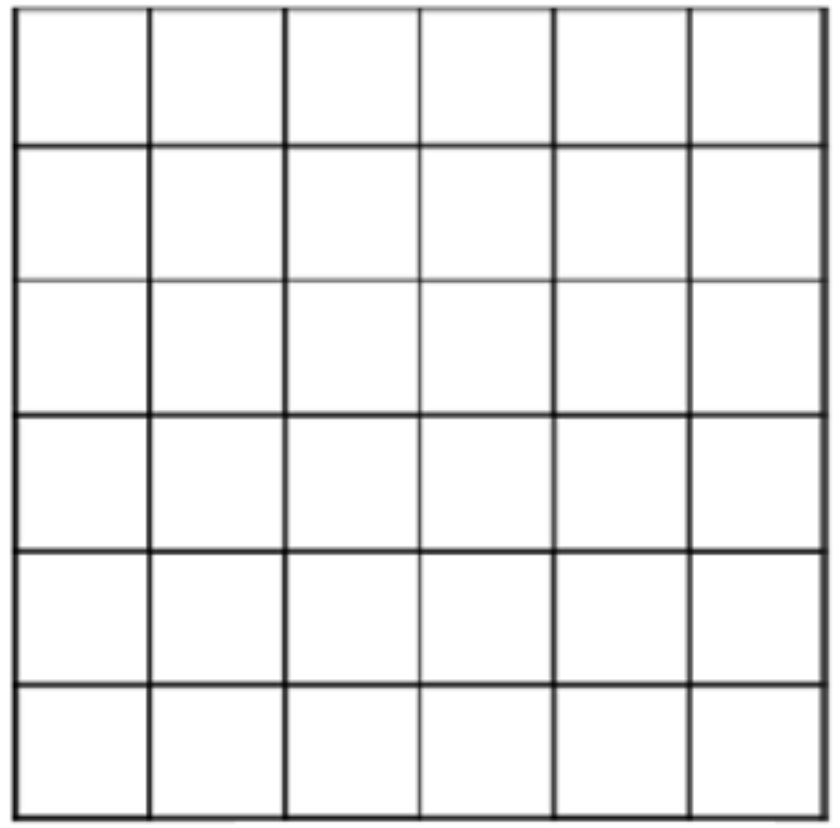
Classical spins on hyperbolic surfaces

Details of the algorithms are skipped.

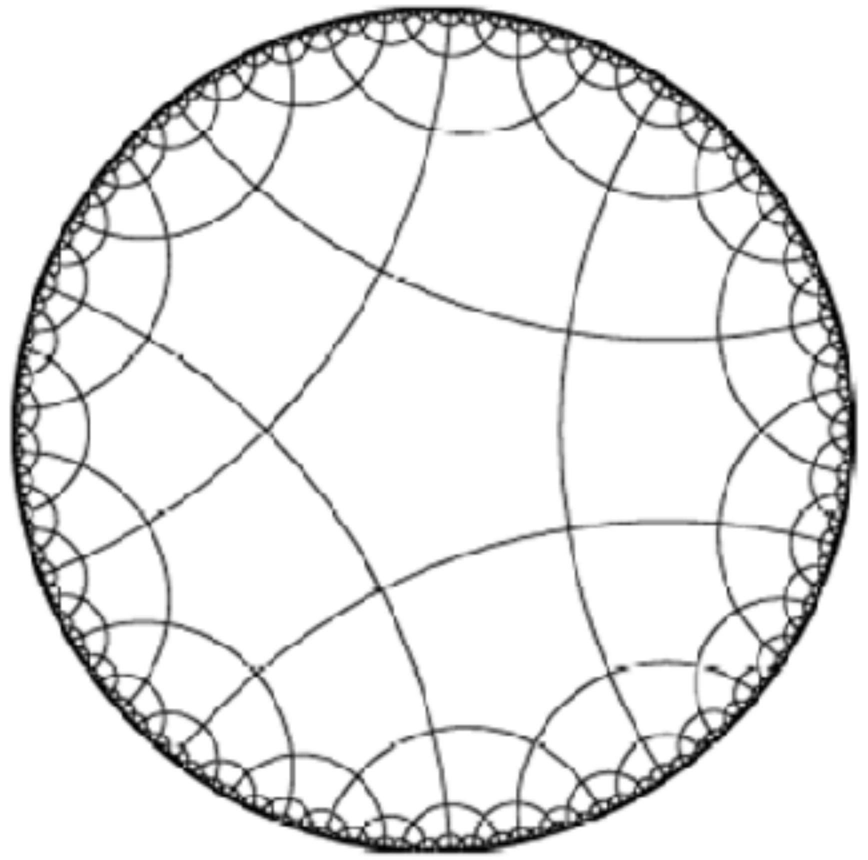
How to assure that our Tensor Network is correct?

Let us check the phase transitions toward the Bethe lattice $(p \to \infty, 4)$:

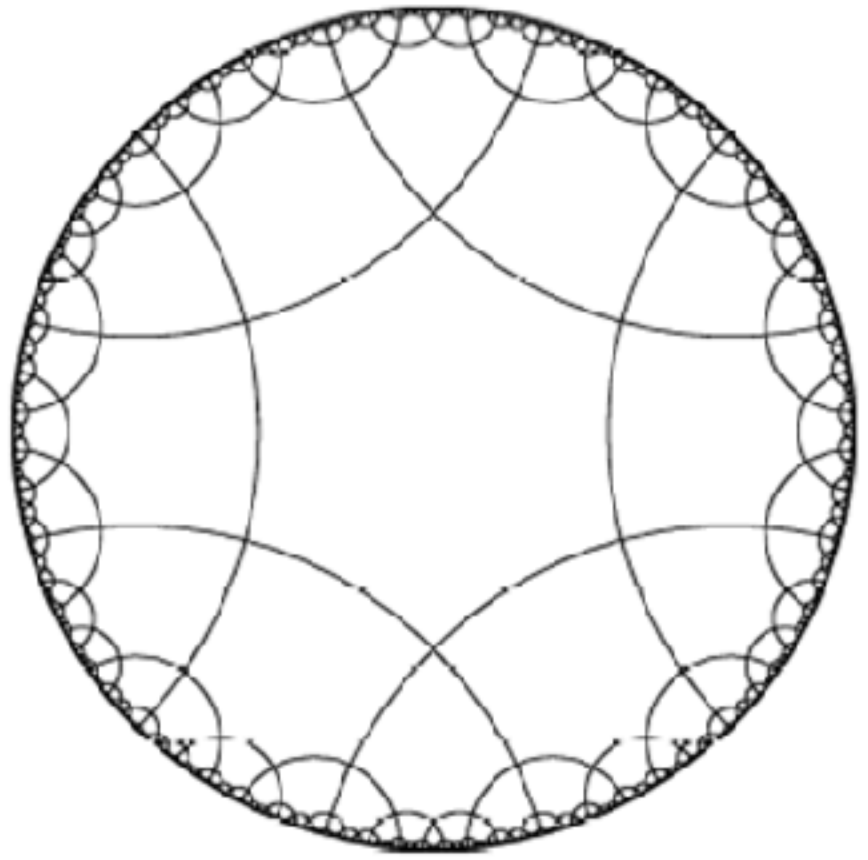
- We know the exact solutions on square (4, 4) & Bethe (∞, 4) lattices for the classical Ising model



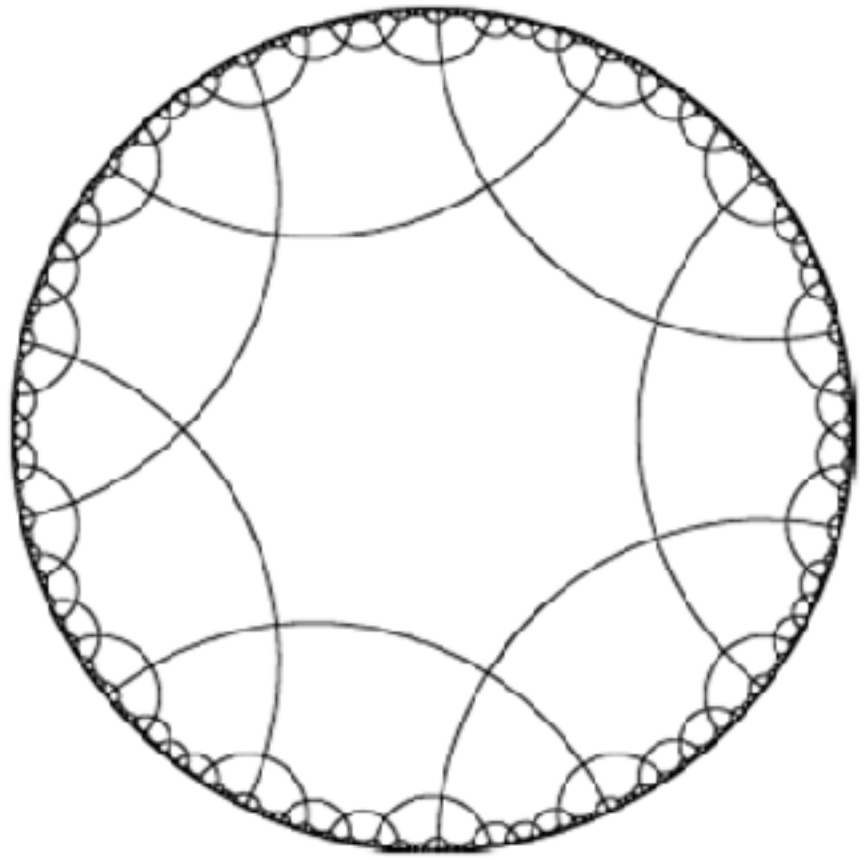
(4, 4) ✔



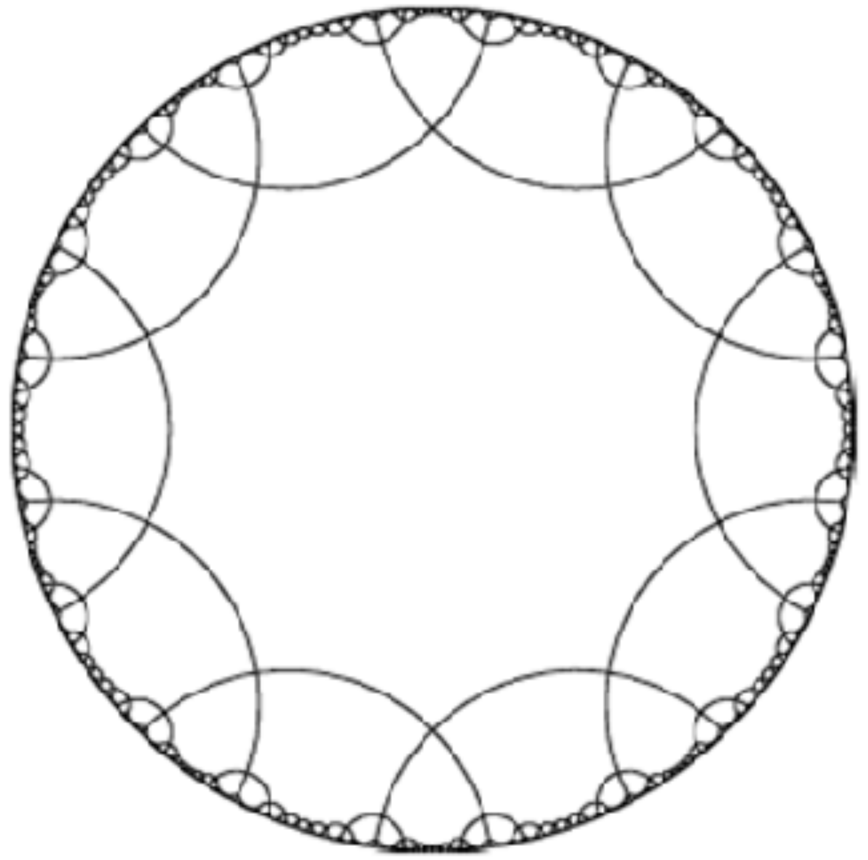
(5, 4)



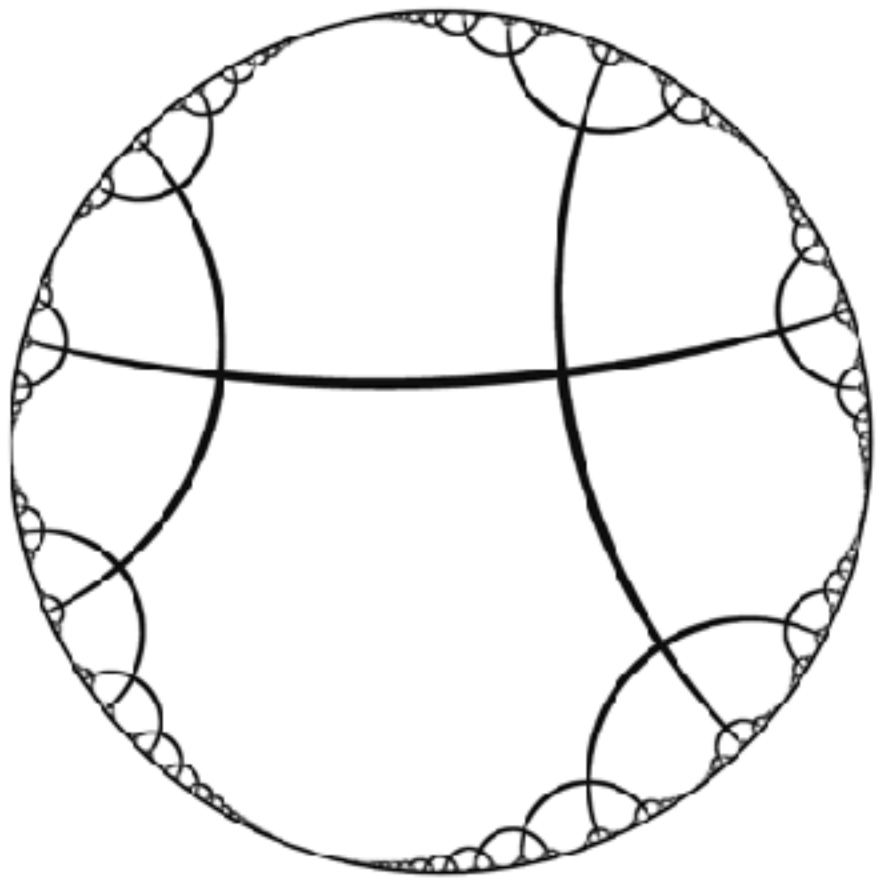
(6, 4)



(7, 4)



(10, 4)

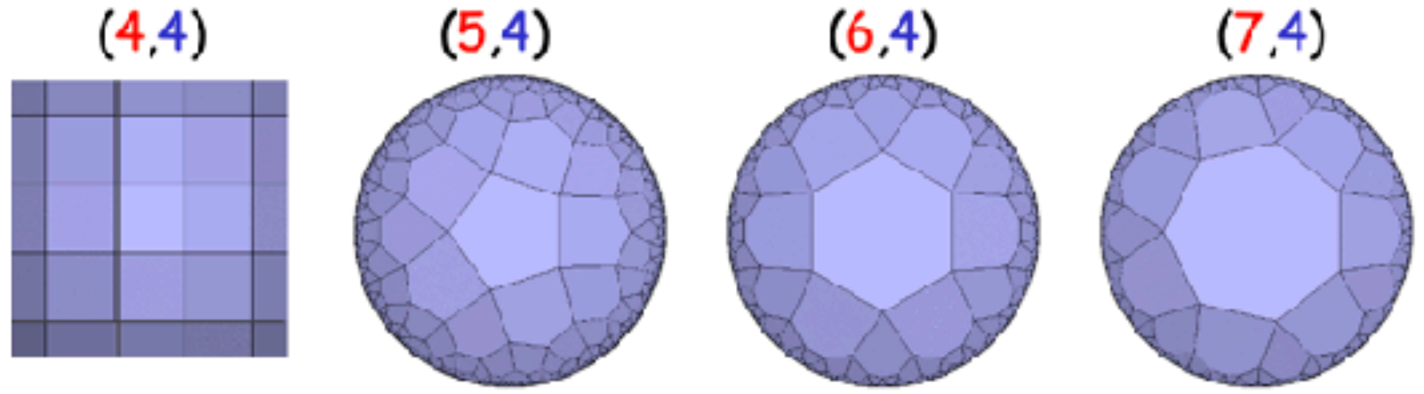


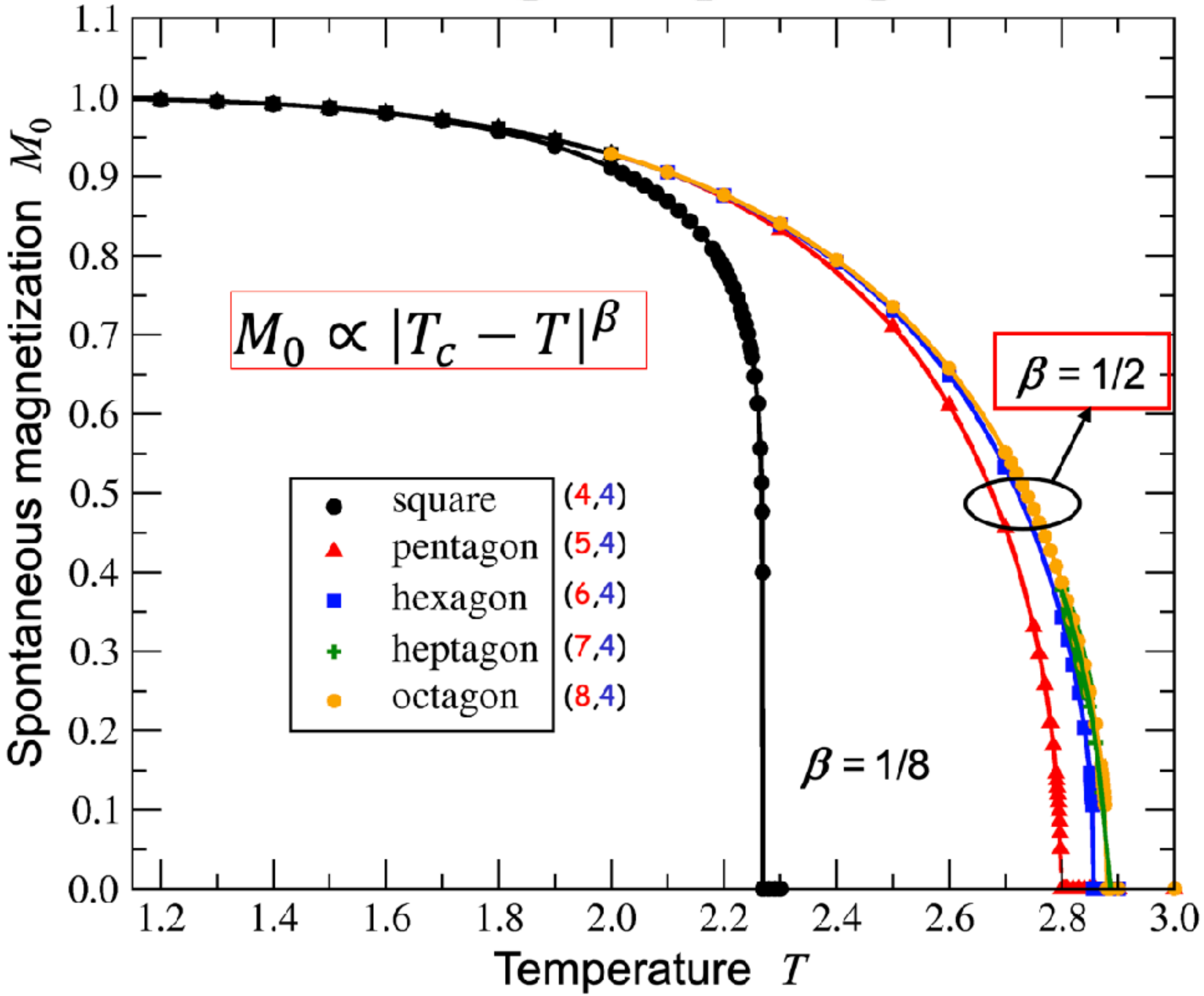
(∞, 4) ✔

Magnetization for various *p*-gons at *q* = 4

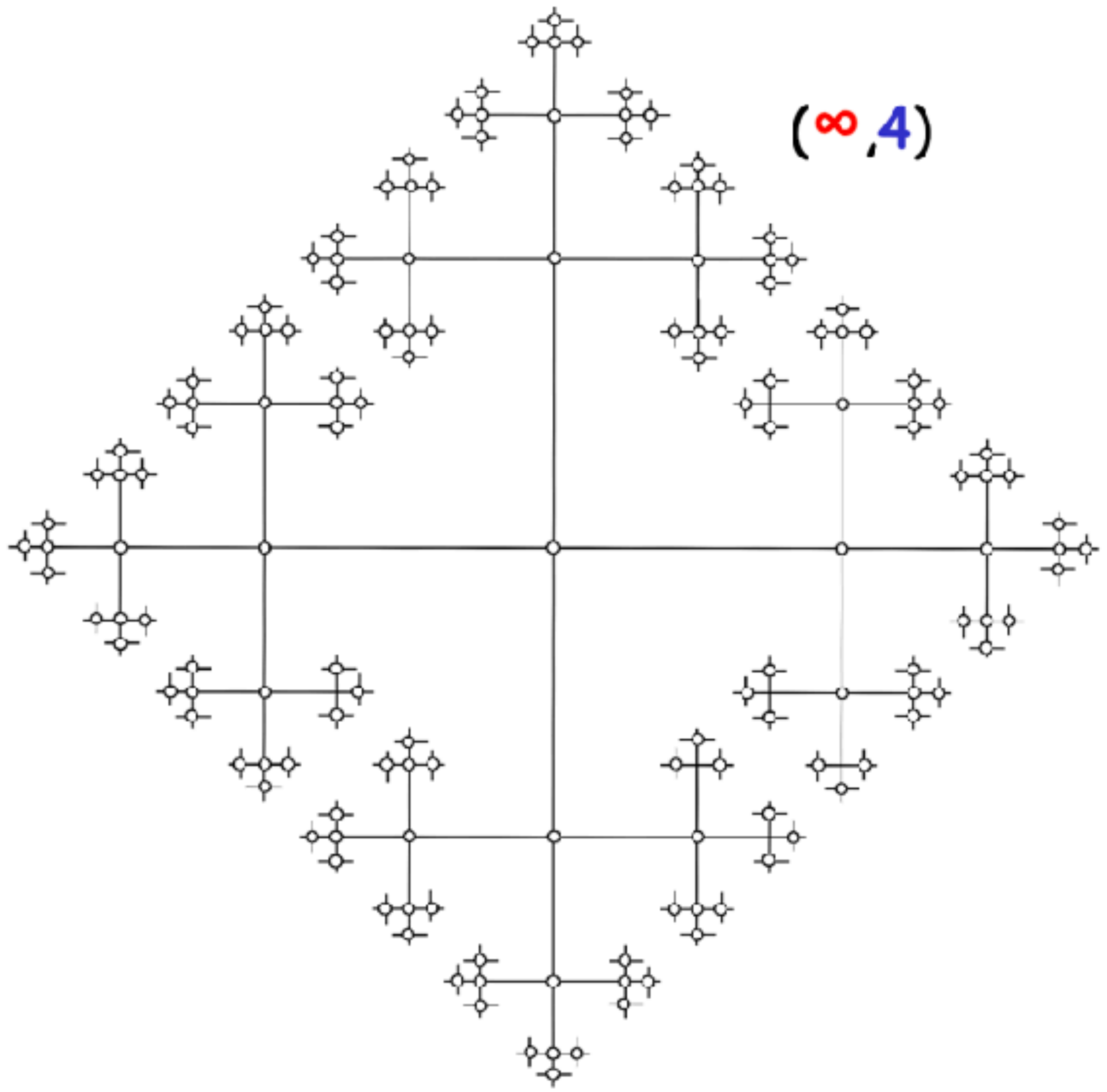
(Classical spin systems on hyperbolic geometry exhibit mean-field universality)

CTMRG study

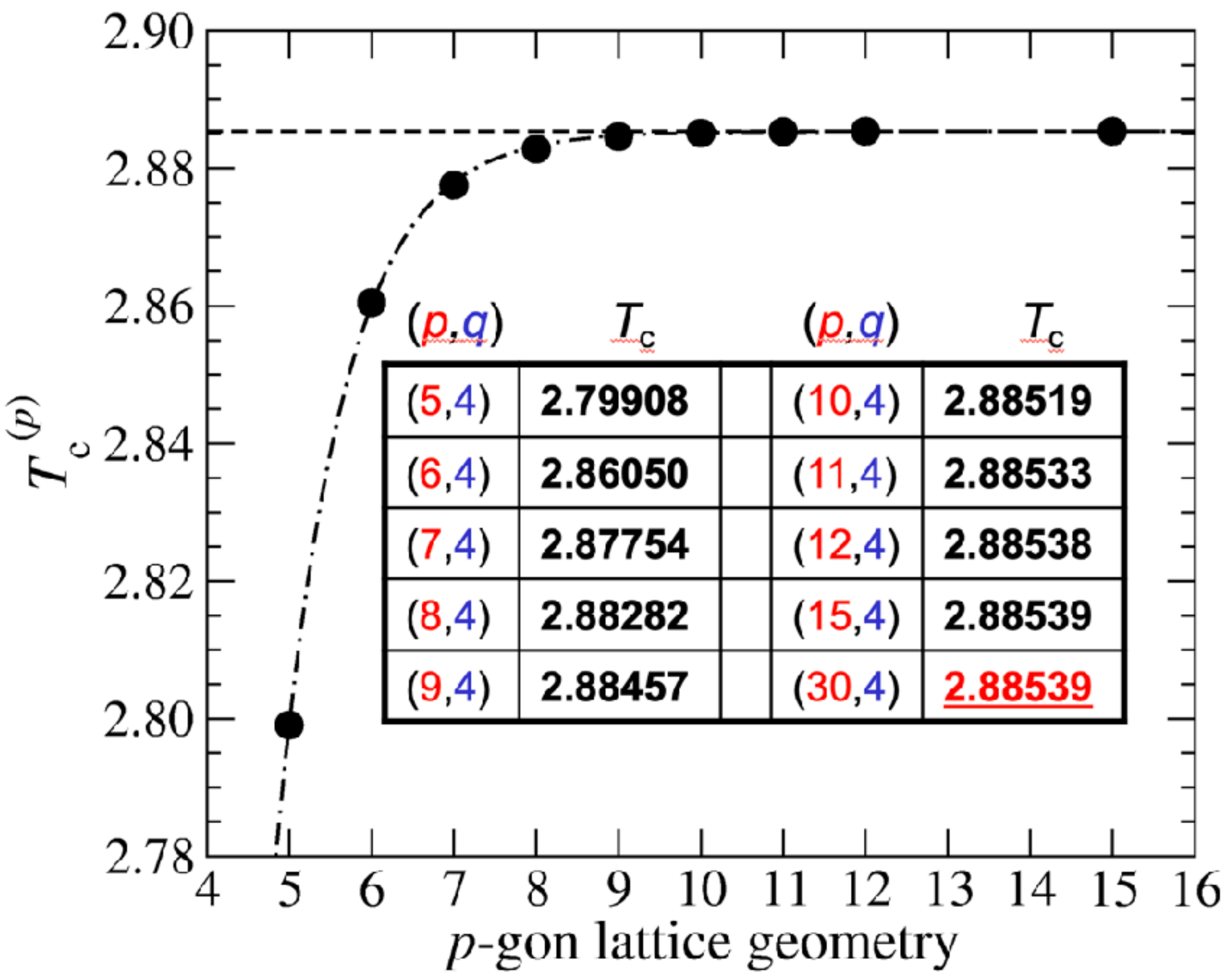




The classical Ising model on (p,4) lattices by CTMRG



Phase transition: T_c versus p



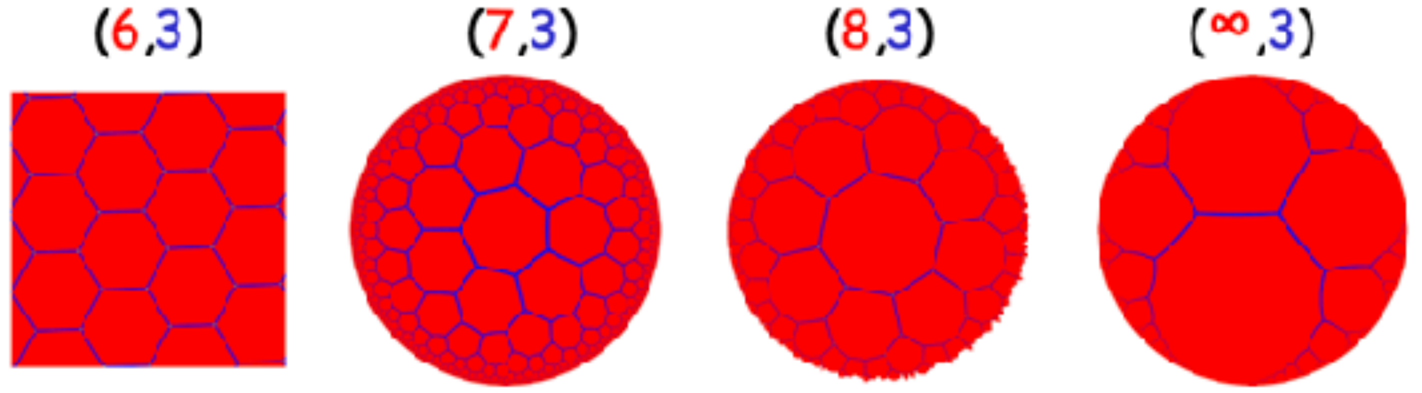
(p,q)	T_c	(p,q)	T_c
(5,4)	**2.79908**	(10,4)	**2.88519**
(6,4)	**2.86050**	(11,4)	**2.88533**
(7,4)	**2.87754**	(12,4)	**2.88538**
(8,4)	**2.88282**	(15,4)	**2.88539**
(9,4)	**2.88457**	(30,4)	**2.88539**

The **classical** Ising model on the **Bethe lattice (∞,4)** is exactly solvable with the critical temperature $T_c = 2 / \ln 2 = \mathbf{2.88539008\ldots}$

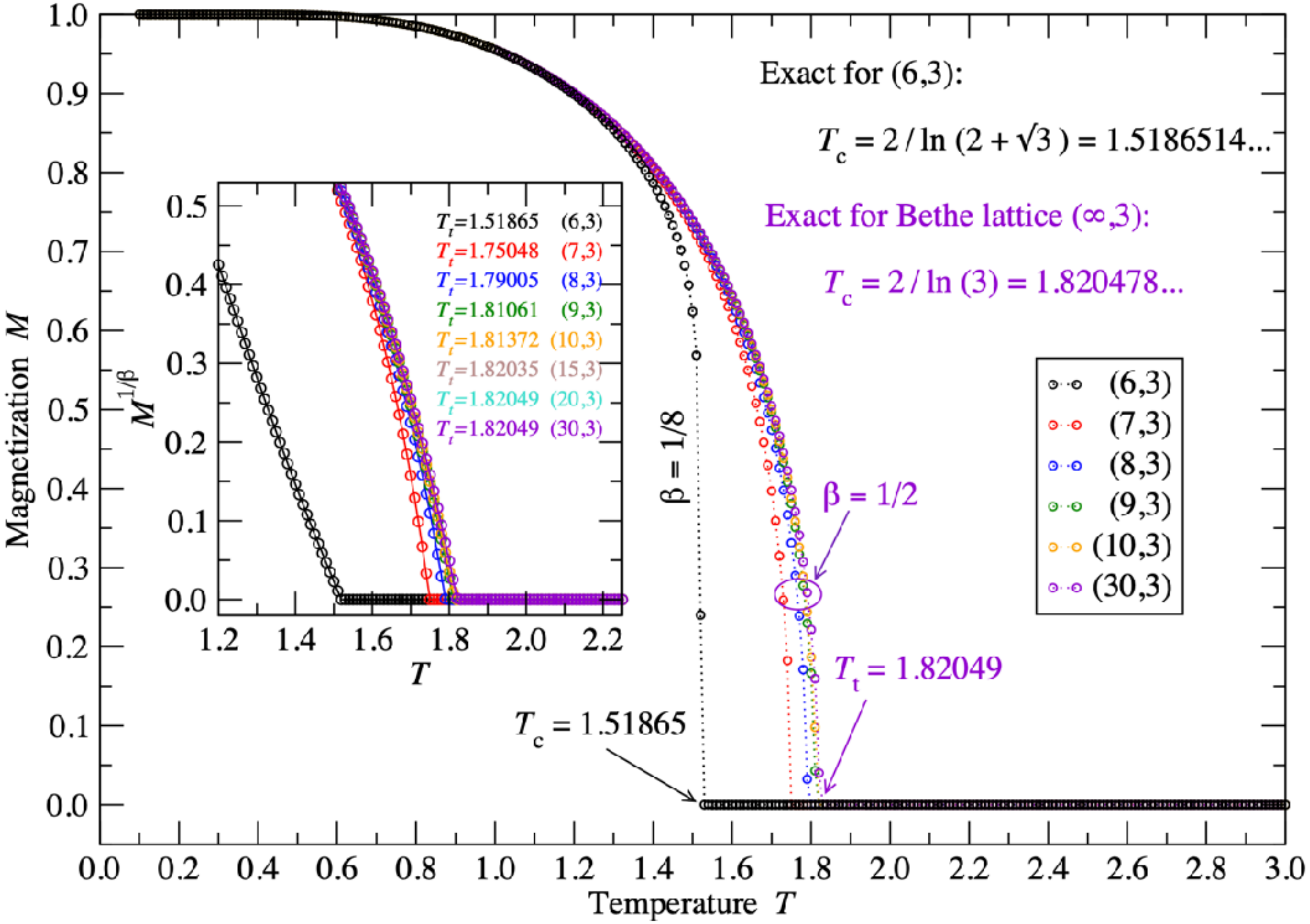
Magnetization for various p-gons at q=3

(mean-field universality)

CTMRG study



Ising model on $(p, 3)$ lattice geometries by CTMRG with $m = 20$



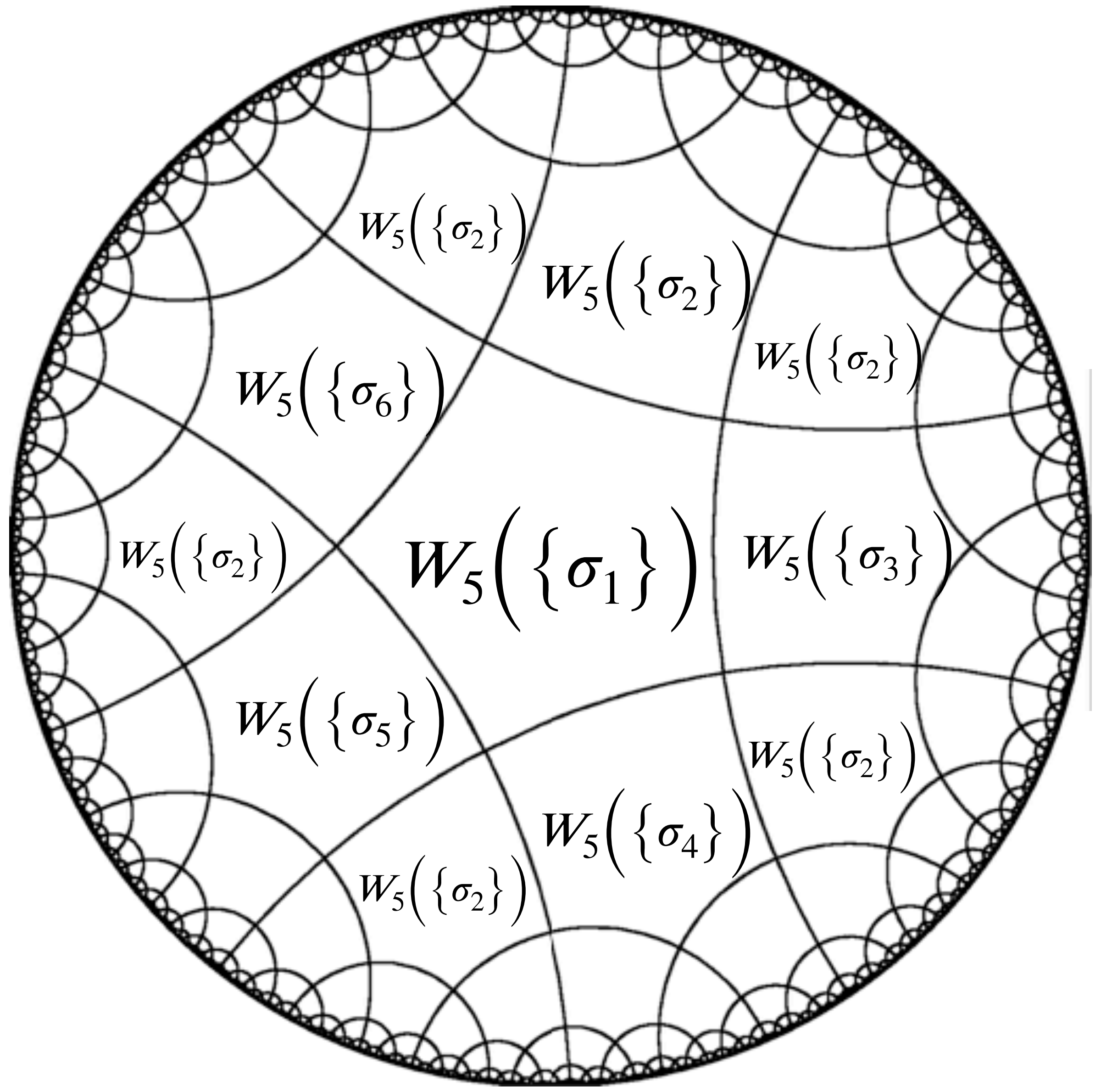
Extension to quantum systems

Details of the algorithms are skipped.

Tensor-Network algorithm

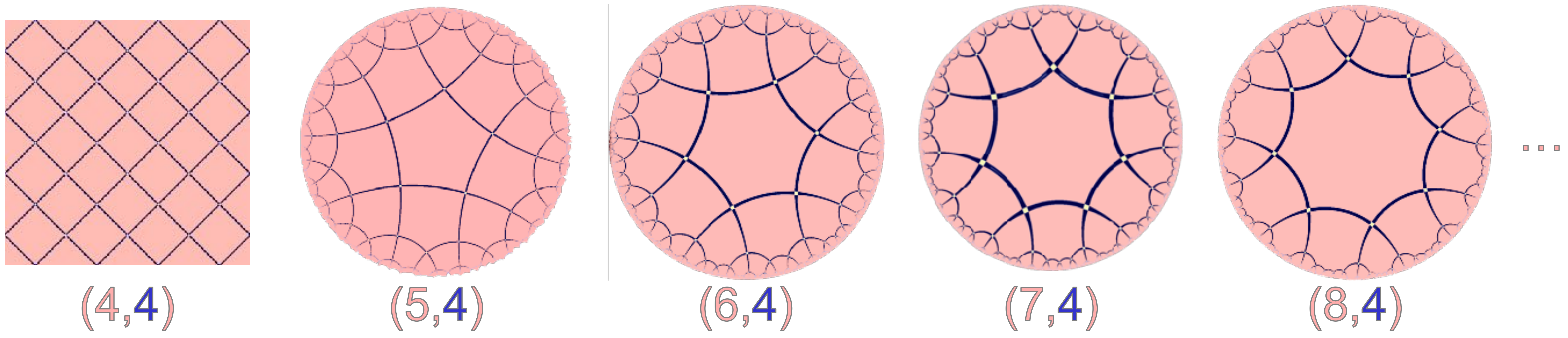
Tensor Product Variational Approximation (TPVA)

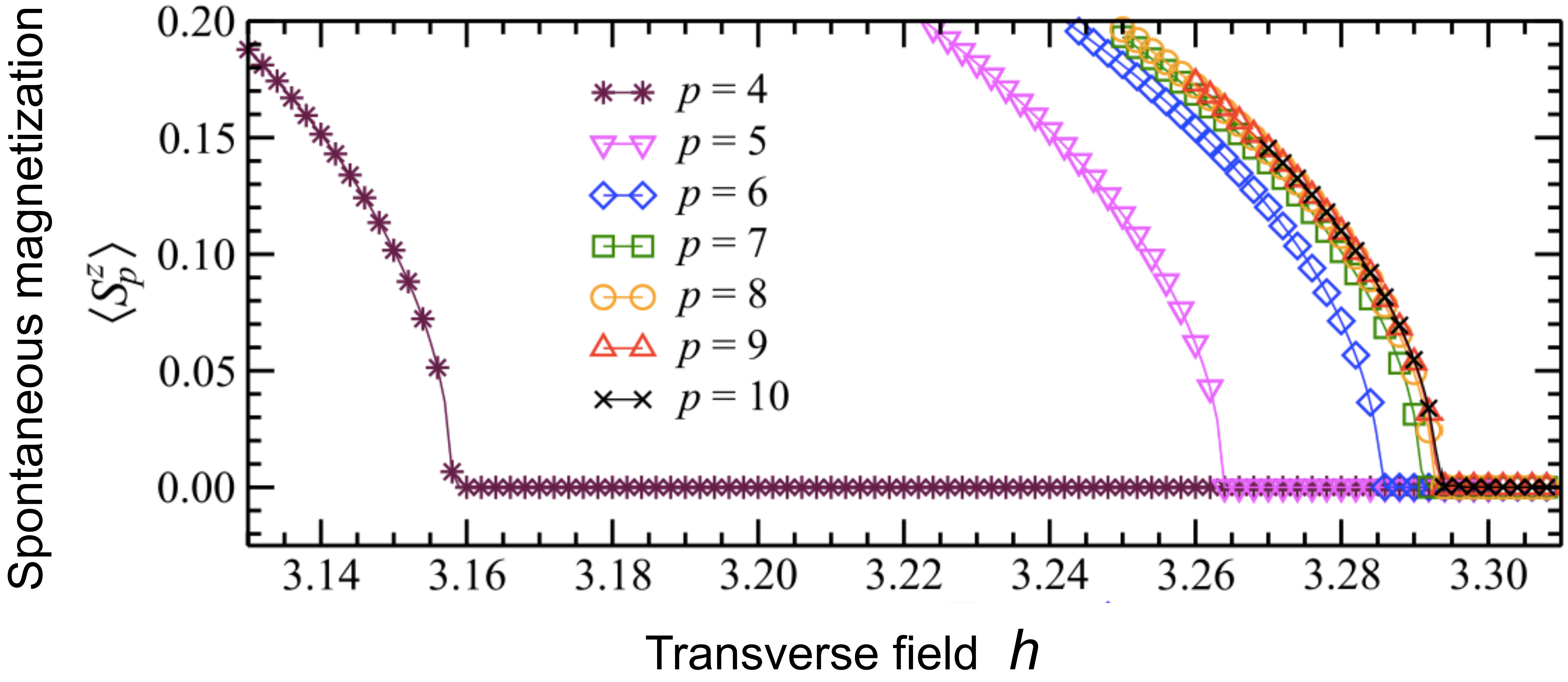
$$|\Psi_p\rangle = \lim_{N\to\infty} \sum_{\sigma_1\sigma_2\cdots\sigma_N} \prod_{\langle k\rangle_p} W_p(\{\sigma_k\})|\sigma_1\sigma_2\cdots\sigma_N\rangle$$



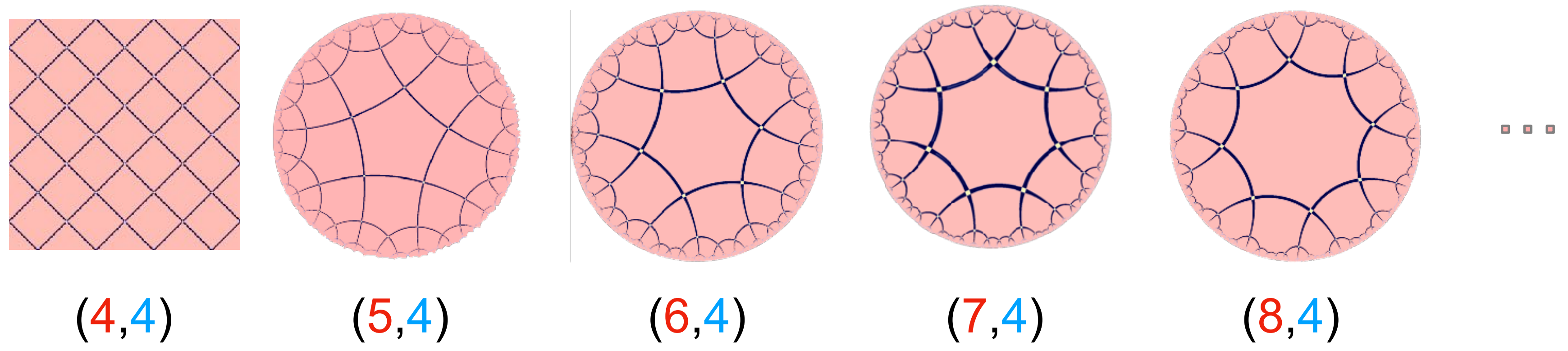
M. Daniška and A. Gendiar, *J. Phys. A: Math. Theor.* **48**, 435002 (2015); **49**, 145003 (2016)

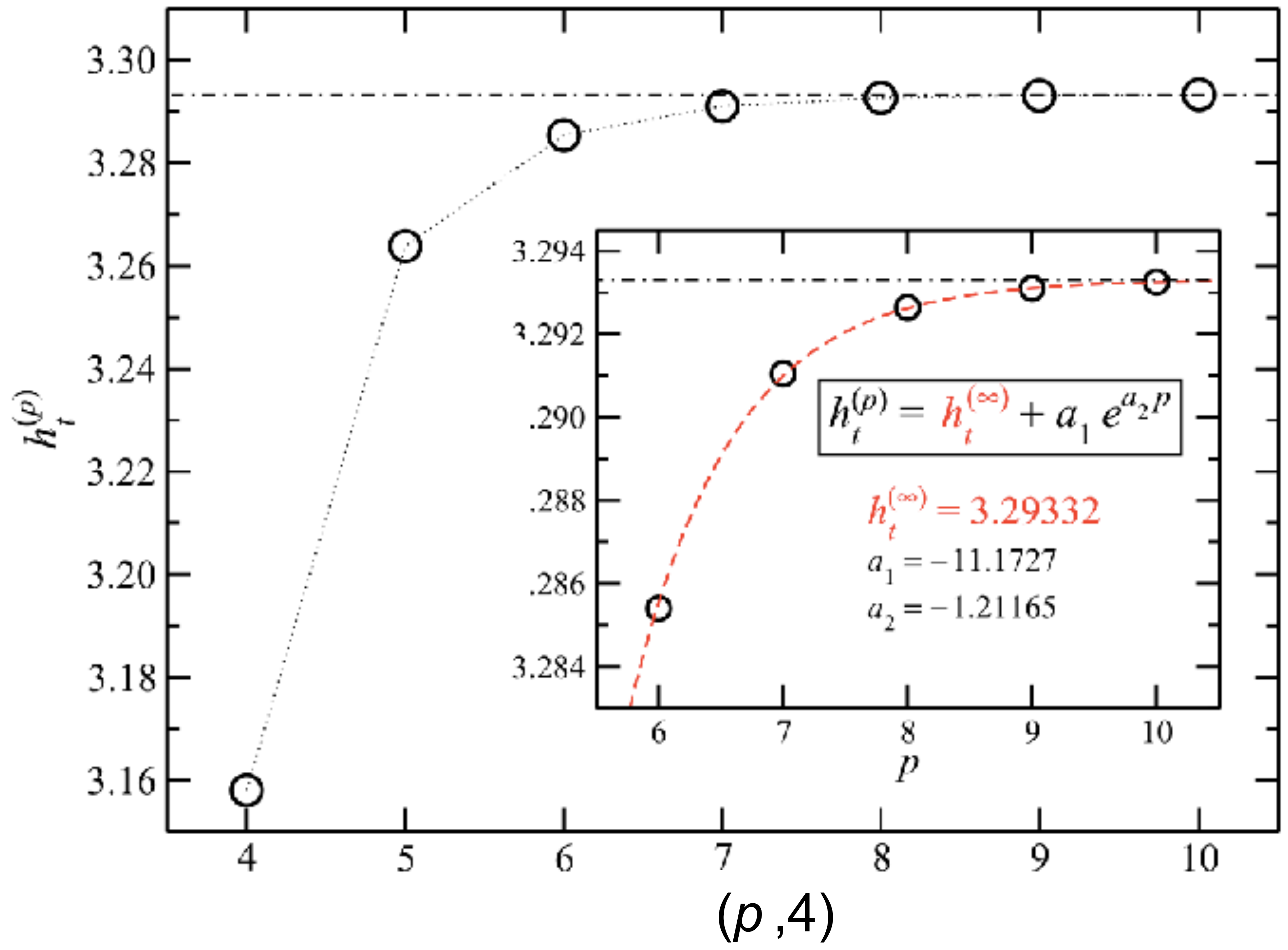
Transverse-field **Ising** model: **Quantum** phase transitions on (*p*,4)

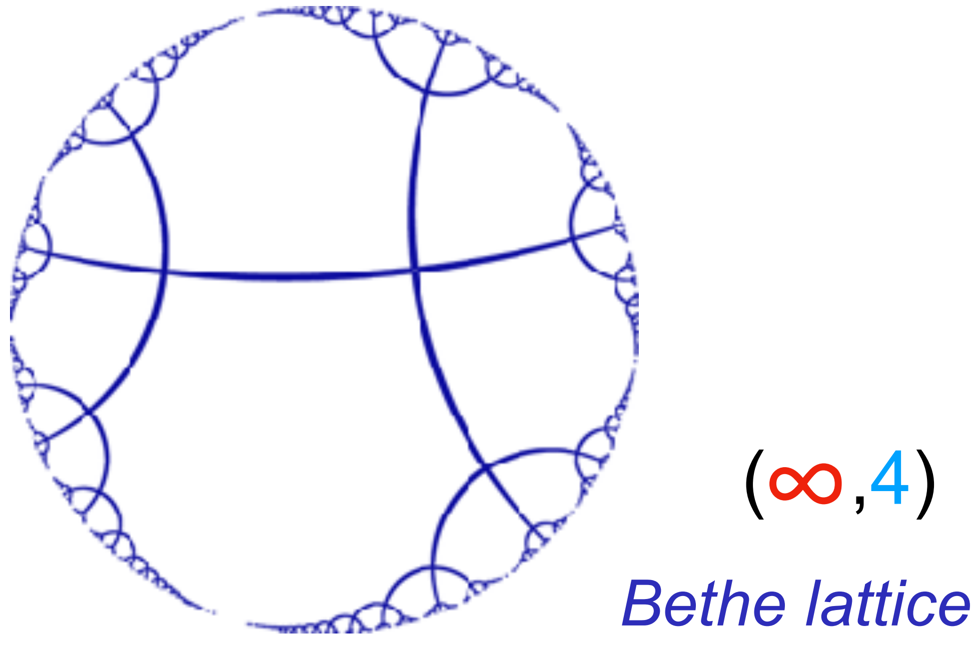




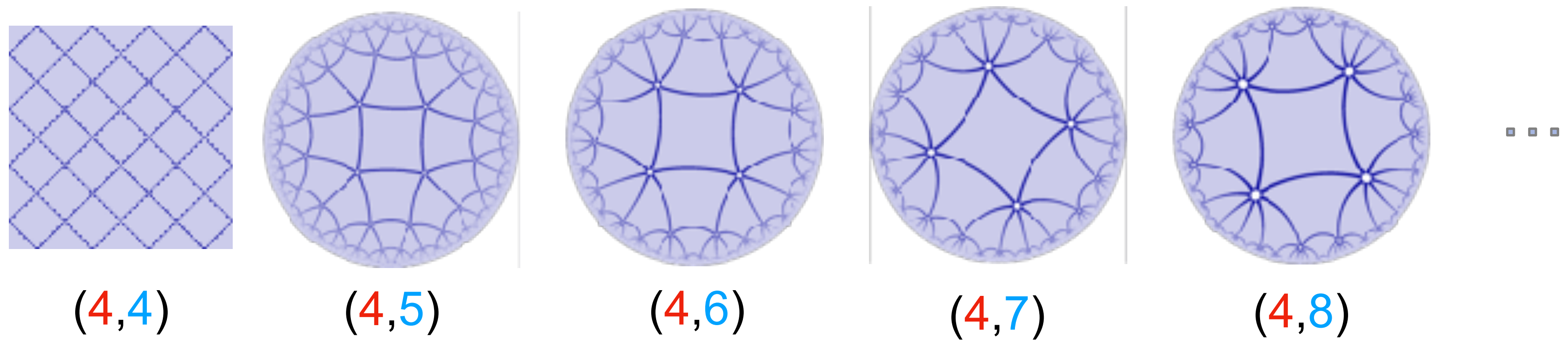
Transverse-field **Ising model** on (*p*,4)

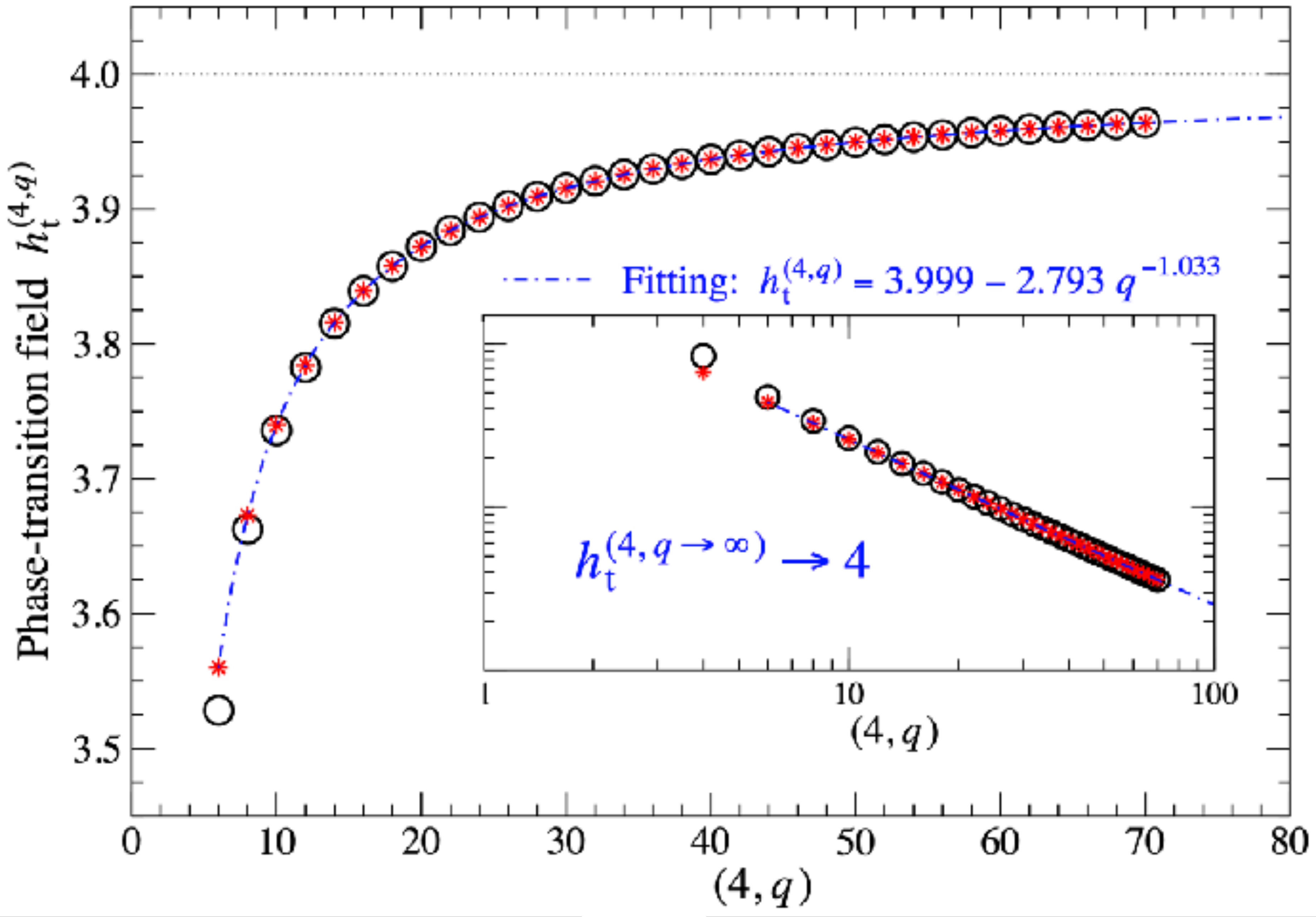


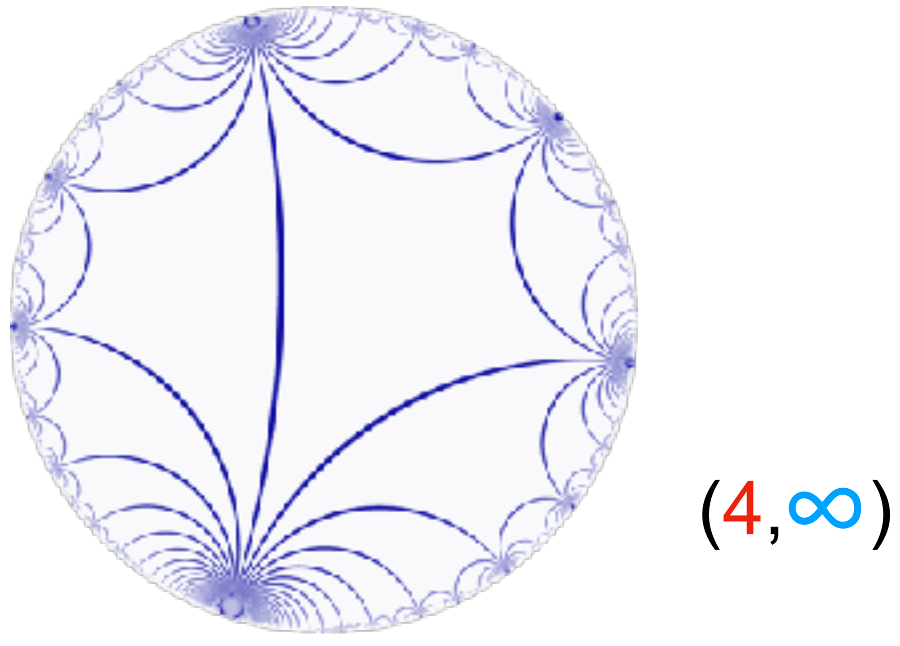




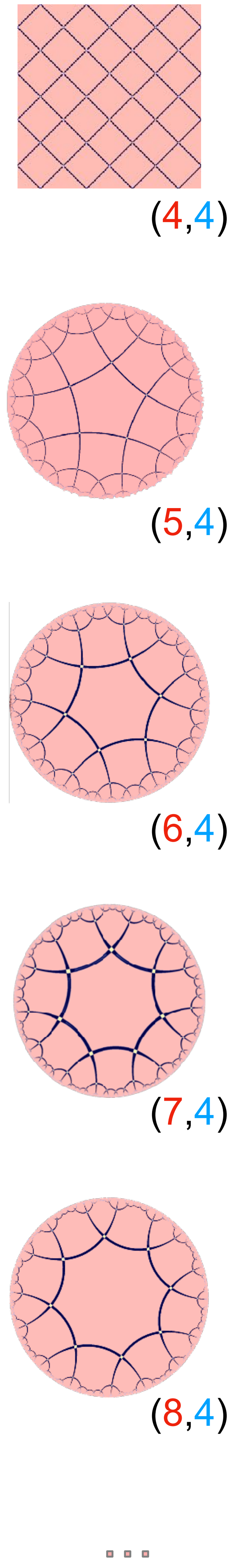
Transverse-field **Ising model** on (4,*q*)



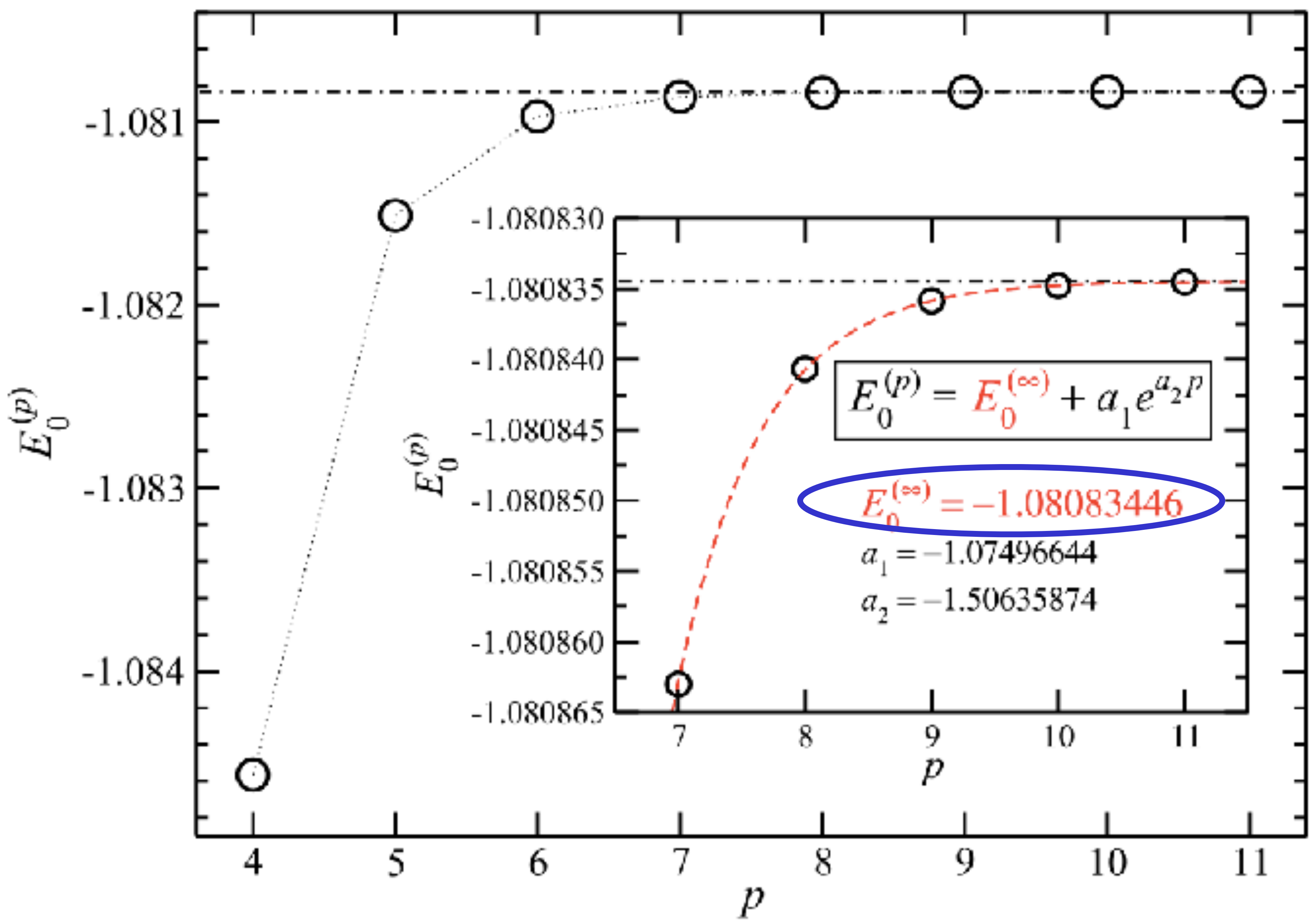




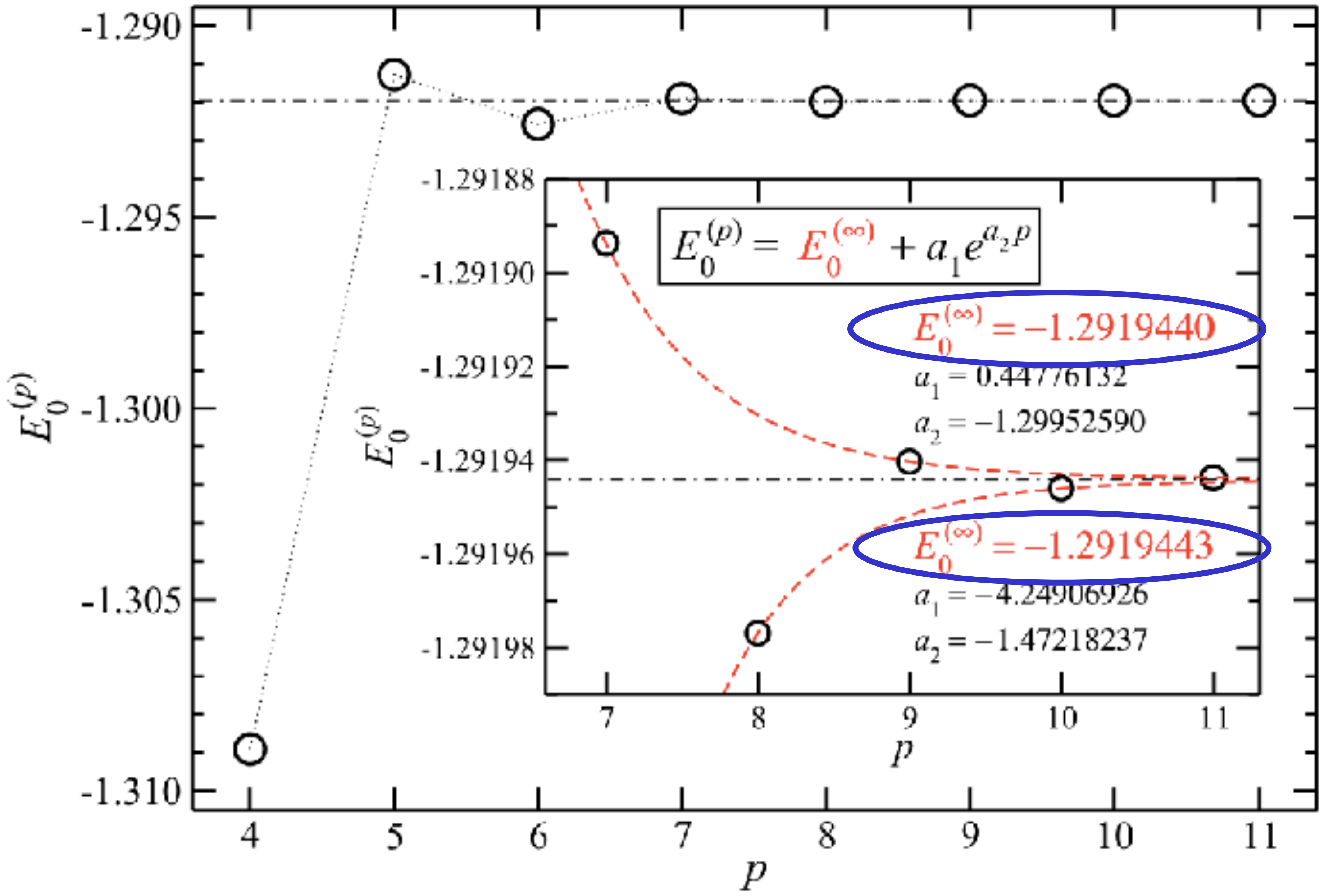
The polygon (p-) scaling in **XY** *and* **Heisenberg** *models on (p,4) lattices*

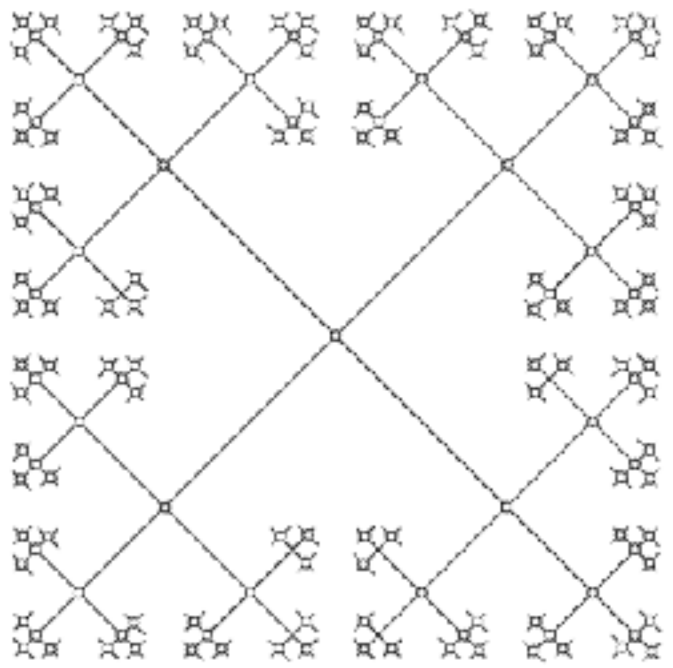


Ground-state energy of **XY** model on (***p***,**4**)



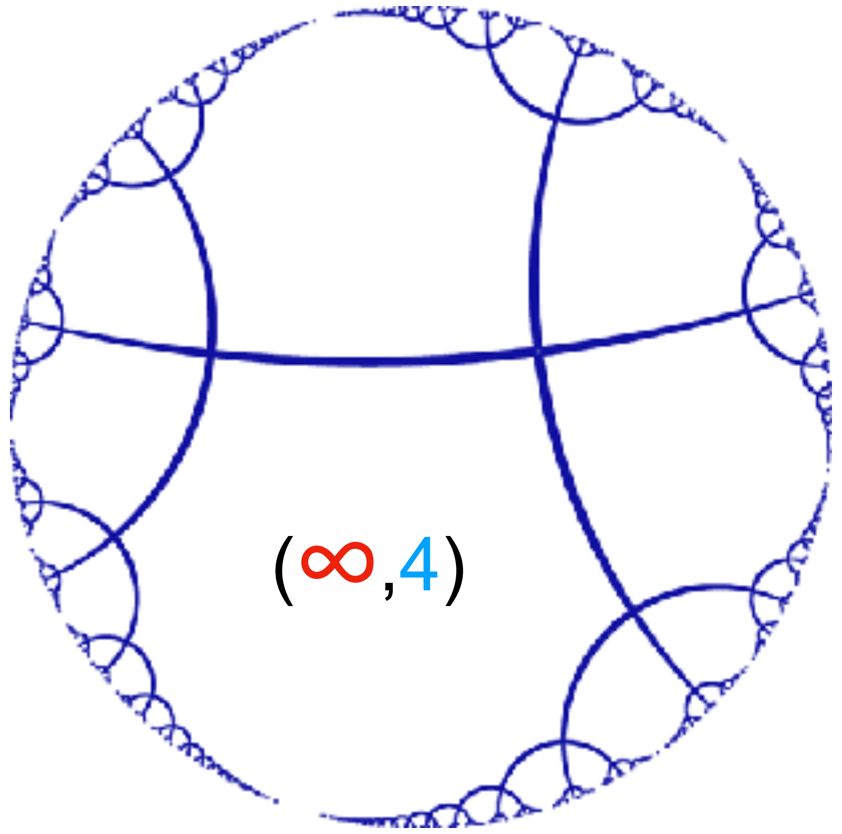
Ground-state energy of **Heisenberg** model on (***p***,**4**)





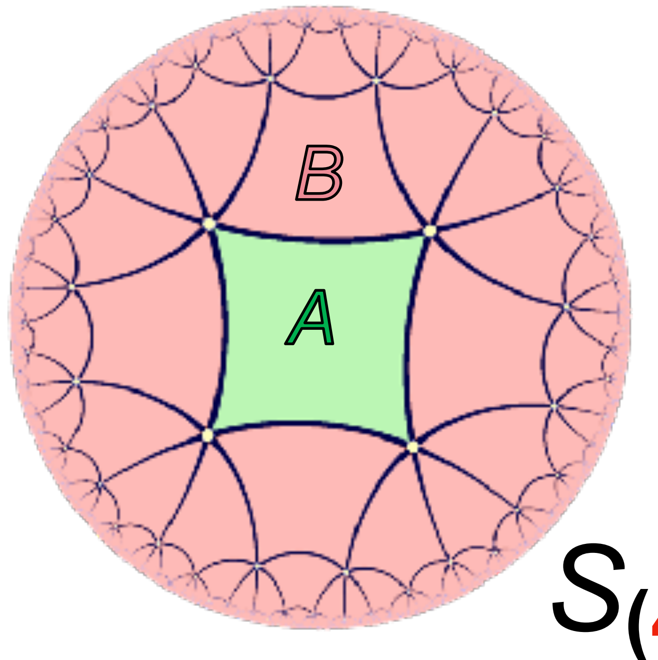
(∞,4) *Bethe lattice*

p	$E_0^{(p)}$ XY	$E_0^{(p)}$ Heisenberg
4	−1.08456618	−1.3089136
5	−1.08151200	−1.2912704
6	−1.08097046	−1.2925639
7	−1.08086301	−1.2918936
8	−1.08084068	−1.2919769
9	−1.08083585	−1.2919403
10	−1.08083478	−1.2919460
11	−1.08083453	−1.2919437
∞	−1.08083446	−1.291944

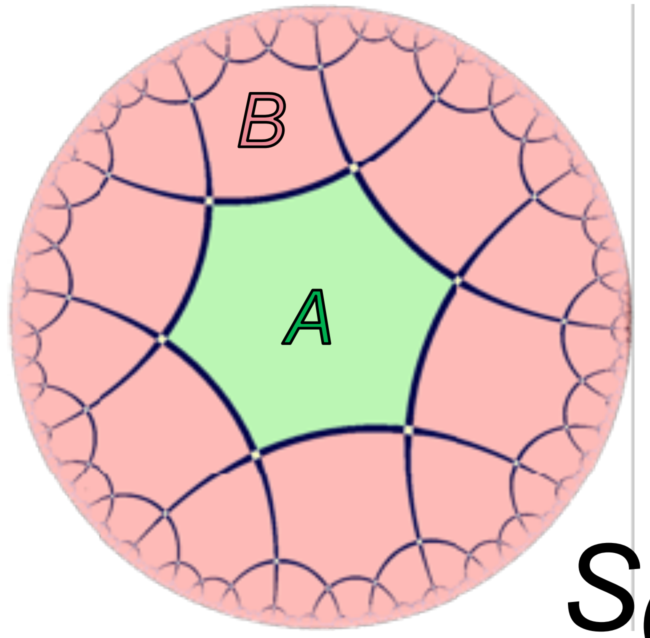


Entanglement entropy between the central subsystem *A* (polygon) in contact with the reservoir *B*

$$S_{(p,q)}[A] = -\mathrm{Tr}_B |\Psi_{AB}\rangle\langle\Psi_{AB}| \quad \text{on} \quad (p,4) \;\&\; (4,q)$$

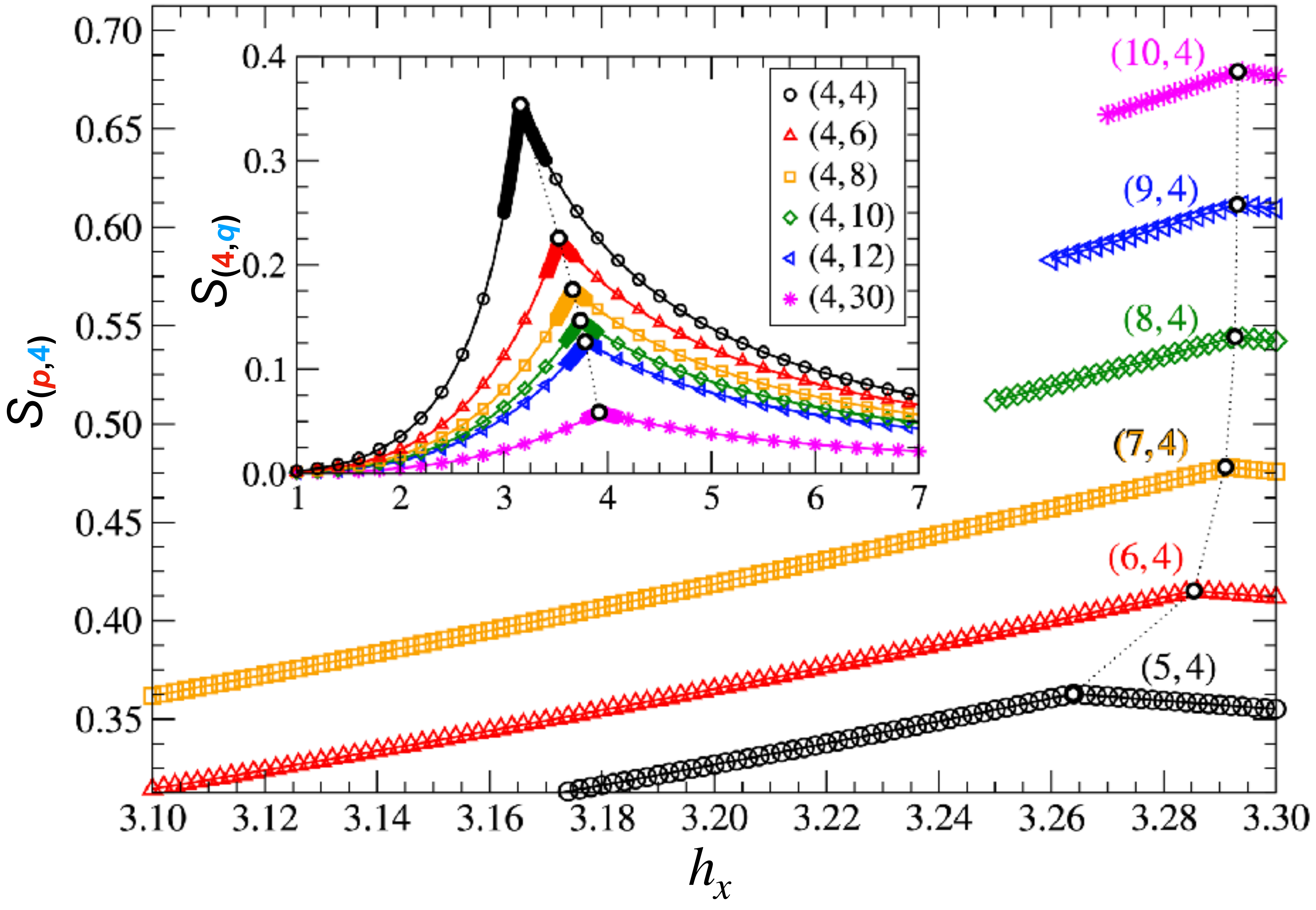


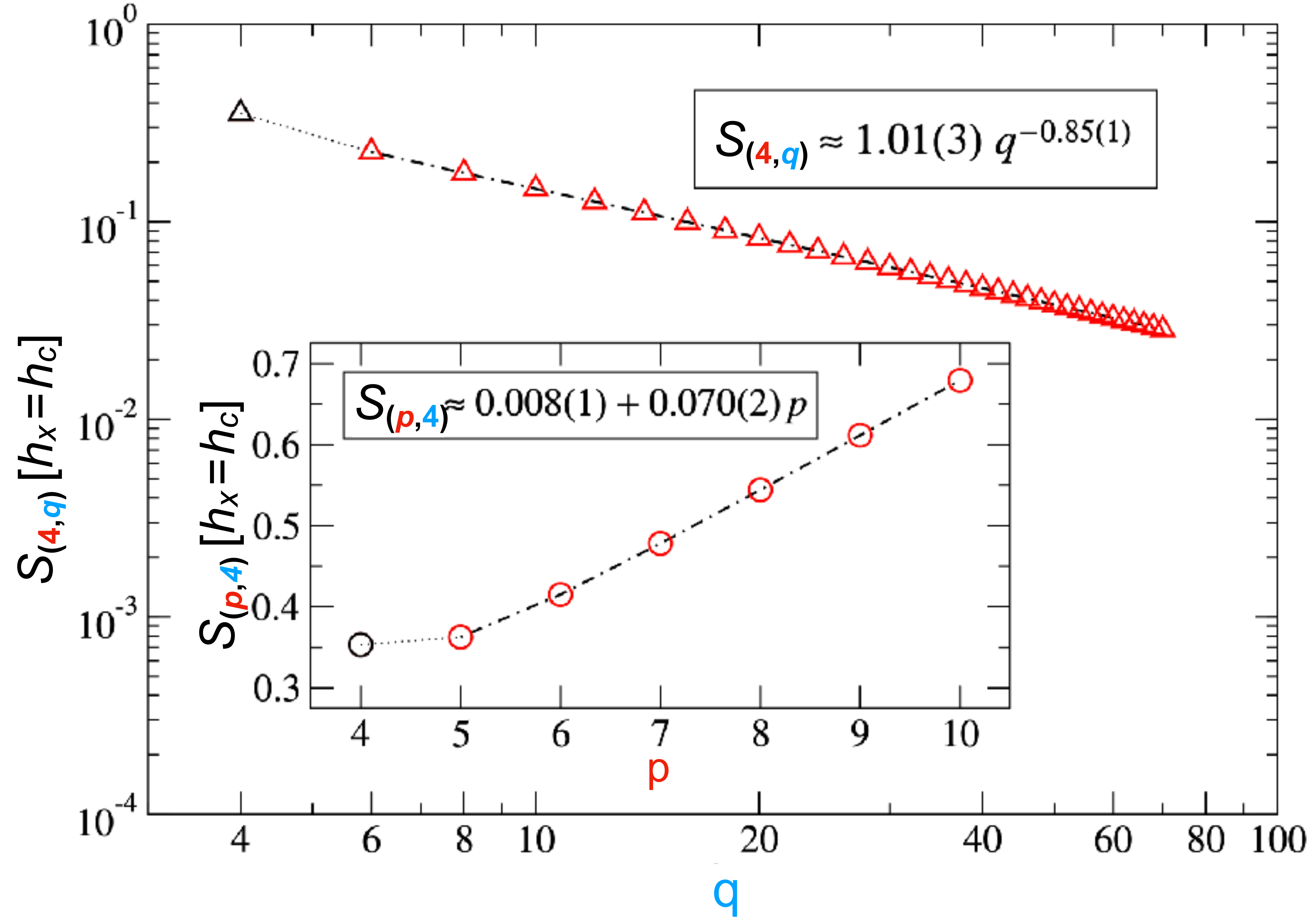
$S_{(4,6)}$



$S_{(6,4)}$

$$H = -\sum_{\{j,k\}} J_z S_j^z S_k^z - \sum_{\{j\}} h_x S_j^x$$

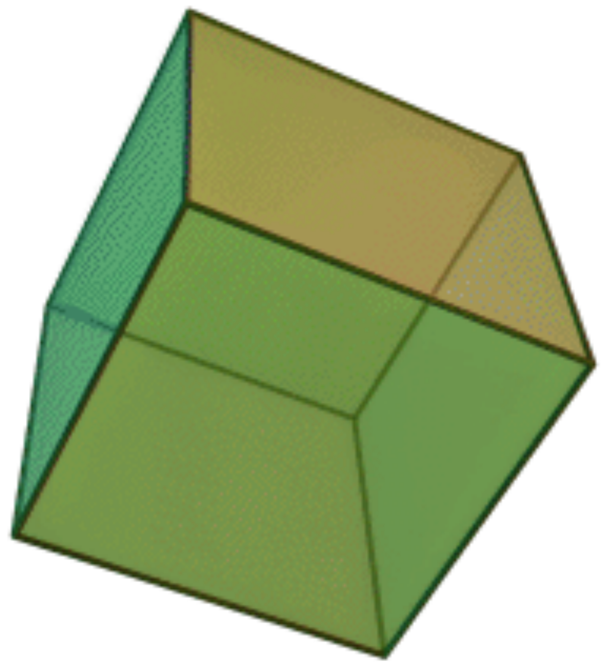




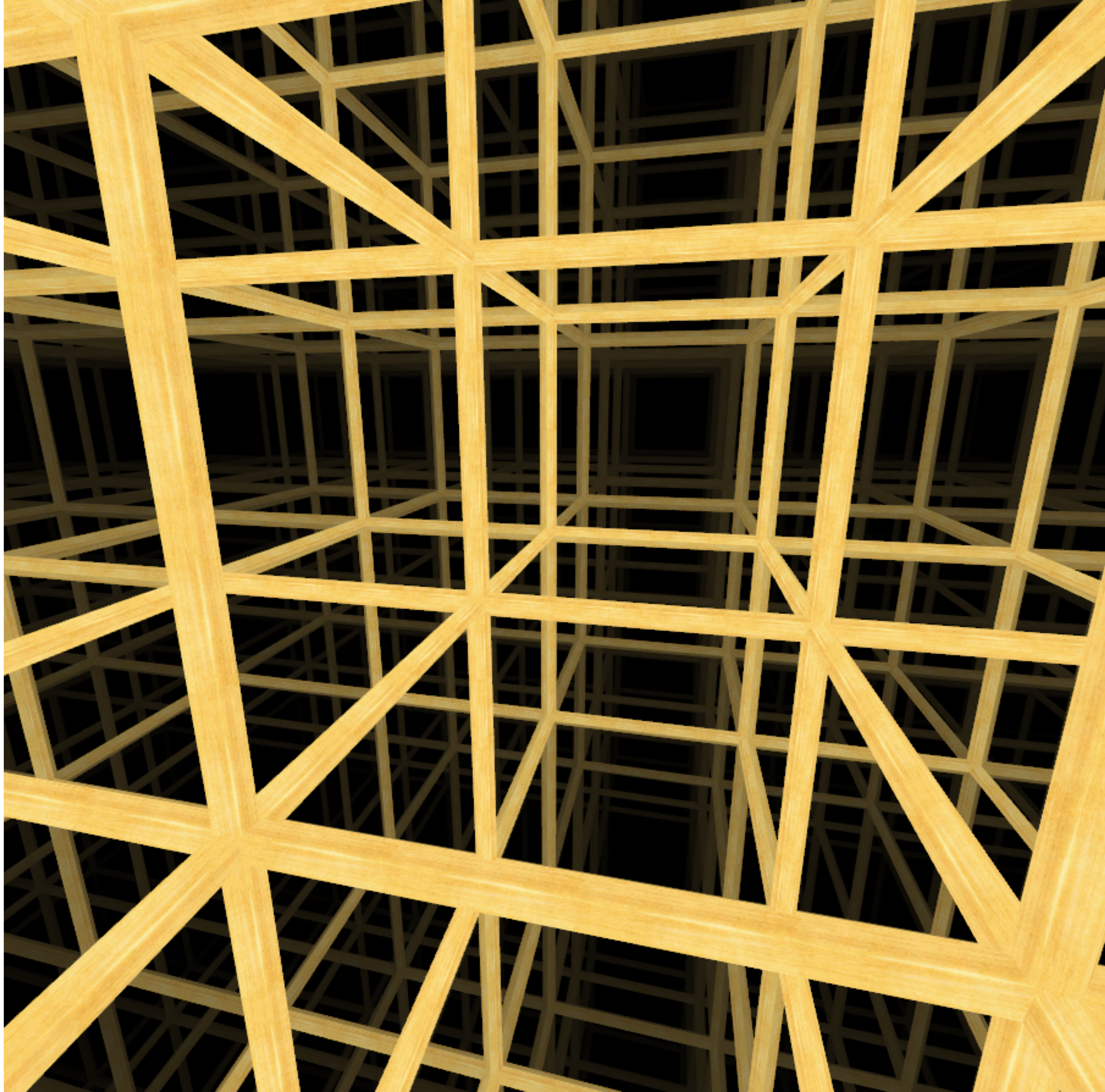
…more to be done?

3D Euclidean space

At each vertex,
there are 8 identical cubes.

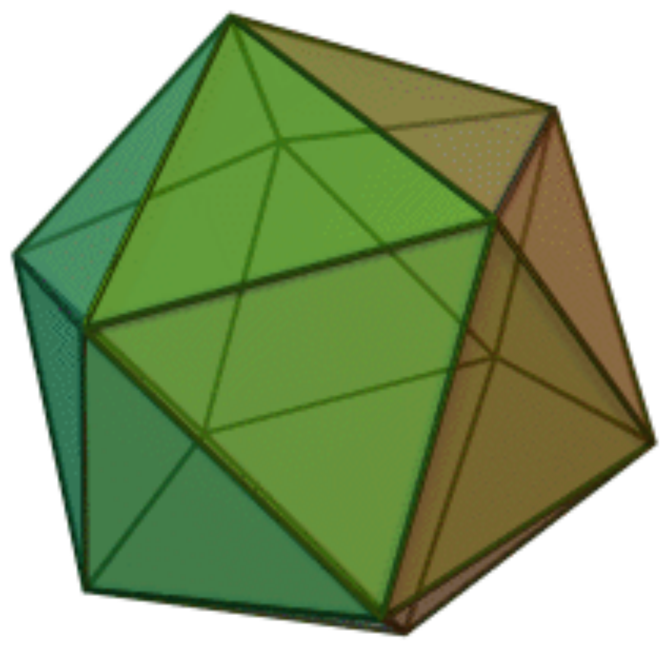


Can we construct a homogeneous **hyperbolic** 3D space (not surface) embedded in infinite dimensions?

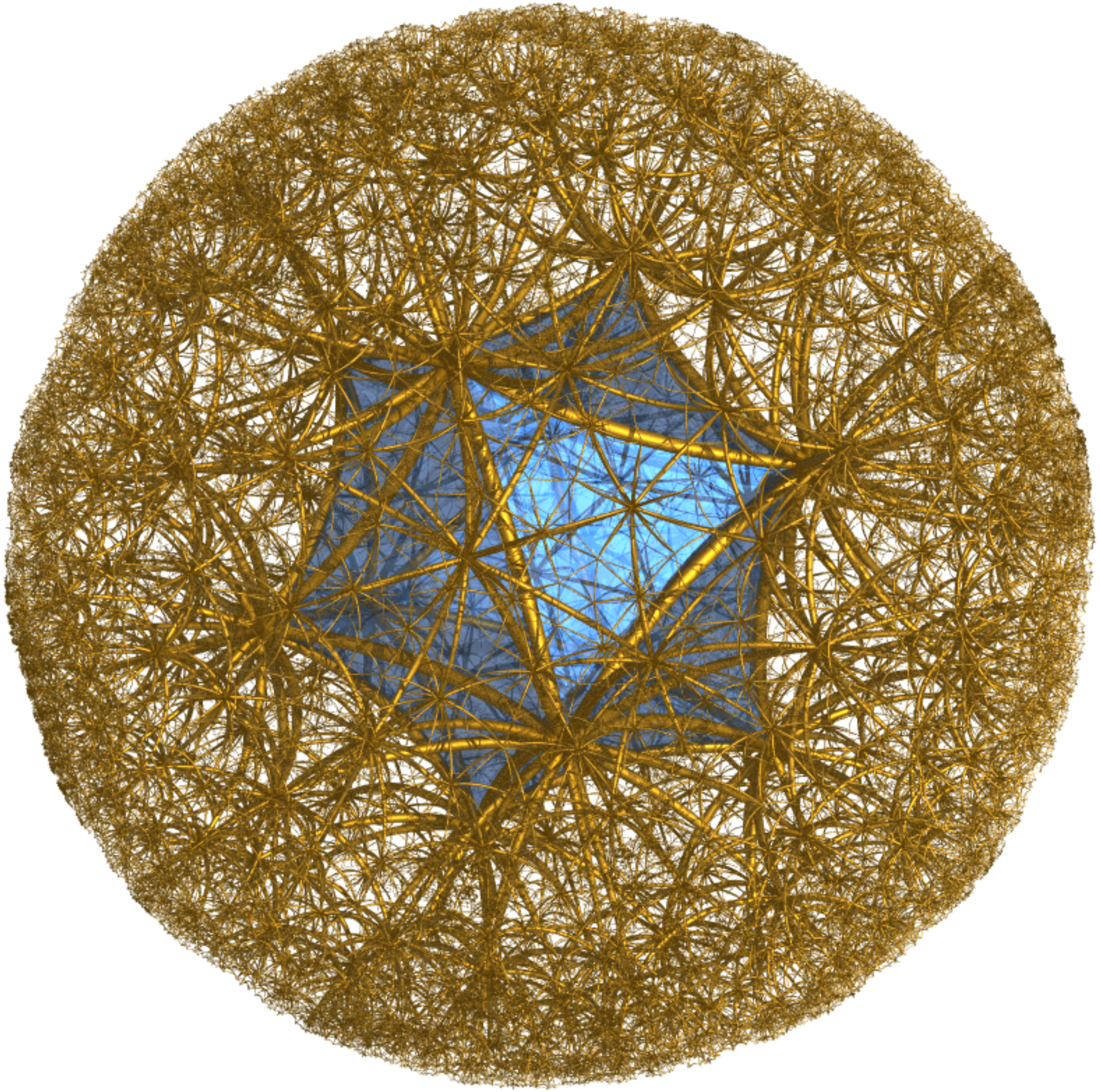


3D Hyperbolic space

At each vertex,
there are 12 identical icosahedra.

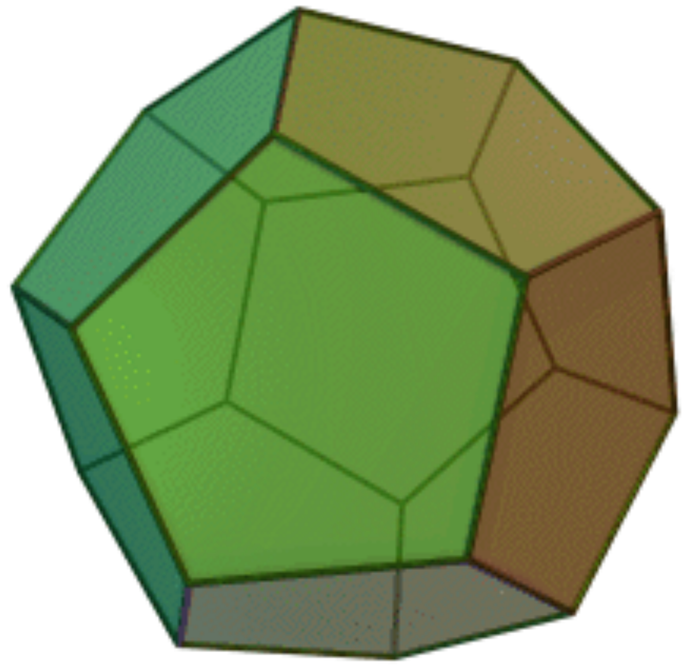


Poincaré sphere representation

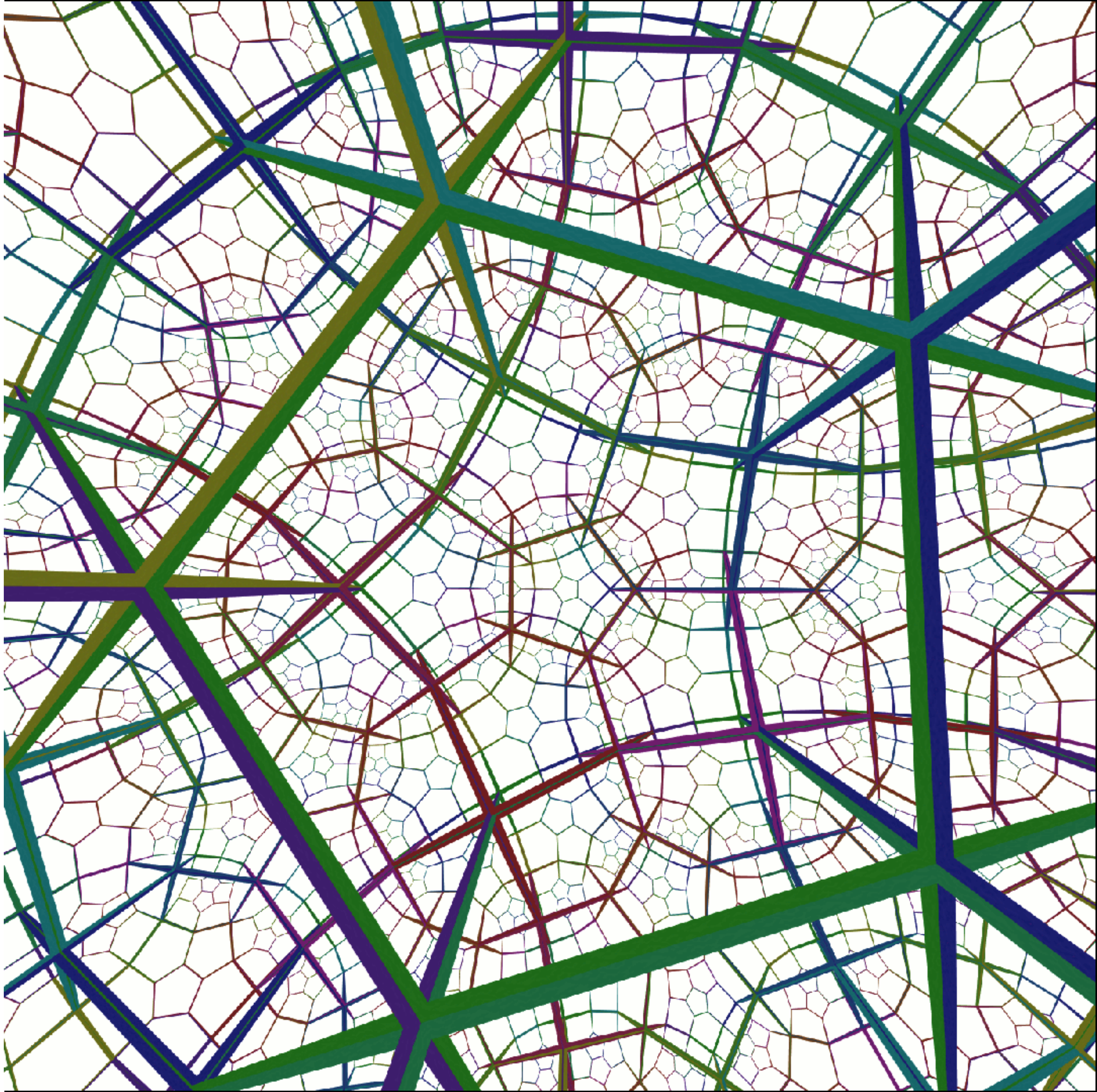


3D Hyperbolic space

At each vertex,
there are 8 identical dodecahedra.

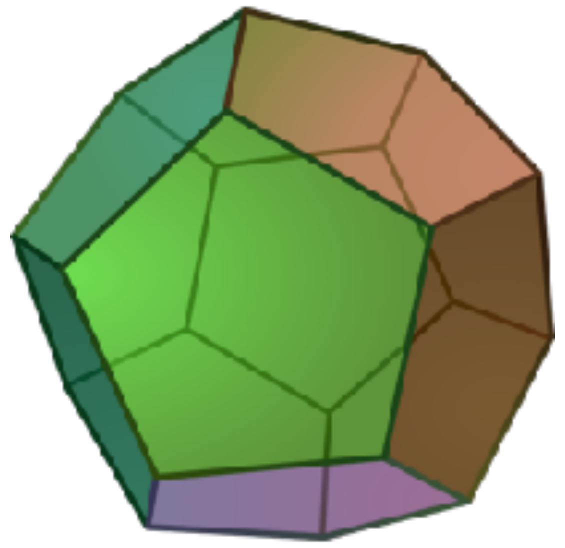


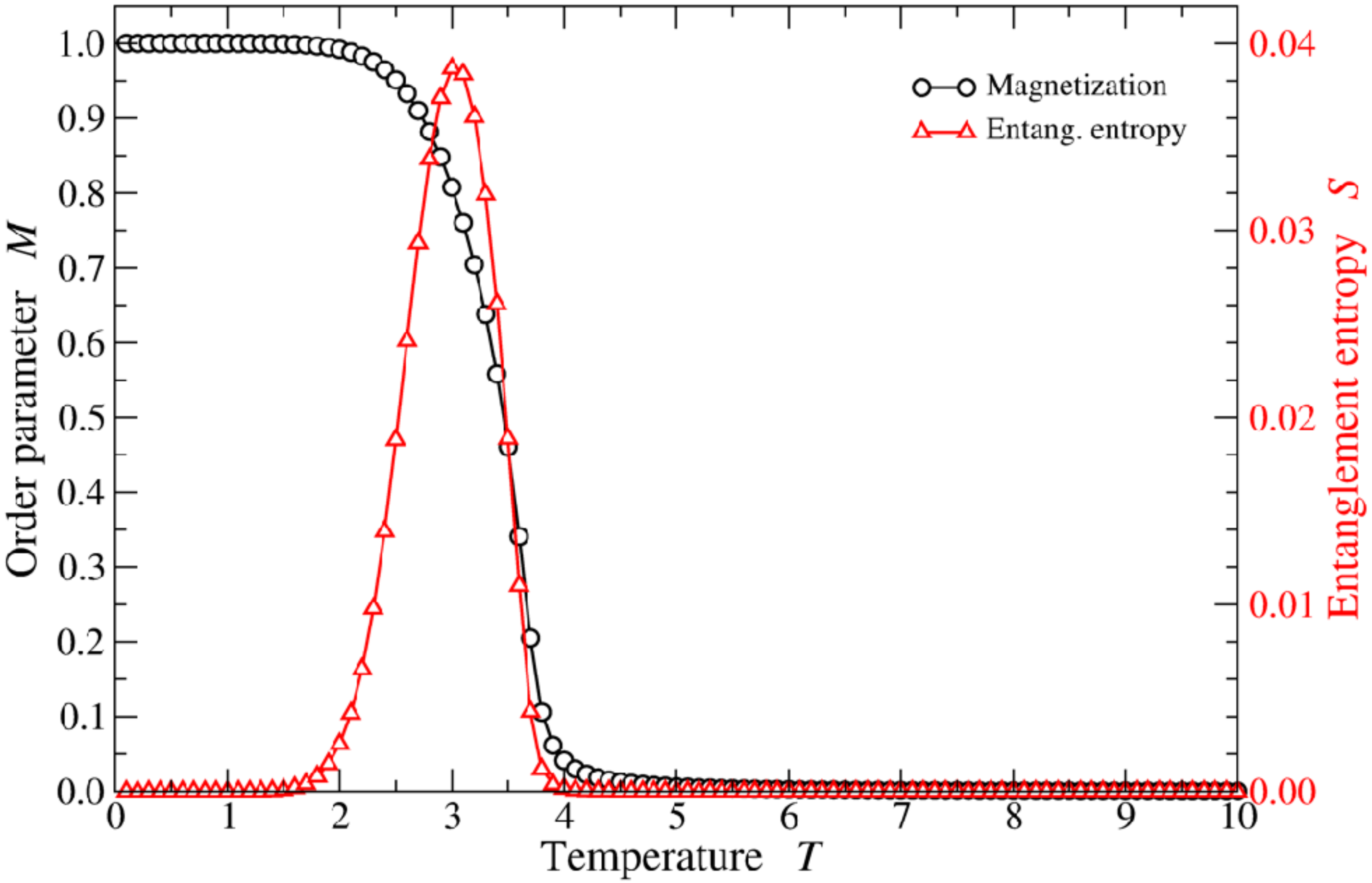
The coders'challenges
to blow-up imagination

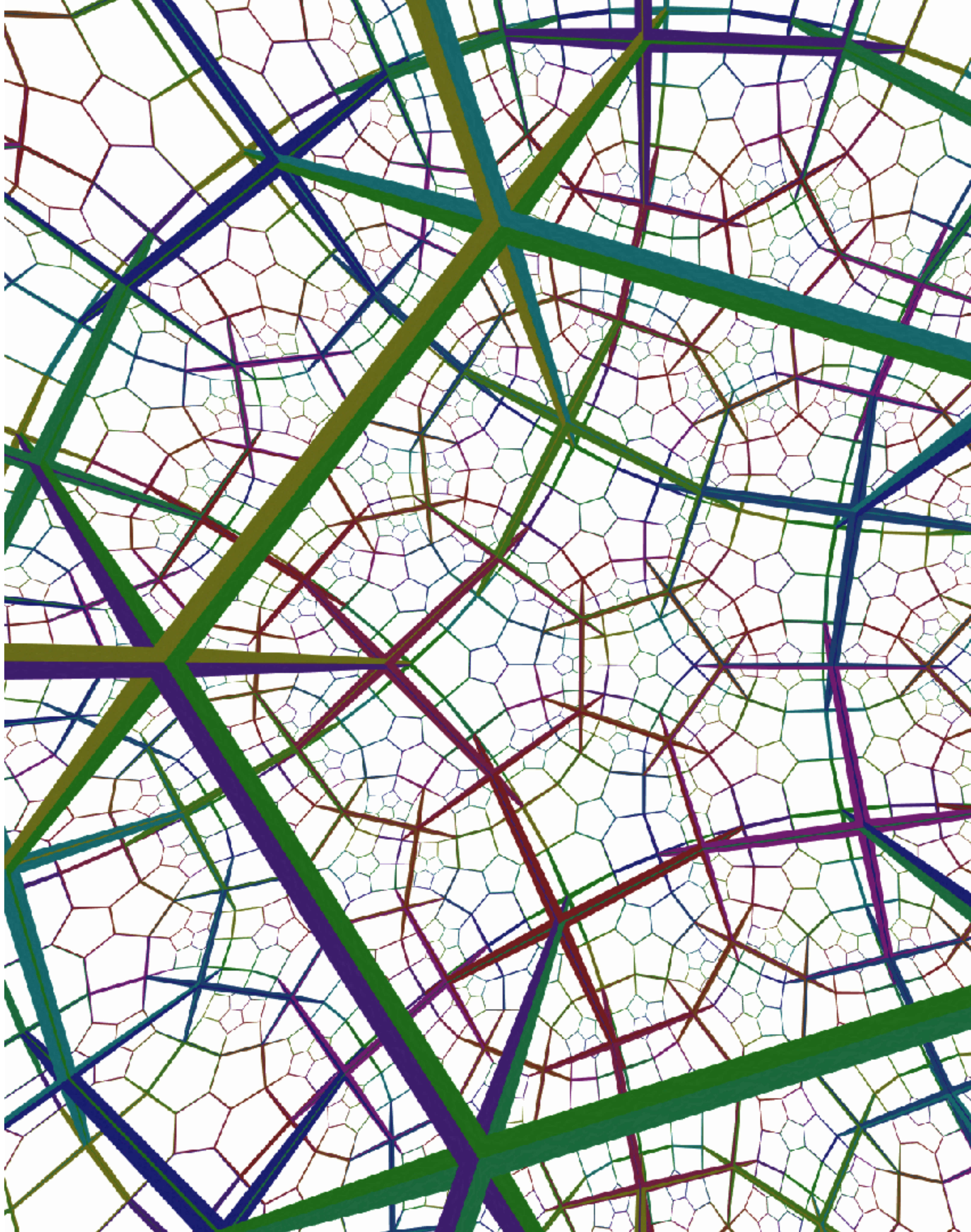


Dodecahedra hyperspace

(The coordination # q = 8)








Thank you