Reduced density matrix construction in interacting quantum field theory

Benasque, 22 Feb 2022

Ivan Kukuljan

Max Planck Institute for Quantum Optics
Munich, Germany

In collaboration with: Patrick Emonts (MPQ)

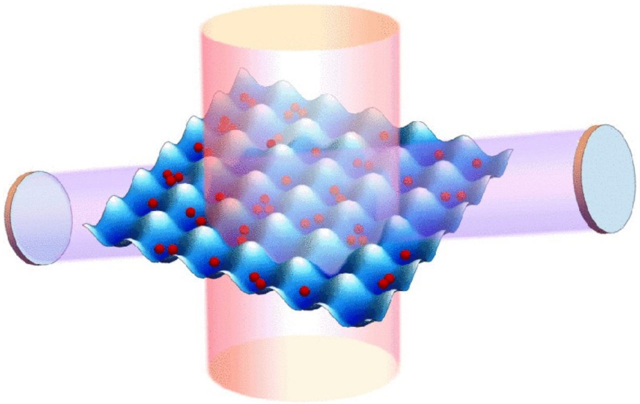
Emonts and Kukuljan, arXiv:2202.11113 [quant-ph]

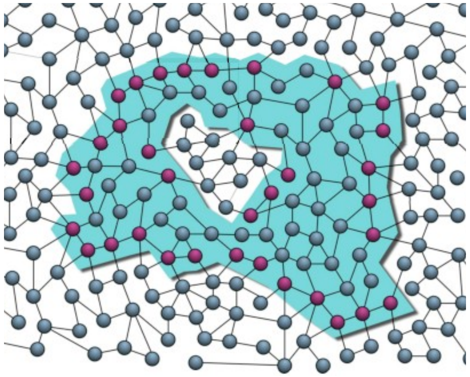
-Entanglement?   -Entanglement!!

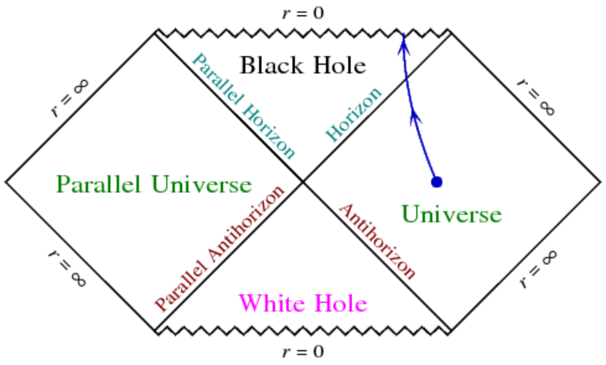
Quantum info
Condensed matter
Atomic physics

Simulability

High energy
Gravity







Entanglement in quantum field theory

$$\rho_V(\alpha_V, \alpha'_V) = N^{-1} \int_{\psi(\vec{x},0^-)=\alpha'_V(\vec{x}),\, x\in V}^{\psi(\vec{x},0^+)=-\alpha_V(\vec{x}),\, x\in V} D\psi\, D\bar{\psi}\, e^{-S_E(\psi,\bar{\psi})}$$

Elegant but difficult to compute

Available methods

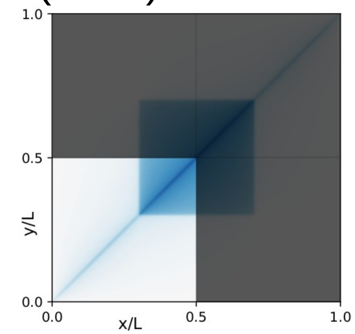
Replica trick (+CFT)

Calabrese and Cardy, J. Phy. A **42**, 504005 (2009)

Covariance matrices

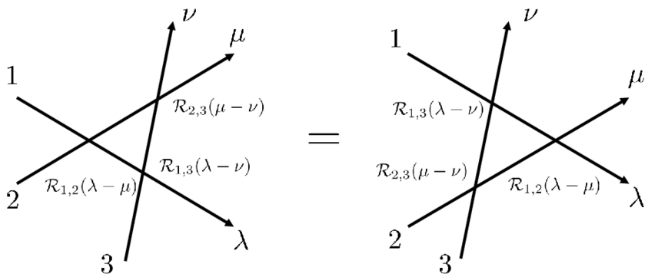
Casini and Huerta, J. Phy. A **42**, 504007 (2009)

Serafini, Quantum Continuous Variables, CRC Press (2017)



Form factors

Castro-Alvaredo and Doyon J. Phys. A: Math. Theor. **41** 275203 (2008)



Tensor networks

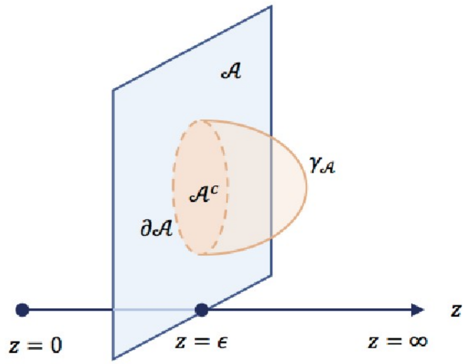
Orús, Annals of Physics **349**, 117 (2014).

$$|\Psi\rangle = \sum_{\boldsymbol{n}} A_1^{n_1} A_2^{n_2} A_3^{n_3} A_4^{n_4} |\boldsymbol{n}\rangle$$

Available methods

Ryu–Takayanagi

Ryu and Takayanagi, PRL 96, 181602 (2006)



Hamiltonian truncation

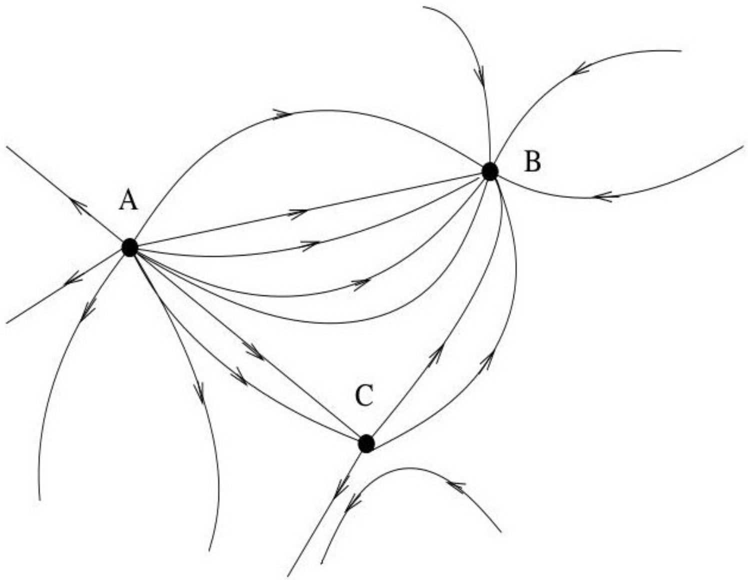
Hamiltonian truncation (HT)

James *et al*, Rep. Prog. Phys. **81** 046002 (2018)
Bajnok&Takacs, Nuclear Physics B **614**, 3 (2001)
Hogervorst *et al*, Phys. Rev. D **91**, 025005 (2015)

- Numerical method for nonperturbative study of strongly coupled QFT
- Based on Renormalisation group and Conformal field theory
- Does not need a discretisation of space
- Works in principle in any dimension, so far efficient in (1+1)D and (1+2)D
- Introduced by Yurov & Zomolodchikov (1991), applied to the sG model By Feverati, Ravanini and Takacs (1998-99)

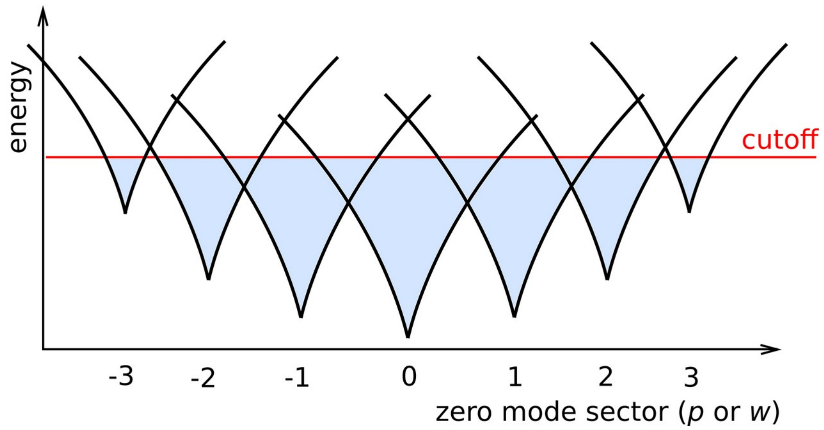
Hamiltonian truncation (HT)

$$H = H_{CFT} + V$$



- Regard the QFT as an RG flow from a CFT point
- Express V and all the observables as matrices in the CFT Hilbert space – exact CFT calculation
- Introduce an energy cutoff = keep only the low-energy physics
- $\rightarrow$ Use numerical linear algebra
- The perturbing operator needs not to be small but has to couple the low-energy of the spectrum to the high-energy part only weakly
- This is achieved by relevant perturbations
- Any solvable QFT can be used instead of a CFT $\rightarrow$ expansion in massive basis

Hamiltonian truncation (HT)



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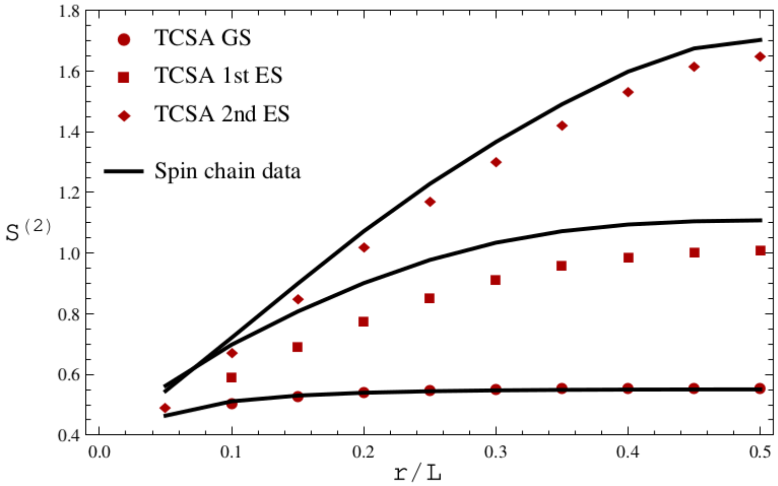
Entanglement?

- CFT and momentum bases not best suited for a bi partition in space
- Earlier works:
  - T. Palmai, Physics Letters B **759**, 439 (2016)
  - Murciano, Calabrese, and Konik arXiv:2112.04412 [cond-mat, physics:quant-ph] (2021)
- Use HT to compute occupation numbers in CFT basis
- Use replica techniques and CFT machinery to find the entanglement content for these states
- Successful for lowest Rényi entropies while von Neumann entropy and negativity out of scope



Our method

Emonts and Kukuljan, arXiv:2202.11113 [quant-ph]

The goal

- Robust, computationally efficient and model independent method to construct reduced density matrices using HT

Splitting the system

General idea

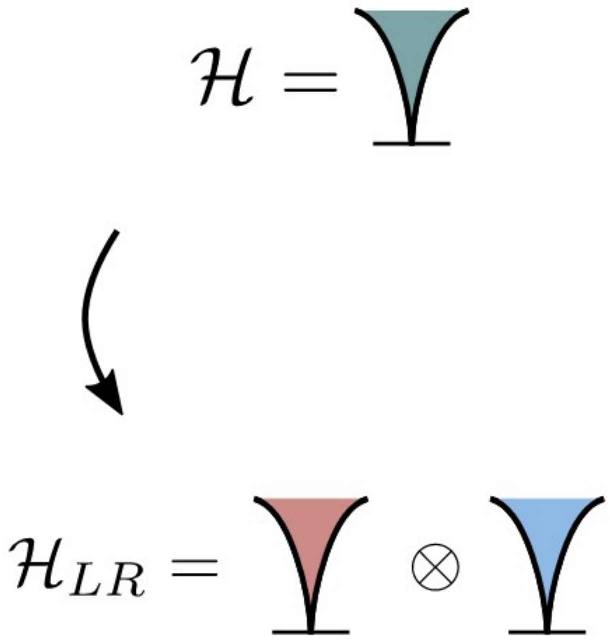
- HT gives a state ρ in the Hilbert space of the full system $\mathcal{H}_F$
- We want to construct Hilbert spaces corresponding to subsystems L,R: $\mathcal{H}_L \otimes \mathcal{H}_R$
- And find a unitary map between them

$$U_T : \mathcal{H}_F \to \mathcal{H}_L \otimes \mathcal{H}_R$$

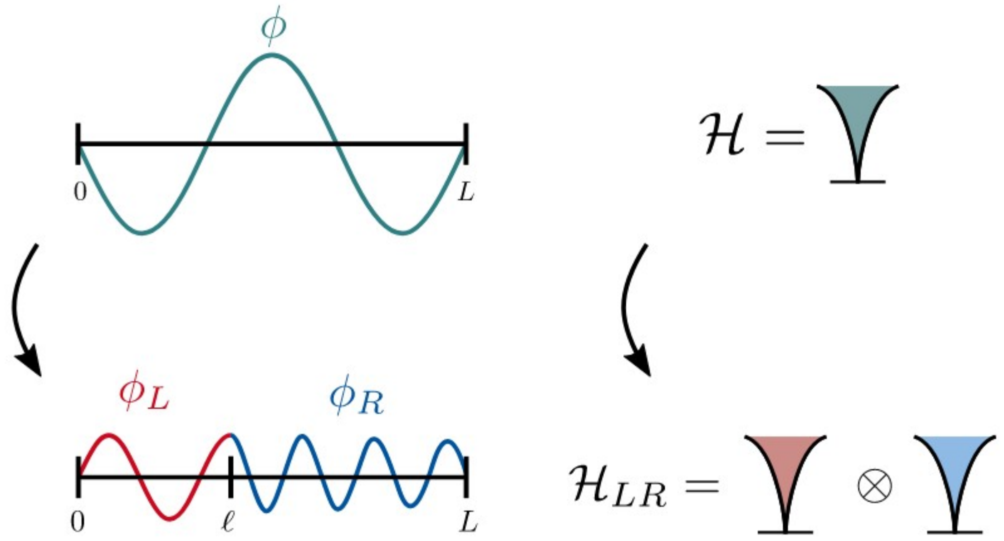
- Such that

$$\rho_{LR} = U_T \rho U_T^\dagger$$

is in the form convenient for taking partial traces



Full and split fields

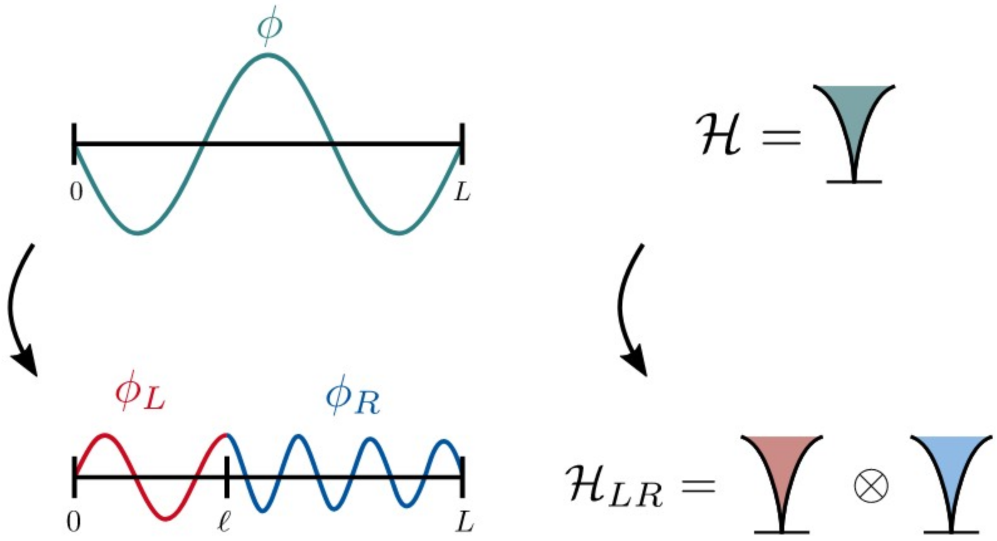


- $\mathcal{H}_F$ is generated by momentum modes of the “full” field

$$\phi(x,t) = 2\sqrt{\frac{\pi}{L}} \sum_{k=1}^{\infty} \frac{1}{\sqrt{p_k}} \left(A_k e^{-ip_k t} + A_k^\dagger e^{ip_k t} \right) \sin(p_k x)$$

$$|\vec{n}_F\rangle \equiv |n_1, n_2, \ldots\rangle \equiv \frac{1}{N_F} \prod_{k>0} \left(A_k^\dagger\right)^{n_k} |0\rangle \qquad \left[A_k, A_l^\dagger\right] = \delta_{k,l}$$

Full and split fields

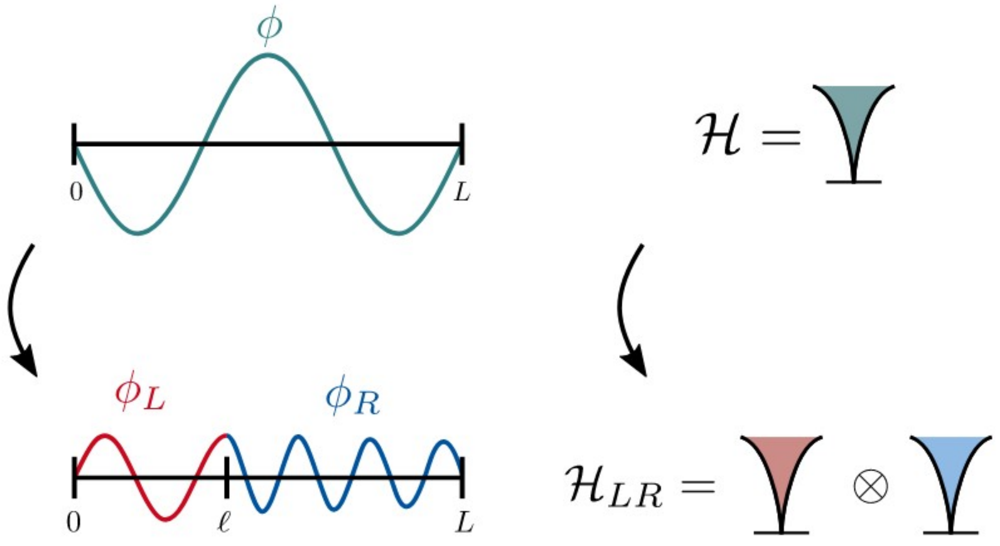


- We introduce a cut and impose a boundary condition (Neumann):

$$\partial_x \phi_L(\ell) = \partial_x \phi_R(\ell) = 0$$

- This gives rise to "split" fields ϕ_L and ϕ_R that live on the subintervals

Full and split fields

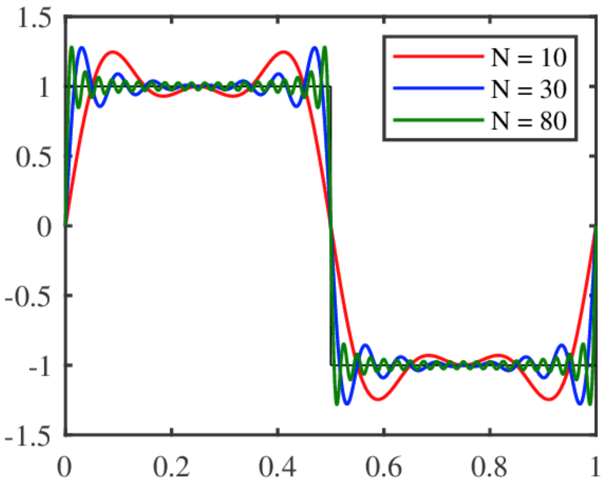


- Their momentum modes span $\mathcal{H}_L \otimes \mathcal{H}_R$

$$\begin{aligned}
\phi_L(x,t) &= 2\sqrt{\frac{\pi}{\ell}} \sum_{m=1}^{\infty} \frac{1}{\sqrt{q_m^{(\ell)}}} \times \\
&\quad \left(a_m^L e^{-iq_m^{(\ell)} t} + a_m^{L,\dagger} e^{iq_m^{(\ell)} t} \right) \sin\left(q_m^{(\ell)} x \right) \\
\phi_R(x,t) &= 2\sqrt{\frac{\pi}{L-\ell}} \sum_{m=1}^{\infty} \frac{1}{\sqrt{q_m^{(L-\ell)}}} \times \\
&\quad \left(a_m^R e^{-iq_m^{(L-\ell)} t} + a_m^{R,\dagger} e^{iq_m^{(L-\ell)} t} \right) \sin\left(q_m^{(L-\ell)} (L-x) \right)
\end{aligned} \tag{9}$$

Meaningfulness and exactness of such construction

- Intuitive argument: Any field configuration of the full field can be approximated arbitrarily closely by the split fields such that the boundary condition at the split is preserved
- This is in spirit similar to expanding a step function in terms of Fourier modes
- Rigorously: Carleson’s theorem establishes the completeness of the functional basis. Symplectic structure of the Bogoliubov transformation establishes the isomorphism of the algebras.



Bogoliubov transformation

- Identifying the fields through the continuity condition

$$\phi(x,t) = \begin{cases} \phi_L(x,t) & \text{if } x < \ell, \\ \phi_R(x,t) & \text{if } \ell < x < L \end{cases}$$

- Gives rise via

$$A_k = \frac{\sqrt{p_k}}{2\sqrt{L\pi}} \int_0^L \mathrm{d}x \left[\phi(x,t) + \frac{i}{p_k}\pi(x,t)\right] \sin(p_k x)$$

- to a Bogoliubov transformation between the modes

$$A_k = \sum_m \gamma_{km}^{+,L} a_m^L + \sum_m \gamma_{km}^{-,L} a_m^{L,\dagger} + \sum_m \gamma_{km}^{+,R} a_m^R + \sum_m \gamma_{km}^{-,R} a_m^{R,\dagger}$$

- This makes it possible to construct the transition matrix

$$(U_T)_{\vec{n}_L \vec{n}_R; \vec{n}_F} = \langle \vec{n}_L, \vec{n}_R | \vec{n}_F \rangle$$

Multimode squeezed vacuum

- The Hilbert spaces $\mathcal{H}_F$ and $\mathcal{H}_L \otimes \mathcal{H}_R$ don't have the same vacuua
- It is therefore important to transform the vacua correctly
- This is achieved by the multimode squeezed vacuum

$$|0\rangle = U\,|0,0\rangle$$

$$U = \exp\left(-\frac{1}{2}\left[a^{\dagger T} \;\; a^T\right] K \ln M \begin{bmatrix} a \\ a^\dagger \end{bmatrix}\right)$$

$$\begin{bmatrix} A \\ A^\dagger \end{bmatrix} = M \begin{bmatrix} a \\ a^\dagger \end{bmatrix} \qquad M = \begin{bmatrix} \gamma^{L,+} & \gamma^{R,+} & \gamma^{L,-} & \gamma^{R,-} \\ \gamma^{L,-} & \gamma^{R,-} & \gamma^{L,+} & \gamma^{R,+} \end{bmatrix}$$

Algorithm

Exponential complexity

- The matrix elements of the transition matrix are exponentially costly to evaluate

$$\langle \vec{n}_L, \vec{n}_R | \vec{n}_F \rangle = \frac{1}{N} \langle 0,0| \left[\prod_{m>0} \left(a_m^R\right)^{n_{m,R}} \left(a_m^L\right)^{n_{m,L}} \right] \times \left[\prod_{k>0} \left[\sum_\sigma \sum_{l>0} \left(\gamma_{kl}^{\sigma,-} a_l^\sigma + \gamma_{kl}^{\sigma,+} a_l^{\sigma\dagger} \right) \right]^{n_k} \right] \times \left[\exp\left(-\sum_{ij} \sum_{\sigma,\chi} a_i^{\sigma\dagger} \chi_{ij}^{\sigma,\xi} a_j^{\xi\dagger} \right) \right] |0,0\rangle\,,$$

- This can be seen by expressing them in the generating functional form

$$\langle \vec{n}_L, \vec{n}_R | \vec{n}_F \rangle = \frac{1}{N} \prod_{m>0} \prod_\sigma \frac{\mathrm{d}^{n_{m,\sigma}}}{\mathrm{d}j_{m,\sigma}^{n_{m,\sigma}}} \prod_{k>0} \frac{\mathrm{d}^{n_k}}{\mathrm{d}J_L^{n_k}} \langle 0,0| e^S e^F e^V |0,0\rangle \Big|_{J_k=0, j_{m,\sigma}=0}$$

$$S = \sum_{m>0} \sum_\sigma j_{m,\sigma} a_m^\sigma$$

$$F = \sum_{k,m>0} \sum_\xi J_k \left(\gamma_{k,m}^{+,\xi} a_m^{\xi\dagger} + \gamma_{k,m}^{-,\xi} a_m^\xi \right)$$

$$V = -\sum_{\kappa,\lambda} \sum_{m,n>0} a_{-m}^\kappa \chi_{m,n}^{\kappa,\lambda} a_{-n}^\lambda.$$

- The complexity scales as

$$\exp\left[(n/2) \left(\log(n/2) - 1/4 \right) \right]$$

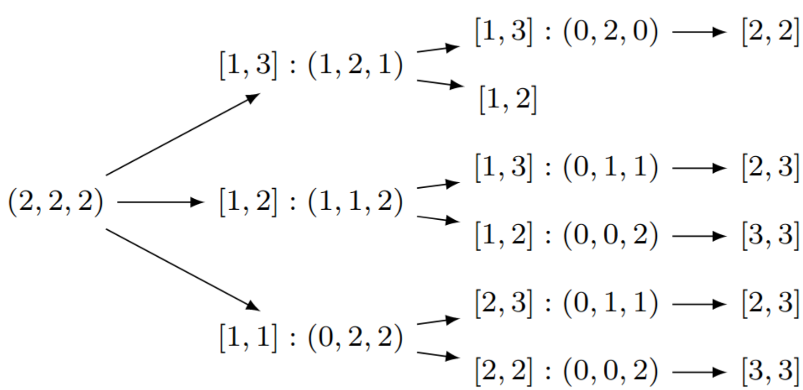
Algorithm - reducing the exponent

- We reduce the complexity by using the symmetry of the expression to bring it to the form

$$\prod_i \frac{d^{n_i}}{d\mathcal{J}_i^{n_i}} e^T \Big|_{\mathcal{J} J_i=0} = \sum_k{}' c_k \prod_l T^{p_{kl}} \left[\mathcal{J}_{l_1}, \mathcal{J}_{l_2}\right]$$

  where the sum runs over pairwise lexicographically ordered sets of tuples and the multiplicities c_k are computed analitically

- Tree based algorithm for efficient evaluation

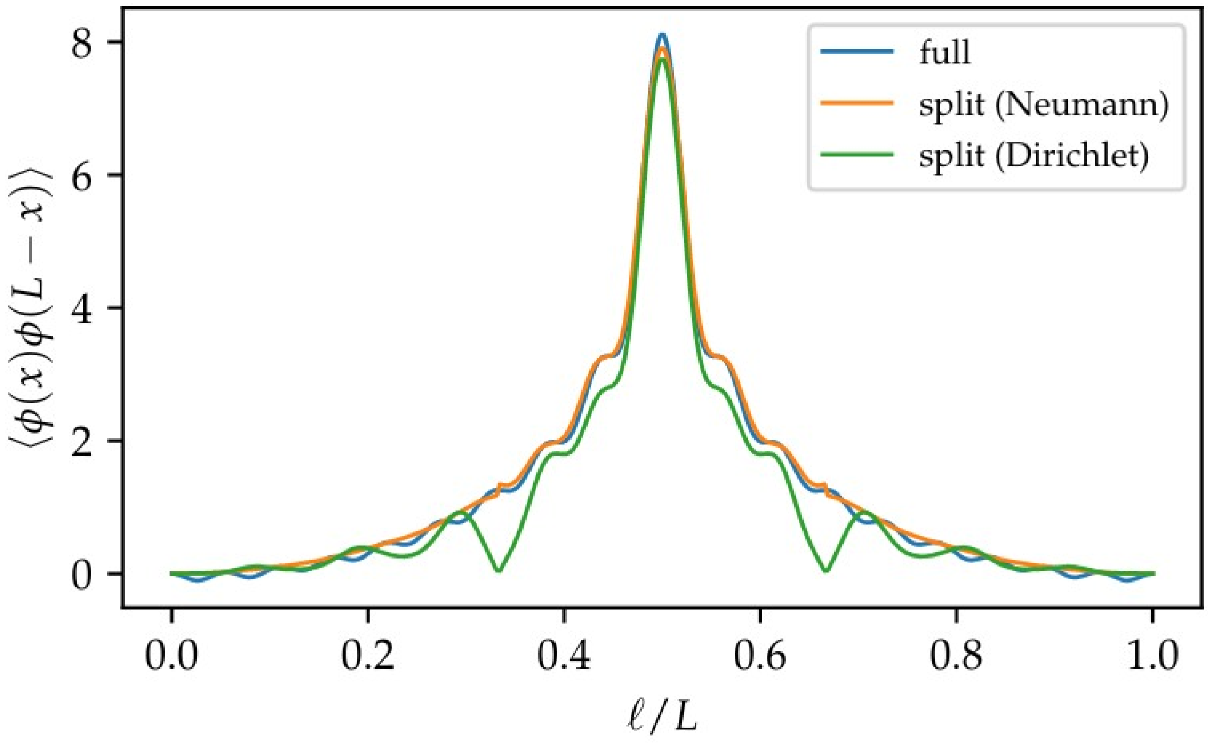


Results

Benchmarking: Klein-Gordon model

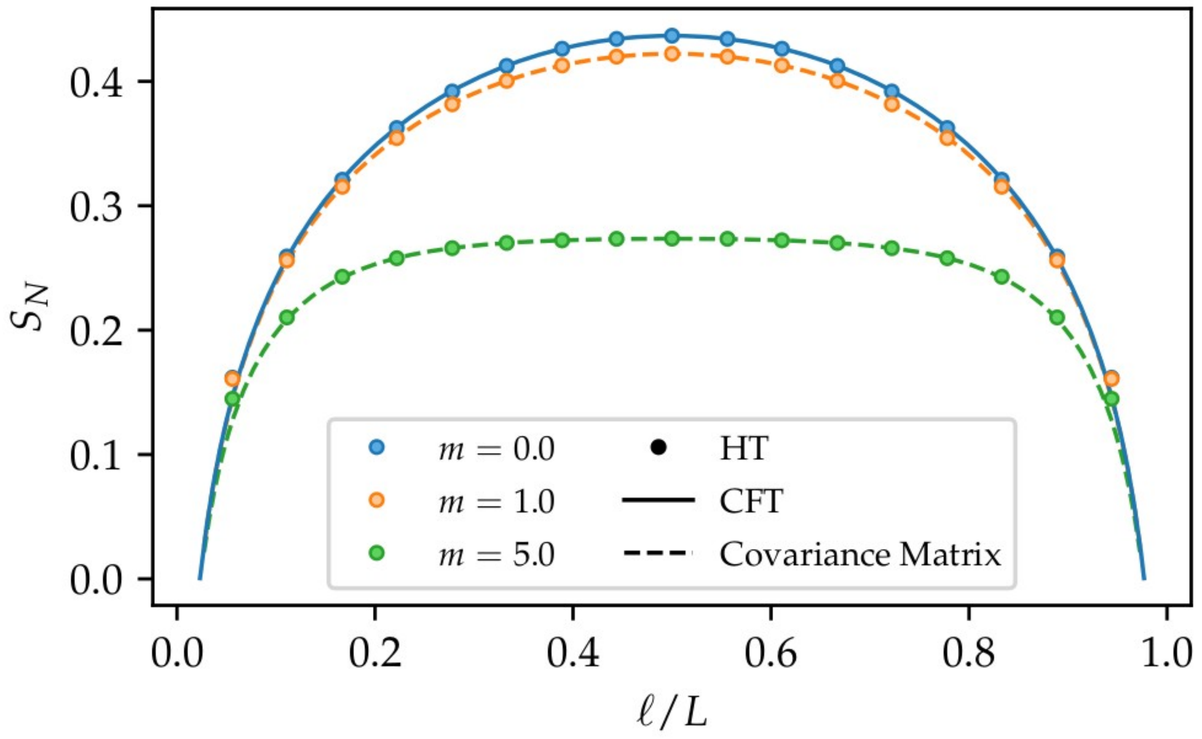
- Correlation functions

$$\langle \phi(x)\phi(L-x)\rangle = \mathrm{Tr}(\phi(x)\phi(L-x)\rho)$$
$$= \mathrm{Tr}\big(\phi_{L/R}(x)\phi_{L/R}(L-x)\rho_{LR}\big)$$



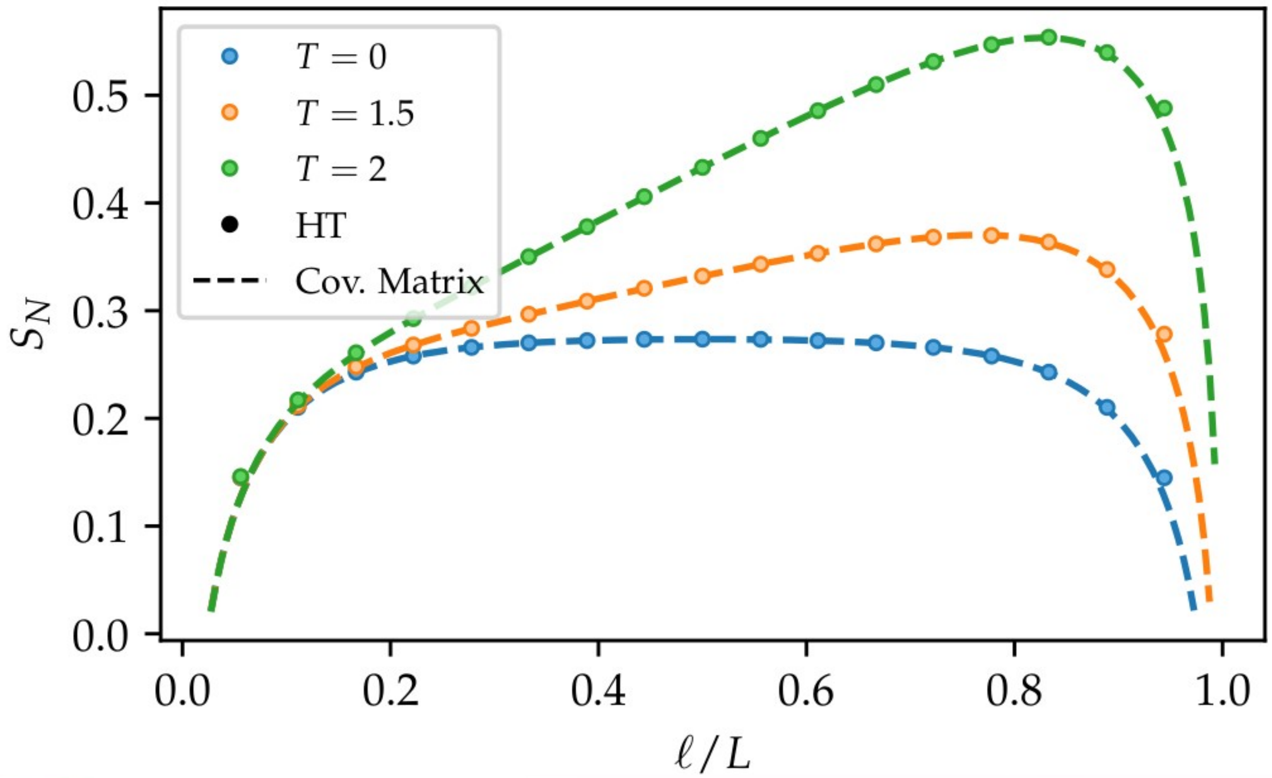
Benchmarking: Klein-Gordon model

- Von Neumann entropy in ground states compared to exact CFT solution and analytic results using covariance matrix formalism



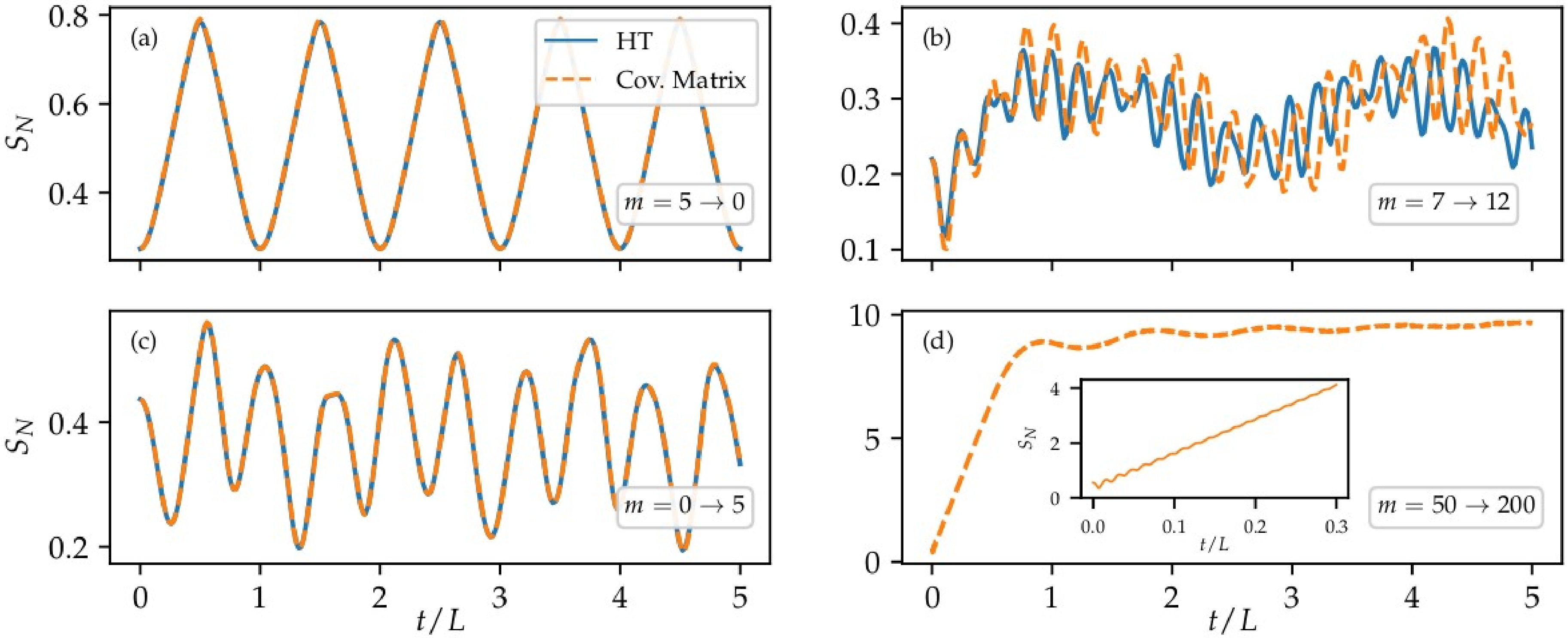
Benchmarking: Klein-Gordon model

- Von Neumann entropy in thermal states compared to the covariance matrix formalism



Benchmarking: Klein-Gordon model

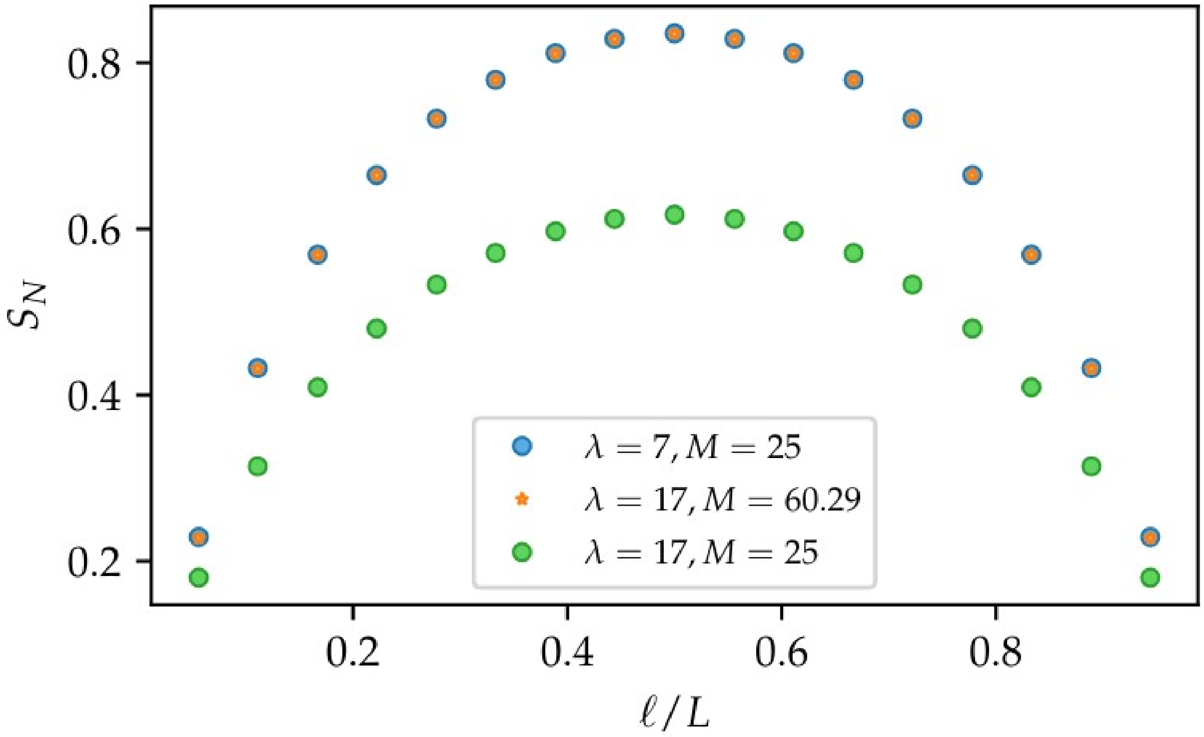
- Real time dynamics after quenches compared to the covariance matrix formalism



Interacting physics: sine-Gordon model

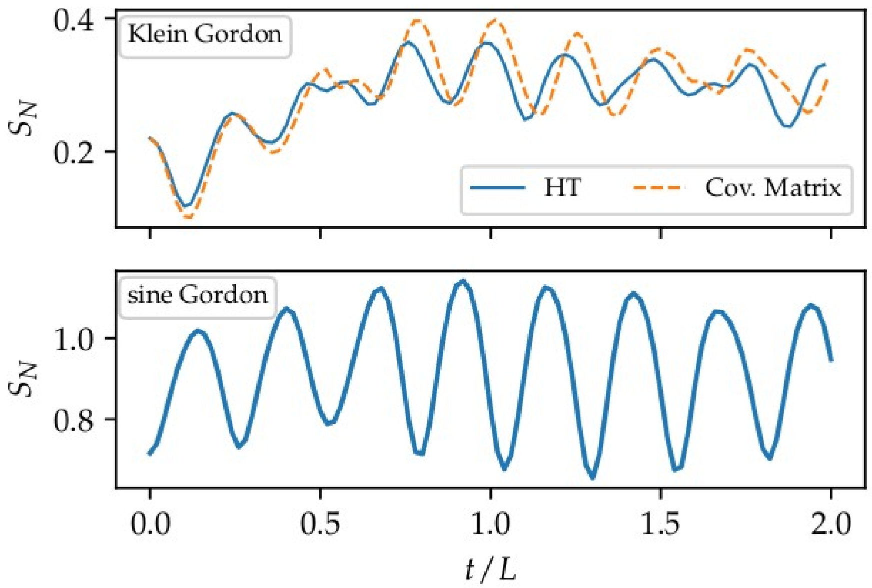
- Ground state scaling: Log law!

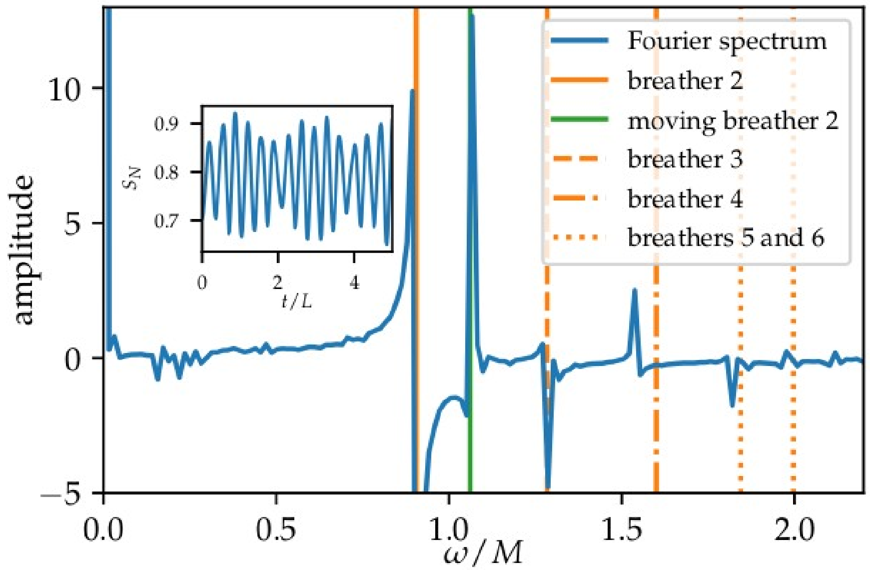
$$H_{\rm sG} = \int \mathrm{d}x \left[\frac{1}{8\pi} \left\{ (\partial_t \phi(x))^2 + (\partial_x \phi(x))^2 \right\} - \frac{m^2}{\beta^2} \cos\left(\frac{\beta}{\sqrt{4\pi}} \phi(x) \right) \right]$$



Interacting physics: sine-Gordon model

- Dynamics, verify predictions by O. Castro-Alvaredo and D. Horvath, SciPost Physics **10** 132 (2021).





Scope

- In case of the free theory, this is an exact construction of reduced density matrices
- Straightforward extension to D>1+1
- Study of the Bisognano-Wichmann theorem
- Symmetry resolved entanglement
- Integrability breaking and other interacting theories, like phi^4

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