Matrix product operator intertwiners as dualities in quantum spin chains

Benasque, Entanglement in Strongly Correlated Systems

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Outline

Symmetries

Dualities

Matrix product operators

Examples

Conclusion

References

- *Dualities in one-dimensional quantum lattice models with categorical symmetries: Hamiltonians and intertwiners*, 2112.09091, LL, Clement Delcamp, Gerardo Ortiz, Frank Verstraete
- *Matrix product operator symmetries and intertwiners in string-nets with domain walls*, SciPost Phys. 10, 053 (2021), LL, Jürgen Fuchs, Jutho Haegeman, Christoph Schweigert, Frank Verstraete

Symmetries

We are interested in global symmetries, represented by operators that commute with the Hamiltonian and form a fusion ring:

$$\mathcal{O}_a \mathbb{H} = \mathbb{H} \mathcal{O}_a, \quad \mathcal{O}_a \mathcal{O}_b = \sum_c N_{ab}^c \mathcal{O}_c$$

If fusion ring is a group, these representations are unitary, $\mathcal{O}_g^\dagger = \mathcal{O}_{g^{-1}}$

Global symmetries decompose the Hilbert space into irreducible representations i:

$$\mathcal{H}_A = \bigoplus_i^n \mathcal{H}_{A,i}$$

This includes symmetry twisted boundary conditions $\to$ tube algebras

Dualities

Dualities relate distinct realizations of the same physics; e.g.

- Wave $\leftrightarrow$ particule duality in quantum mechanics
- Holographic duality (AdS/CFT, CS/WZW)
- **High $\leftrightarrow$ low temperature Ising model (Kramers-Wannier)**

We characterize a duality as follows:

1. **local, symmetric** operators $\rightarrow$ dual **local, symmetric** operators ($\mathbb{H}_A \rightarrow \mathbb{H}_B$)
2. **local order** operators $\rightarrow$ dual **non-local disorder** operators
3. implemented as an isometry between dual Hilbert spaces

Duality as an isometry

Hilbert space and Hamiltonian split into sectors, which have to match between models:

$$\mathcal{H}_A = \bigoplus_i^n \mathcal{H}_{A,i} \quad \text{and} \quad \mathcal{H}_B = \bigoplus_i^n \mathcal{H}_{B,i}\,,$$
$$\mathbb{H}_A = \bigoplus_i^n \mathbb{H}_{A,i} \quad \text{and} \quad \mathbb{H}_B = \bigoplus_i^n \mathbb{H}_{B,i}\,.$$

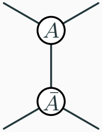
although they need not be the same size (different degeneracies). Dualities are isometries defined by

$$\mathbb{U}_i : \mathcal{H}_{A,i} \times \mathcal{H}_{A,i}^{\text{aux}} \to \mathcal{H}_{B,i} \times \mathcal{H}_{B,i}^{\text{aux}}$$
$$\text{s.t.} \quad \mathbb{U}_i(\mathbb{H}_{A,i} \otimes \mathbb{1}_{A,i})\mathbb{U}_i^\dagger = \mathbb{H}_{B,i} \otimes \mathbb{1}_{B,i}$$

Matrix product operators

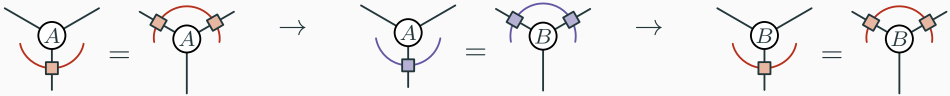
Defining the Hamiltonian as a sum of local terms

$$\mathbb{h}_{A,\mathrm{i}} = \sum_k A^k_{i'l'} \bar{A}^k_{il} \, |i', l'\rangle\langle i, l| \equiv$$



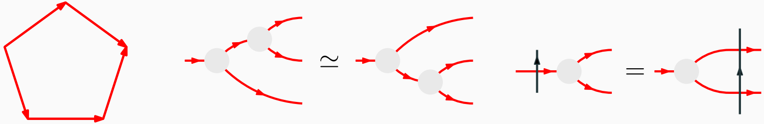
Symmetries are represented as MPOs, dualities are represented as MPO intertwiners:

MPO symmetry — MPO intertwiner — dual MPO symmetry

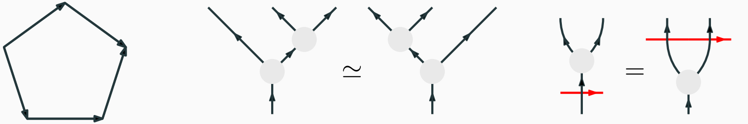


MPO symmetries

MPO symmetries are described by $(\mathcal{C}, \mathcal{D})$-bimodule category $\mathcal{M}$:



$^0F^0F = {}^0F^0F^0F$ $\quad$ $^1F^1F = {}^0F^1F^1F$ $\quad$ $^2F^1F = {}^1F^2F^2F$



$^4F^4F = {}^4F^4F^4F$ $\quad$ $^3F^3F = {}^4F^3F^3F$ $\quad$ $^2F^3F = {}^3F^2F^2F$

$\mathcal{C}$ describes the symmetries, $\mathcal{D}$ describes the representations

Dual models

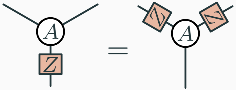
- **Dual models are characterized by the same fusion category $\mathcal{D}$, with the same recoupling theory, but different choices of module category $\mathcal{M}$.**
- Consequence: algebra of symmetric operators is the same
- Dual models have equivalent but distinct realizations of the symmetries $\mathcal{C}$, completely determined by the choice of $\mathcal{M}$: $\mathcal{C} = \mathcal{D}^*_{\mathcal{M}}$
- MPO intertwiners relating dual models can be constructed from the categorical data

Example: Ising model

We consider the transverse field Ising model (note half-integer sites):

$$\mathbb{H}_A = -J \sum_{\mathsf{i}} (X_{\mathsf{i}-\frac{1}{2}} X_{\mathsf{i}+\frac{1}{2}} + g Z_{\mathsf{i}+\frac{1}{2}})$$

It has a global $\mathbb{Z}_2$ symmetry represented by tensor products of Pauli Z operators:



This model has two dualities:

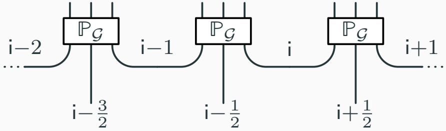
- Kramers-Wannier duality
- Jordan-Wigner transformation

Kramers-Wannier duality

Gauge the global $\mathbb{Z}_2$ symmetry: add $\mathbb{Z}_2$ gauge d.o.f. at integer sites in between matter d.o.f., subject to

$$\mathcal{G}_{i+\frac{1}{2}} := Z_{\mathsf{i}} Z_{\mathsf{i}+\frac{1}{2}} Z_{\mathsf{i}+1} \stackrel{!}{=} \mathbb{1}$$

Can be written as an MPO[1]:

$O_{\mathsf{KW}} =$ 

where $\mathbb{P}_{\mathcal{G}} = (\mathbb{1} + \mathcal{G})/2$.

[1]Haegeman, Van Acoleyen, Schuch, Cirac, Verstraete, PRX (2015)

Kramers-Wannier duality

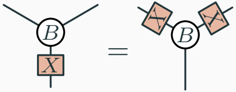
Acting on the Hamiltonian, we find

$$O_{\mathsf{KW}}\mathbb{H}_A = \mathbb{H}_B O_{\mathsf{KW}}$$

with

$$\mathbb{H}_B = -J\sum_{\mathsf{i}}(X_{\mathsf{i}} + gZ_{\mathsf{i}}Z_{\mathsf{i}+1}), \quad \text{compare to} \quad \mathbb{H}_A = -J\sum_{\mathsf{i}}(X_{\mathsf{i}-\frac{1}{2}}X_{\mathsf{i}+\frac{1}{2}} + gZ_{\mathsf{i}+\frac{1}{2}})$$

$\mathbb{H}_B$ is the Kramers-Wannier dual of $\mathbb{H}_B$, with dual global $\mathbb{Z}_2$ symmetry



Jordan-Wigner transformation

Mapping of spins to fermions:

$$S_\mathsf{i}^+ = \frac{1}{2}(X_\mathsf{i} + iY_\mathsf{i}) \mapsto K_\mathsf{i} c_\mathsf{i}^\dagger\,, \quad S_\mathsf{i}^- = \frac{1}{2}(X_\mathsf{i} - iY_\mathsf{i}) \mapsto K_\mathsf{i} c_\mathsf{i}\,,$$

where

$$K_\mathsf{i} = \exp\big(i\pi \sum_{\mathsf{j}=-\infty}^{\mathsf{i}-1} c_\mathsf{j}^\dagger c_\mathsf{j}\big)$$

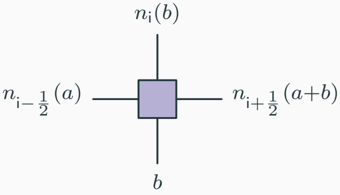
ensures correct commutation relations. Resulting Hamiltonian is

$$\mathbb{H}_C = -J \sum_\mathsf{i} \big(c^\dagger_{\mathsf{i}-\frac{1}{2}} c_{\mathsf{i}+\frac{1}{2}} + c^\dagger_{\mathsf{i}-\frac{1}{2}} c^\dagger_{\mathsf{i}+\frac{1}{2}} + \text{h.c.} - g(2c^\dagger_{\mathsf{i}+\frac{1}{2}} c_{\mathsf{i}+\frac{1}{2}} - 1)\big)$$

Jordan-Wigner transformation

Defining $|n_i(a)\rangle \equiv (c_i^\dagger)^{n(a)}|\varnothing\rangle$, we have $|n_i(a)\rangle|n_j(b)\rangle = (-1)^{ab}|n_j(b)\rangle|n_i(a)\rangle$. We propose the following MPO tensor for the Jordan-Wigner transformation:

$$\sum_{a,b=0,1}$$



$$\equiv \sum_{a,b=0,1} |n_{i-\frac{1}{2}}(a)\rangle|n_i(b)\rangle\langle n_{i+\frac{1}{2}}(a+b)|\langle b|$$

This tensor has even parity, and satisfies

$$\sum_{a,b} |n_{i-\frac{1}{2}}(a)\rangle|n_i(b)\rangle\langle n_{i+\frac{1}{2}}(a+b)|\langle b|X_i = \sum_{a,b} K_i(c_i^\dagger + c_i)|n_{i-\frac{1}{2}}(a)\rangle|n_i(b)\rangle\langle n_{i+\frac{1}{2}}(a+b+1)|\langle b|$$

Jordan-Wigner transformation

The evenness of these MPO tensors allows us to write

$$O_{\mathsf{JW}} X_{\mathsf{i}} = K_{\mathsf{i}}(c_{\mathsf{i}}^{\dagger} + c_{\mathsf{i}}) O'_{\mathsf{JW}}$$

where O'_{JW} has antiperiodic boundary conditions. This allows us to write

$$O_{\mathsf{JW}} \mathbb{H}_A = \mathbb{H}_C O_{\mathsf{JW}}$$

Any model with global $\mathbb{Z}_2$ symmetry admits these Kramers-Wannier and Jordan-Wigner dualities, and the MPOs implementing them are universal.

Examples

Recovering well known examples:

1. $\mathcal{D} = \mathsf{Vec}_{\mathbb{Z}_2}$: $\mathbb{Z}_2$ symmetry
   - $\mathcal{M} = \mathsf{Vec}$: transverse field Ising model
   - $\mathcal{M} = \mathsf{Vec}$: Kramers-Wannier dual
   - $\mathcal{M} = \mathsf{sVec}/\langle\psi \simeq \mathbb{1}\rangle$: free fermion
2. $\mathcal{D} = \mathsf{Ising}$: $\mathbb{Z}_2$ symmetry + Kramers-Wannier self-duality
   - $\mathcal{M} = \mathsf{Ising}$: critical transverse field Ising model
   - $\mathcal{M} = \mathsf{Ising}/\langle\psi \simeq \mathbb{1}\rangle$: massless free fermion
3. $\mathcal{D} = \mathsf{Ising}^{\boxtimes 2}$: $(\mathbb{Z}_2 + \text{Kramers-Wannier self-duality})^{\otimes 2}$
   - $\mathcal{M} = \mathsf{Ising}^2$: two decoupled critical transverse field Ising models
   - $\mathcal{M} = \mathsf{Ising}$: critical XY model
   - $\mathcal{M} = \mathsf{Ising}/\langle\psi \simeq \mathbb{1}\rangle$: massless Dirac fermion

Examples

More exotic exapmles:

1. $\mathcal{D} = \text{Rep}(\text{U}_q(\mathfrak{sl}_2))$: quantum deformed $SU(2)$ symmetry
    - $\mathcal{M} = \text{Rep}(\text{U}_q(\mathfrak{sl}_2))$: solid-on-solid (SOS) models
    - $\mathcal{M} = \text{Vec}$: 6-vertex model (XXZ)
2. $\mathcal{D} = \mathcal{H}_3$: exotic fusion category, "Haagerup subfactor"
    - $\mathcal{M} = \mathcal{H}_3$: ?[2][3]
    - $\mathcal{M} = \mathcal{M}_{3,2}$: ?
    - $\mathcal{M} = \mathcal{M}_{3,1}$: ?

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[2]Vanhove, LL, Van Damme, Wolf, Osborne, Haegeman, Verstraete, *A critical lattice model for a Haagerup conformal field theory* , 2110.03532

[3]Huang, Lin, Ohmori, Tachikawa, Tezuka, *Numerical evidence for a Haagerup conformal field theory*, 2110.03008

Conclusion

Quantum lattice models have 2 aspects:

Topological

- Symmetries, sectors
- (String) order parameters
- Dualities
- ...

Geometrical

- Correlation functions
- Criticality, scaling dimensions
- Integrability
- ...

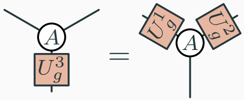
Outlook:

- **Higher dimensions**[4][5]

[4]Haegeman, Van Acoleyen, Schuch, Cirac, Verstraete, *Gauging quantum states*, PRX (2015)
[5]Delcamp, *Tensor network approach to electromagnetic duality in (3+1)d topological gauge models*, 2112.08324

Symmetric tensor networks

Take a three-leg tensor A that is symmetric under some (finite) group G



with U_g^i representations of G. These representations decompose into irreps as

$$U_g^i = \bigoplus_{j_i} D^{j_i}(g)$$

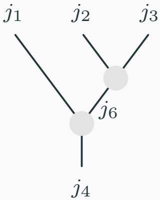
The Wigner-Eckart theorem then states that A must be built from Clebsch-Gordan coefficients:

$$A_{(j_1 m_1)(j_2 m_2)}^{(j_3 m_3)} = \bigoplus_{j_i \in U^i} A^{j_3}_{j_1 j_2} C^{j_1 j_2 j_3}_{m_1 m_2 m_3}$$

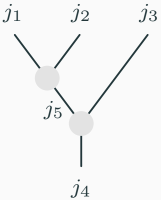
Clebsch-Gordan coefficients are recoupled using F-symbols:

$$\sum_{m_6} C^{j_2 j_3 j_6}_{m_2 m_3 m_6} C^{j_1 j_6 j_4}_{m_1 m_6 m_4} = \sum_{j_5, m_5} \left(F^{j_1 j_2 j_3}_{j_4}\right)^{j_6}_{j_5} C^{j_1 j_2 j_5}_{m_1 m_2 m_5} C^{j_1 j_6 j_4}_{m_1 m_6 m_4}$$

which up to a phase are the $6j$ symbols; as a picture,



$$= \sum_{j_5} \left(F^{j_1 j_2 j_3}_{j_4}\right)^{j_6}_{j_5}$$

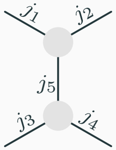


F-symbols allow arbitrary symmetric tensor network contractions;
Clebsch-Gordan coefficients only required to translate to normal tensors

Bond algebra

Application of this: consider the algebra of symmetric operators generated by

$$\mathbb{b}_{a,\mathsf{i}} \equiv \sum_{\{j_i\}} b_a(j_1, j_2, j_3, j_4, j_5)$$



that we refer to as the bond algebra, with elements

$$\{\mathbb{id}, \mathbb{b}_{a,\mathsf{i}}, \mathbb{b}_{b,\mathsf{j}}\mathbb{b}_{c,\mathsf{k}}, \mathbb{b}_{a,\mathsf{i}}\mathbb{b}_{b,\mathsf{j}}\mathbb{b}_{c,\mathsf{k}} \ldots\}$$

i.e. we consider all possible products of $\mathbb{b}_{a,\mathsf{i}}$ on different sites. This bond algebra contains all symmetric Hamiltonians.

Bond algebra

Going to some basis $\{\mathcal{O}_x\}$ of the bond algebra, we get the following operator product expansion:

$$\mathcal{O}_x \mathcal{O}_y = \sum_z f_{xy}^z(F) \mathcal{O}_z,$$

with a structure factor **that only depends on** F.

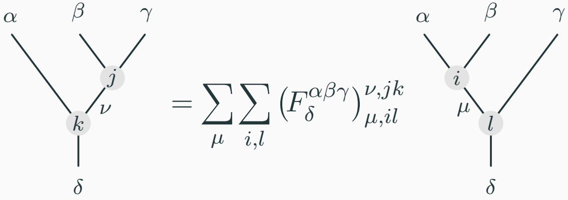
Claim: there exist distinct sets of "Clebsch-Gordan coefficients" that are recoupled by the same F-symbol, that generate isomorphic bond algebrass, which define **dual** Hamiltonians!

How do we find these different generalized Clebsch-Gordan coefficients?

Category theory

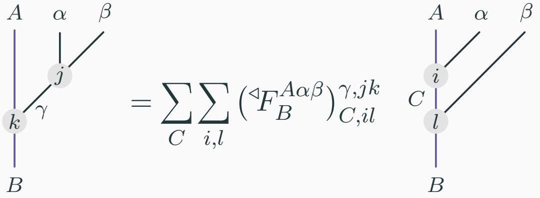
Fusion category $\mathcal{D}$

- Simple objects $\alpha, \beta, \gamma \in \mathcal{D}$
- Fusion rules: $\alpha \otimes \beta = \bigoplus_\gamma N_{\alpha\beta}^{\gamma}\gamma$
- F-symbol that generalize $6j$'s
- Usual pentagon: $FF = \sum FFF$



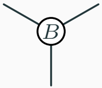
Right $\mathcal{D}$-module category $\mathcal{M}$

- Simple objects $A, B, C \in \mathcal{M}$
- Action rules: $A \triangleleft \alpha = \bigoplus_B N_{A\alpha}^{B} B$
- ${}^{\triangleleft}F$-symbol that generalize CG's
- Mixed pentagon: ${}^{\triangleleft}F{}^{\triangleleft}F = \sum F{}^{\triangleleft}F{}^{\triangleleft}F$



Generalized Clebsch-Gordan coefficients

Define generalized Clebsch-Gordan coefficients as



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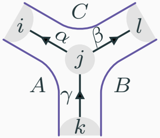
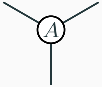


$$:= \left({}^{\triangleleft}F^{A\alpha\beta}_{B}\right)^{\gamma,jk}_{C,il} \Big/$$



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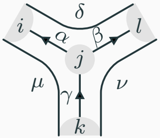
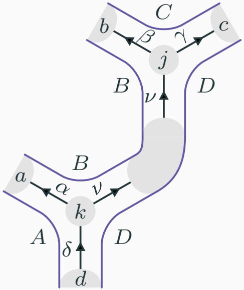
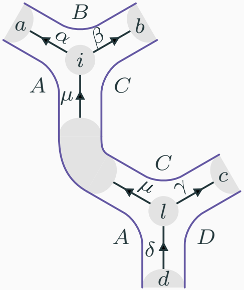


$$:= \left(F^{\mu\alpha\beta}_{\nu}\right)^{\gamma,jk}_{\delta,il}$$

Their recoupling condition is the mixed/usual pentagon equation:

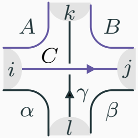


$$= \sum_{\mu} \sum_{i,l} \left(F^{\alpha\beta\gamma}_{\delta}\right)^{\nu,jk}_{\mu,il}$$



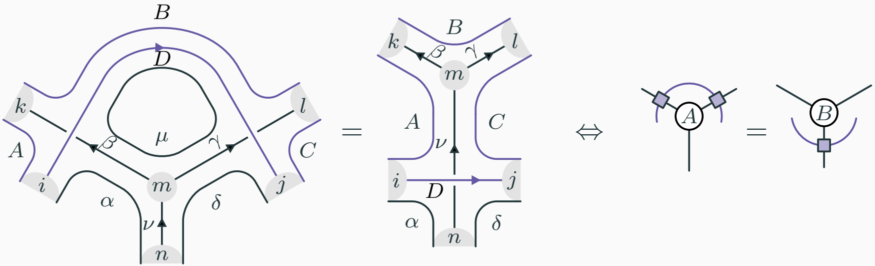
MPO intertwiners

Define the MPO intertwiner tensors as



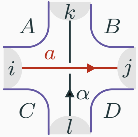
$$:= \left({}^{\triangleleft}F_{B}^{C\alpha\gamma} \right)_{A,ik}^{\beta,lj}$$

The intertwining condition is again the mixed pentagon equation:



MPO symmetries

Given some generalized CG coefficients, MPO symmetry tensors can be constructed as



$$= \left({}^{\bowtie}F^{aC\alpha}_{B}\right)^{A,jk}_{D,il}$$

where ${}^{\bowtie}F$ can be computed from F and ${}^{\triangleleft}F$, such that