Anyons, Flux Crystals and Emergent Fractionalized Excitations in Quantum Spin-Orbital Liquids

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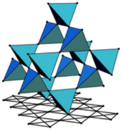


References for this talk

- '*Microscopic models for Kitaev's sixteenfold way of anyon theories*', **SC**, Urban F. P. Seifert, Matthias Vojta, Lukas Janssen, and Hong-Hao Tu, Phys. Rev. B **102**, 201111(R) (2020)
- '*Flux crystals, Majorana metals, and flat bands in exactly solvable spin-orbital liquids*', **SC**, Lukas Janssen, Matthias Vojta, Hong-Hao Tu, and Urban F. P. Seifert, Phys. Rev. B **103**, 075144 (2021)
- '*Fractionalized Fermionic Quantum Criticality in Spin-Orbital Mott Insulators*', Urban F.P. Seifert, Xiao-Yu Dong, **SC**, Matthias Vojta, Hong-Hao Tu, and Lukas Janssen, Phys. Rev. Lett. **125**, 257202 (2020)

Outline

Kitaev Honeycomb Model

QSOLs: Anyons

QSOLs: Flux Crystals

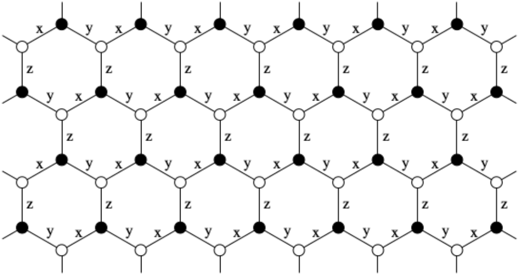
- Onsite spin magnetic field
- Nearest neighbour interactions

QSOLs: Emergent Fractionalized Quasiparitcles

Kitaev honeycomb model

- Model Hamiltonian [1]

$$H = -J_x \sum_{x\text{-links}} \sigma_j^x \sigma_k^x - J_y \sum_{y\text{-links}} \sigma_j^y \sigma_k^y - J_z \sum_{z\text{-links}} \sigma_j^z \sigma_k^z$$



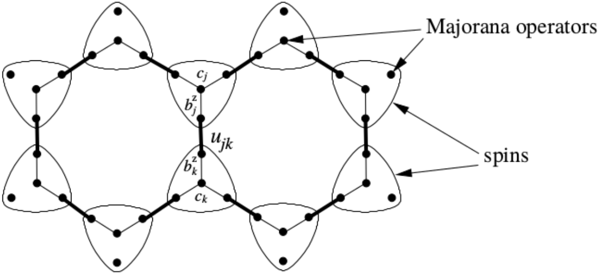
[1] Kitaev. "Anyons in an exactly solved model and beyond". 2006. *Ann. Phys.* 321 1

Kitaev honeycomb model: solution

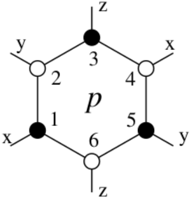
- Majorana decomposition: $\sigma^\alpha = ib^\alpha c$ for $\alpha = x, y, z$

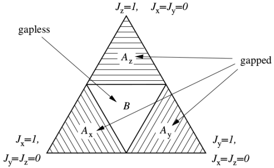
$$\tilde{H} = \sum_{\langle jk \rangle_\alpha} iJ_\alpha u_{jk} c_j c_k$$

where $u_{jk} = ib_j^\alpha b_k^\alpha$ is a Z_2 gauge field which commutes with $\tilde{H}$.
Fermion parity constraint: $b^x b^y b^z c = 1$



Loop operators and phase diagram





Phase Diagram

$$W_p = \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z = \prod_{\langle jk \rangle \in p} (-iu_{jk})$$

Ground state flux configuration is zero flux (Lieb's theorem), i.e.,

$$W_p = +1 \ \forall p.$$

Gapless phase features Dirac excitations.

Anyonic excitations

- In the gapless phase, on adding appropriate weak magnetic field peturbation, Kitaev showed that there exist non-Abelian anyonic excitations ($W_p = -1$) characterized by an anyon topological spin of $\theta_a = \pi/8$ and a Chern number $\nu = \pm 1$.

Anyonic qp	CFT primary field a	h_a	$\theta_a = 2\pi h_a$
Vacuum	$\mathbb{I}$	0	0
Fermion	ψ	1/2	π
Non-Abelian Anyon	σ	1/16	$\pi/8$

Kitaev's sixteenfold way is a classification of topological orders of a Z_2 gauge theory coupled to free or weakly interacting fermions (gapped spectrum) with a spectral Chern number of ν.
Classification: ν mod 16 into sixteen distinct types.

Why quantum spin-orbital models?

- Spin-orbital models were first introduced in the context of Kugel-Khomskii-type models for transition metal oxides (Kugel and Khomskii 1982).
- $\mathrm{SU}(4)$-symmetric point in the parameter space could possibly explain the disordered ground state in the spin-orbital system $\mathrm{Ba_3CuSb_2O_9}$ observed experimentally (Nakatsuji et al., Science **336** (2012)).
- Frustrated inter-orbital interactions: The double perovskite $\mathrm{Ba_2YMoO_6}$, which has effective $j_{\mathrm{eff}} = 3/2$ moments as a result of degenerate t_{2g} orbitals and spin-orbit coupling, and does not order down to low temperatures (Vries et al. PRL **104** (2010)).

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Main Idea

- Combining systems of Chern number 1 (in some non-trivial way) to obtain systems of higher Chern number.
- A useful mathematical tool for achieving this is the Clifford Algebra of order n.

$$\{\Gamma^\alpha, \Gamma^\beta\} = 2\delta_{\alpha\beta} \qquad \Gamma^{\alpha\beta} = \frac{i}{2}[\Gamma^\alpha, \Gamma^\beta]$$

  where α, β can take any value form 1 to $2n-1$. In the simplest representation, Γ matrices are $2^{n-1} \times 2^{n-1}$ matrices. It has a Majorana representation [2] as follows:

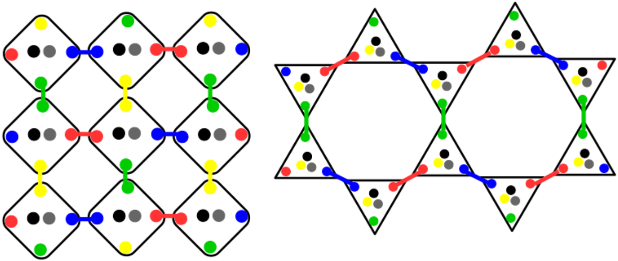
$$\Gamma^\alpha = ib^\alpha c \qquad \Gamma^{\alpha\beta} = ib^\alpha b^\beta$$

  Thus there are $2n$ Majorana operators.
- Clifford Algebra of order 2 is satisfied by Pauli matrices.

[2] Wu, Arovas **and** Hung. "Γ-matrix generalization of the Kitaev model". 2009. *Phys. Rev. B* 79

A possible way of achieving $\nu = 2, 3$

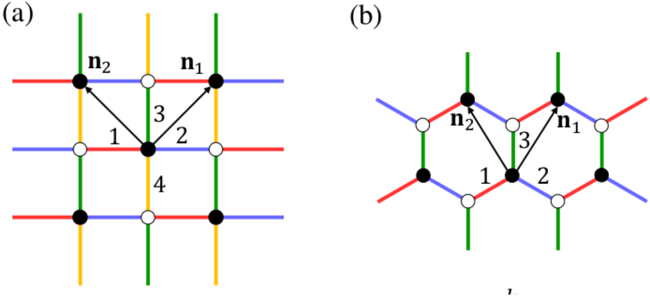
- For $\nu = 2, 3$ we use Clifford Algebra of order 3, which has 5 4×4 Γ matrices and requires six Majorana fermions.



- Coloured dots $\rightarrow$ Majorana operators, Black rounded square and triangle $\rightarrow$ spins. Left diagram could corresponds to $\nu = 2$ since it has two itinerant Majorana modes. Similarly right to $\nu = 3$.

Lattice Systems

- Thus, in order to realize sixteenfold way models we need to use square and honeycomb lattice for even and odd ν respectively.



Model

- Model Hamiltonian (for Chern number ν):

$$H = -\sum_{\langle ij\rangle_\gamma} J_\gamma \left(\Gamma_i^\gamma \Gamma_j^\gamma + \sum_{\beta=\gamma_\mathrm{m}+1}^{2q+3} \Gamma_i^{\gamma\beta} \Gamma_j^{\gamma\beta} \right),$$

  where $\nu = 2q$ $(2q+1)$ and $\gamma_\mathrm{m} = 4$ (3) for square (honeycomb) lattice.
- In Majorana representation, we have:

$$\tilde{H} = \sum_{\langle ij\rangle_\gamma} J_\gamma u_{ij} \left(i c_i c_j + \sum_{\beta=\gamma_\mathrm{m}+1}^{2q+3} i b_i^\beta b_j^\beta \right),$$

- Itinerant Majorana modes decouple once we fix the static Z_2 gauge field configuration.
- Fermion parity constraint:

$$D_j \equiv i^{q+2} b_j^1 b_j^2 \dots b_j^{2q+3} c_j = -1,$$

Anyons in quantum spin-orbital liquid

$$H = -\sum_{\langle ij \rangle_\gamma} J_\gamma (\vec{\sigma}_i \cdot \vec{\sigma}_j) \otimes (\tau_i^\gamma \tau_j^\gamma), \qquad (1)$$

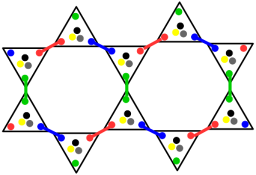
Spin-orbital model on a honeycomb lattice with Heisenberg coupling in the spin sector and a Kitaev coupling in the orbital sector (Yao and Lee, PRL **107** (2011)).

Four dimensional representation of Clifford Algebra:

$(\Gamma^\alpha)_{\alpha=1,\ldots,5} = (\sigma^y \otimes \tau^x, \sigma^y \otimes \tau^y, \sigma^y \otimes \tau^z, \sigma^x \otimes \mathbb{1}, \sigma^z \otimes \mathbb{1})$.

Majorana representation: $\Gamma^\alpha = ib^\alpha c$.

$$\tilde{H} = \sum_{\langle ij \rangle_\gamma} J_\gamma u_{ij} \left(ic_i c_j + ib_i^4 b_j^4 + ib_i^5 b_j^5 \right),$$



3 Dirac cones in the gapless B phase.

$\Longrightarrow \nu = 3$ after breaking TRS.

Anyons in quantum spin-orbital liquid

Anyonic qp	CFT primary field a	h_a	$\theta_a = 2\pi h_a$
Vacuum	$\mathbb{I}$	0	0
Fermion	ψ	1/2	π
Non-Abelian Anyon	σ	1/16	$\pi/8$

Table: Kitaev Spin Liquid

Anyonic qp	CFT primary field a	h_a	$\theta_a = 2\pi h_a$
Vacuum	$\mathbb{I}$	0	0
Fermion	ψ	1/2	π
Non-Abelian Anyon	σ	3/16	$3\pi/8$

Table: $\nu = 3$ Quantum Spin-Orbital Liquid

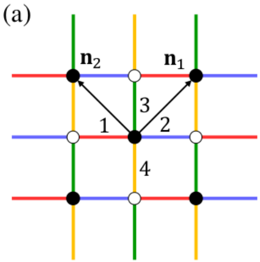
Odd ν theories have non-abelian anyons while even ν theories have abelian anyons. (For more details, see Phys. Rev. B **102**, 201111(R) (2020)).

Spin-orbital liquid on a square lattice with $\nu = 2$

A spin-orbital model on the square lattice with an XY coupling in the spin sector and a Kitaev coupling in the orbital sector (Nakai et al., PRB **85** (2012)).

$$H = -\sum_{\langle ij \rangle_\gamma} J_\gamma (\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y) \otimes (\tau_i^\gamma \tau_j^\gamma),$$

$$(\tau^\gamma)_{\gamma=1,\ldots,4} = (\tau^x, \tau^y, \tau^z, \mathbb{1}).$$



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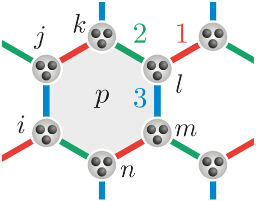
Onsite spin magnetic field

(Chulliparambil et al., Phys. Rev. B **103**, 075144 (2021))

Plaquette operators ($\mathbb{Z}_2$ flux operator): $W_p = \mathbb{1} \otimes \tau_i^x \tau_j^y \tau_k^z \tau_l^x \tau_m^y \tau_n^z$.
Coupling only spin to magnetic field.

$$\mathcal{H}_h^{(3)} = -\vec{h} \cdot \sum_i \vec{\sigma}_i \otimes \mathbb{1}.$$

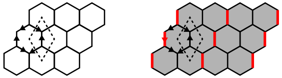


Majorana representation:

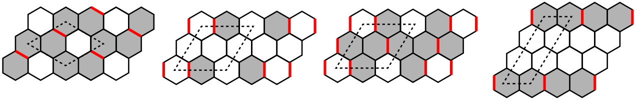
$$\tilde{\mathcal{H}}_h^{(3)} = \sum_i \left(h^x c_i^y c_i^z + h^y c_i^z c_i^x + h^z c_i^x c_i^y \right).$$

Dispersion relation depends only on $|\vec{h}|$. Ground state flux configuration?

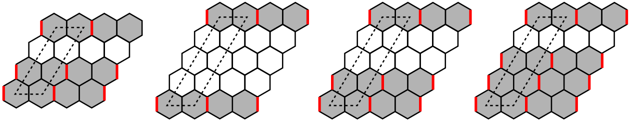
Onsite spin magnetic field: Ground state flux configuration



(a) Flux free **(b)** π flux **(c)** 1/2-flux stripy **(d)** 1/3-flux crystal



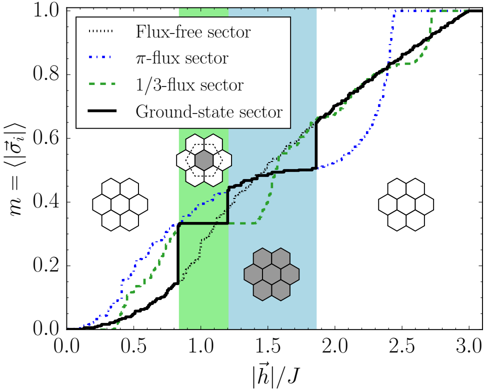
(e) 2/3-flux crystal **(f)** 1/4-flux crystal **(g)** 3/4-flux crystal **(h)** 1/3-flux stripy

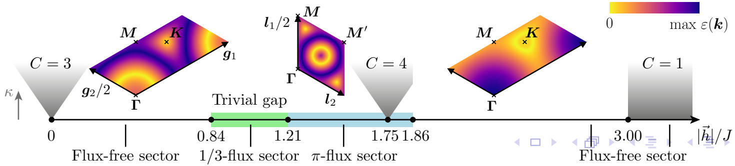


(i) 2/3-flux stripy **(j)** 1/4-flux stripy **(k)** 2/4-flux stripy **(l)** 3/4-flux stripy

Onsite spin magnetic field: Average magnetization per site

Majorana fermi surfaces, Dirac cones,..





Nearest neighbour interactions:

1. Spatially isotropic $\bar{\Gamma}$ interaction

$$\mathcal{H}_{\bar{\Gamma}}^{(3)} = \bar{\Gamma} \sum_{\langle ij \rangle_\gamma} \left[\sigma_i^\alpha \sigma_j^\beta + \sigma_i^\beta \sigma_j^\alpha + \sigma_i^\gamma \sigma_j^\alpha + \sigma_i^\alpha \sigma_j^\gamma + \sigma_i^\gamma \sigma_j^\beta + \sigma_i^\beta \sigma_j^\gamma \right] \otimes \tau_i^\gamma \tau_j^\gamma,$$

Majorana representation:

$$\tilde{\mathcal{H}}_J^{(3)} + \tilde{\mathcal{H}}_{\bar{\Gamma}}^{(3)} = \sum_{\langle ij \rangle} u_{ij} \left[J \sum_\alpha c_i^\alpha c_j^\alpha - \bar{\Gamma} \sum_{\alpha<\beta} \left(c_i^\alpha c_j^\beta + c_i^\beta c_j^\alpha \right) \right].$$

Using appropriate unitary transformation:

$$\mathcal{H}^{(3)} + \mathcal{H}_{\bar{\Gamma}}^{(3)} = \sum_{\langle ij \rangle_\mu} i u_{ij} \big((J - 2\bar{\Gamma}) d_i^1 d_j^1 + (J + \bar{\Gamma})(d_i^2 d_j^2 + d_i^3 d_j^3) \big),$$

Majorana flat bands at $J = 2\bar{\Gamma}$ and $J = -\bar{\Gamma}$!

Nearest neighbour interactions:

2. Bond-dependent diagonal K interaction

$$\mathcal{H}_K^{(3)} = \sum_{\langle ij\rangle_\gamma} \Big[-K\sigma_i^\gamma \sigma_j^\gamma \Big] \otimes \tau_i^\gamma \tau_j^\gamma,$$

Majorana representation:

$$\mathcal{H}_J^{(3)} + \mathcal{H}_K^{(3)} = \sum_{\langle ij\rangle_\gamma} iu_{ij} \Big[(J+K) c_i^\gamma c_j^\gamma + \sum_{\beta\neq\gamma} J c_i^\beta c_j^\beta \Big].$$

- Finite K spoils $SO(3)$ symmetry. As $K/J \to \infty$, γ-type Majoranas are localized on γ-type bonds leading to gapped dispersion and flat bands.
- $\epsilon^\alpha(\vec{k}) = \pm|f(\vec{k}) + 2Ke^{-i\vec{k}\cdot\delta_\alpha}|$ with $\delta_\alpha = \vec{n}_1, \vec{n}_2, 0$ for $\alpha = x, y, z$ respectively.

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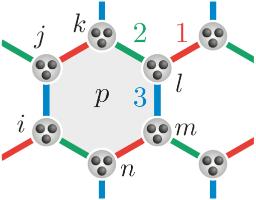
Néel antiferromagent on a honeycomb lattice

Seifert et al. PRL **125**,257202 (2020)

$$\text{QSOL:} \qquad H_K = -\sum_{\langle ij\rangle_\gamma} J_\gamma(\vec{\sigma}_i \cdot \vec{\sigma}_j) \otimes (\tau_i^\gamma \tau_j^\gamma).$$

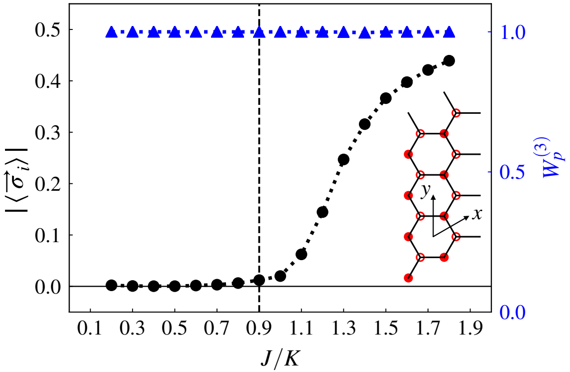
$W_p = \mathbb{1} \otimes \tau_i^x \tau_j^y \tau_k^z \tau_l^x \tau_m^y \tau_n^z$.
Antiferromagnetic
interactions in the spin degree of freedom :

$$H_J = J \sum_{\langle ij\rangle} \vec{\sigma}_i \cdot \vec{\sigma}_j \otimes \mathbb{1}_i \mathbb{1}_j, \qquad J > 0.$$



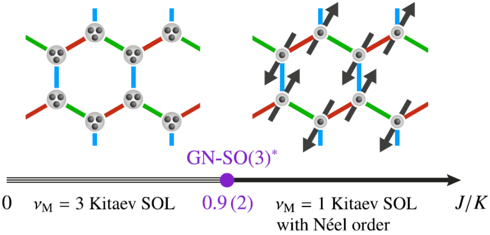
$[H_J, W_p] = 0 \; \forall \; p, \implies$ static
$\mathbb{Z}_2$ gauge fields. In the Majorana form,
$H_J \mapsto \frac{J}{4} \sum_{\langle ij\rangle} (c_i^\top \vec{L} c_i) \cdot (c_j^\top \vec{L} c_j)$, with the $SO(3)$ generators
$L^\alpha_{\beta\gamma} = -i\epsilon^{\alpha\beta\gamma}$. Not exactly solvable!

Continuous phase transition



- For $J << K$ semimetal with three flavours of gapless Dirac excitations corresponding to $\nu = 3$ Kitaev spin-orbital liquid.
- For $J >> K \implies \langle\vec{n}\rangle = \langle\vec{\sigma}_{i,A} - \vec{\sigma}_{j,B}\rangle/2 \neq 0$, a flavour of gapless Dirac excitation ($\nu = 1$ Kitaev spin-orbital liquid) along with anti-ferromagnetic Néel order.

Fractionalized fermionic quantum criticality



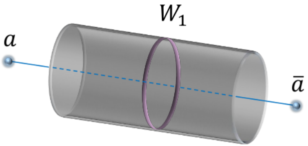
The criticality was studied using $\epsilon-$expansion, large N methods and DMRG to reveal Gross-Neveu-$SO(3)$* criticality. (Phys. Rev. Lett. **125**, 257202 (2020)).

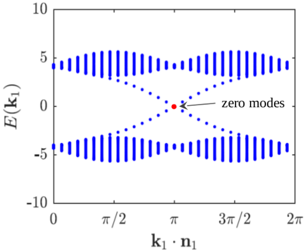
Thank you for your attention..

- QSOLs, in addition to Kitaev Spin Liquids are also excellent starting points for the study of anyons (both Abelian and non-Abelian) and exotic topological orders (Phys. Rev. B **102**, 201111(R) (2020)).
- On adding onsite magnetic fields and nearest neighbour interactions, we observe first order phase transition into non-trivial flux crystals featuring Majorana fermi surfaces, quadratic band touching points and flat bands (Phys. Rev. B **103**, 075144 (2021)).
- Nearest neighbour antiferromagnetic spin interactions lead to novel continuous phase transition characterized by novel Gross Neveu* quantum criticalities (Phys. Rev. Lett. **125**, 257202 (2020)).

Edge States and Topological Spin of Anyons

On solving the QSOL models on a cylinder one can show that there exist exactly ν pairs of gapless Majorana edge modes in the gapless phase in presence of weak magnetic field perturbation that gaps the system.





Types of Anyonic Quasiparticles [3]

Anyonic qp	CFT primary field a	h_a	$\theta_a = 2\pi h_a$
Vacuum	$\mathbb{I}$	0	0
Fermion	ψ	1/2	π
Non-Abelian Anyon	σ	$\nu/16$	$\pi\nu/8$

Table: Odd ν theories and $SO(\nu)_1$ CFT

Anyonic qp	CFT primary field a	h_a	$\theta_a = 2\pi h_a$
Vacuum	$\mathbb{I}$	0	0
Fermion	ψ	1/2	π
Abelian Anyons	λ_1, λ_2	$\nu/16$	$\pi\nu/8$

Table: Even ν theories and $SO(\nu)_1$ CFT

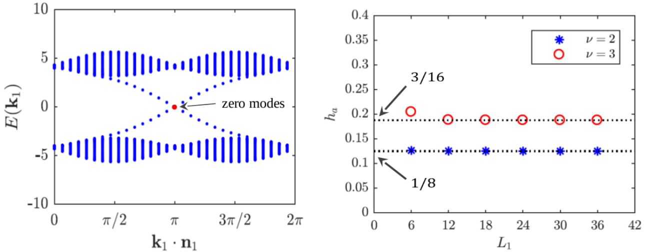
[3] Francesco, Mathieu **and** Sénéchal. *Conformal Field Theory*. 1997

Topological Spin of Anyon Quasiparticle

- Contribution to energy from edge states:

$$E_a - E_0 = \frac{2\pi v}{L_1}(h_a + h_{\bar{a}})$$

- The topological spin of the anyonic quasiparticle $\theta = 2\pi h_a$ [1]; [4].
- Our results match with the CFT predicted value of anyonic topological spin of $\theta = \nu\pi/8$.



[4] Tu, Zhang **and** Qi. "Momentum polarization: An entanglement measure of topological spin and chiral central charge". 2013. *Phys. Rev. B* 88

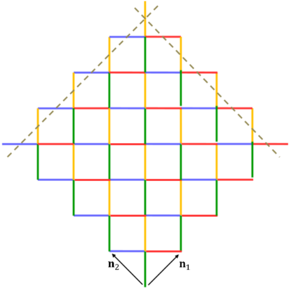
Topological Ground State Degeneracy On A Torus

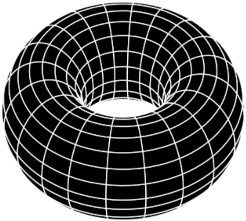
- Four candidate degenerate states:

$$Q|\Psi_F(\{u_0^{\pm\pm}\})\rangle \otimes |\{u_0^{\pm\pm}\}\rangle = |\Psi_F(\{u_0^{\pm\pm}\})\rangle \otimes |\{u_0^{\pm\pm}\}\rangle,$$

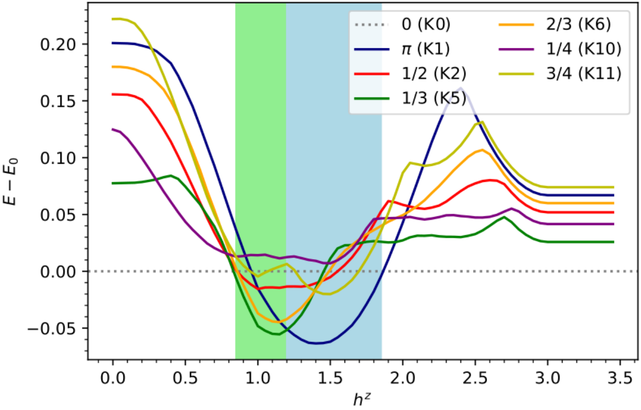
- Fermion parity

$$\begin{aligned}
Q|\Psi_F(\{u_0^{--}\})\rangle \otimes |\{u_0^{--}\}\rangle &= |\Psi_F(\{u_0^{--}\})\rangle \otimes |\{u_0^{--}\}\rangle, \\
Q|\Psi_F(\{u_0^{-+}\})\rangle \otimes |\{u_0^{-+}\}\rangle &= |\Psi_F(\{u_0^{-+}\})\rangle \otimes |\{u_0^{-+}\}\rangle, \\
Q|\Psi_F(\{u_0^{+-}\})\rangle \otimes |\{u_0^{+-}\}\rangle &= |\Psi_F(\{u_0^{+-}\})\rangle \otimes |\{u_0^{+-}\}\rangle, \\
Q|\Psi_F(\{u_0^{++}\})\rangle \otimes |\{u_0^{++}\}\rangle &= (-1)^{\nu}|\Psi_F(\{u_0^{++}\})\rangle \otimes |\{u_0^{++}\}\rangle
\end{aligned}$$





Onsite spin magnetic field on honeycomb: Ground state flux configuration



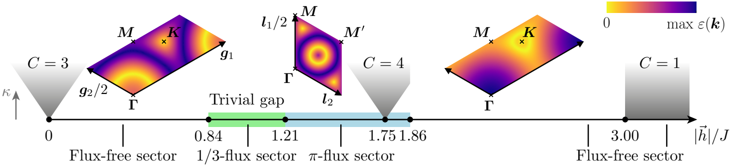
Onsite spin magnetic field: Flux free sector ($|\vec{h}|/J < 0.84$)

- At $\vec{h} = 0$, $C = 3$.
- As we increase $|\vec{h}|$, two of the three Dirac cones are shifted away from zero energy $\rightarrow$ Fermi surface. (Majorana metal!)

$$\varepsilon_{1,2}(\vec{k}) = 2|\vec{h}| \pm |f(\vec{k})|, \quad \varepsilon_{3,4}(\vec{k}) = -2|\vec{h}| \pm |f(\vec{k})|,$$
$$\text{and} \quad \varepsilon_{5,6}(\vec{k}) = \pm|f(\vec{k})|.$$

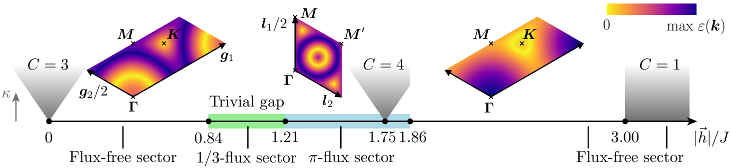
where $f(\vec{k}) = 2J(1 + e^{i\vec{k}\cdot\vec{n}_1} + e^{i\vec{k}\cdot\vec{n}_2})$, where $\vec{n}_{1,2} = (\pm\frac{1}{2}, \frac{\sqrt{3}}{2})$ are the honeycomb lattice vectors.

- $C = 1$.



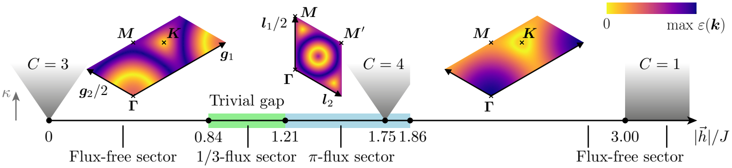
Onsite spin magnetic field: 1/3- flux sector

- Trivial gap for all three Majoranas.



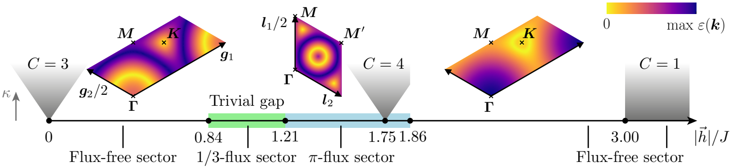
Onsite spin magnetic field: π- flux sector

- Gapless spectrum features two Dirac cones and a Fermi surface resulting in $C = 2$.
- This Fermi surface is formed by the intersection of a Dirac node at $\boldsymbol{M}'/2 = (\pi, \pi/\sqrt{3})/2$, centered at some nonzero elevated energy with its particle-hole-symmetric counterpart.
- At $|\vec{h}| = 1.75J$, at which the Fermi surface shrinks to an isolated point.
- All four Dirac cones now give rise to $C = 4$.



Onsite spin magnetic field: Flux free sector for large $|\vec{h}|/J$

- At $|\vec{h}| = 3J$ two Fermi surfaces shrink to isolated points with quadratic dispersion at the $\Gamma = (0, 0)$ point in BZ/2.
- At $h^z > 3J$, two of the three bands are completely filled and empty, respectively, and only the Dirac cone at $\boldsymbol{K}$ remains, yielding $C = 1$.



Nearest neighbour interactions on a honeycomb lattice:

3. Bond-dependent off-diagonal Γ interaction

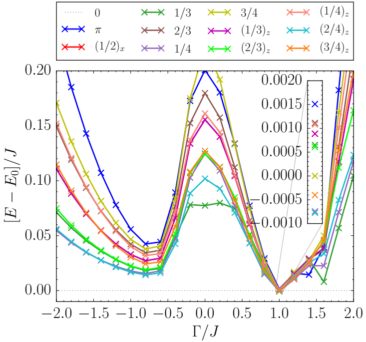
$$\mathcal{H}_\Gamma^{(3)} = \sum_{\langle ij \rangle_\gamma} \Gamma \Big[\Big(\sigma_i^\gamma \sigma_j^\alpha + \sigma_i^\alpha \sigma_j^\gamma + \sigma_i^\gamma \sigma_j^\beta + \sigma_i^\beta \sigma_j^\gamma \Big) \Big] \otimes \tau_i^\gamma \tau_j^\gamma, \qquad (2)$$

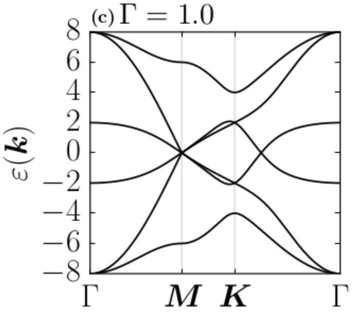
Majorana representation:

$$\mathcal{H}_J^{(3)} + \mathcal{H}_\Gamma^{(3)} = \sum_{\langle ij \rangle_{\alpha(\beta\gamma)}} i u_{ij} \left[J c_i^\alpha c_j^\alpha - \Gamma (c_i^\beta c_j^\gamma + c_i^\gamma c_j^\beta) \right]. \qquad (3)$$

- 0-flux is the ground state flux configuration except at $\Gamma / J = 1$ where several other fluxes are close in energy.

Ground state flux configuration study and nodal lines at $\Gamma = 1.0$:





Nearest neighbour interactions on a honeycomb lattice:

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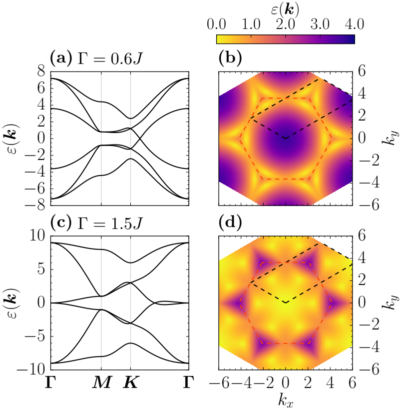

The spectrum becomes fully gapped at $\Gamma = 1.6J$.

QSOL on a square lattice

This model has a chiral gapless quantum spin orbital liquid with $\nu = 2$ when $J_\gamma = 1$ ((Phys. Rev. B **102**, 201111(R) (2020))).

$$H = -\sum_{\langle ij\rangle_\gamma} J_\gamma(\sigma_i^x\sigma_j^x + \sigma_i^y\sigma_j^y)\otimes(\tau_i^\gamma\tau_j^\gamma),$$

where $\gamma = 1,2,3,4$ denotes the four inequivalent bonds in a two-site unit cell and $(\tau^\gamma) = (\tau^x, \tau^y, \tau^z, \mathbb{1})$ for $\nu = 3$.

$$\tilde{H} = \sum_{\langle ij\rangle_\gamma} J_\gamma u_{ij}\left(ic_i^x c_j^x + ic_i^y c_j^y\right),$$

Conserved Plaquette operators:

$$W_p = \sigma_k^z\sigma_n^z \otimes \tau_i^x\tau_j^y\tau_k^x\tau_n^y,$$
$$W_{p'} = \sigma_k^z\sigma_n^z \otimes \tau_k^y\tau_l^x\tau_m^y\tau_n^x.$$

Flux crystals on a square lattice: Onsite spin magnetic field

These terms couple the spin degree of freedom to magnetic field [5].

$$\mathcal{H}_h^{(2)} = -h^z \sum_i \sigma_i^z \otimes \mathbb{1}.$$

Majorana representation:

$$\tilde{\mathcal{H}}_h^{(2)} = h^z \sum_i c_i^x c_i^y,$$

[5] Chulliparambil **andothers**. "Flux crystals, Majorana metals, and flat bands in exactly solvable spin-orbital liquids". 2021. *Phys. Rev. B* 103

Onsite spin magnetic field: Average magnetization per site