Partition function of the Levin-Wen model

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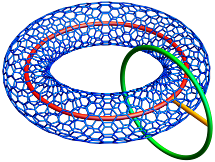

in collaboration with:

- Anna Ritz-Zwilling and Jean-Noël Fuchs (LPTMC, Paris)
- Benoît Douçot (LPTHE, Paris)
- Steven H. Simon (Rudolf Peierls Centre for Theoretical Physics, Oxford)

J. Vidal, Phys. Rev. B **105**, L041110 (2022) / arXiv:2108.13425

Topological quantum order in condensed matter in three dates

- 1989 : High-T_c superconductors, FQHE (X.-G. Wen, F. Wilczek, A. Zee)
- 1997 : Fault-tolerant quantum computation (A. Kitaev, J. Preskill)
- 2005 : String-net condensation (M. Levin, X.-G. Wen)

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Concepts coming from

- Lattice gauge theories (Wegner, Wilson, Kogut)
- Conformal field theories (Pasquier, Verlinde, Moore, Seiberg)
- Topological quantum field theories (Witten)
- Knot theory (Kauffman, Jones, Turaev, Viro)

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In this talk

Two-dimensional gapped quantum systems with:

- Topology-dependent ground-state degeneracy
- Anyonic excitations

Outline

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The Levin-Wen model in a nutshell

The Levin-Wen (string-net) model

- Microscopic lattice model — M. Levin and X.-G. Wen, Phys. Rev. B **71**, 045110 (2005)
- Trivalent graph (honeycomb lattice, two-leg ladder,...)

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Input: Unitary Fusion Category $\mathcal{C}$

- Set of objects (strings, labels, charges, superselection sectors, particles...)
- Fusion rules: $a \times b = \sum_c N_c^{ab} c$
- Associativity of the fusion rules $\rightarrow$ F-symbols

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Output: Unitary Modular Tensor Category D($\mathcal{C}$) (Drinfel'd center)

- Doubled achiral topological phase
- Other objects, other fusion rules, other...
- Fusion $+$ Braiding $+$ Modularity $\rightarrow$ Anyon theory

The Levin-Wen model in a nutshell

Local constraints and Hilbert space

- Degrees of freedom are the objects of $\mathcal{C}$
- Degrees of freedom are defined on the links of the trivalent graph
- Hilbert space $\mathcal{H}$: set of configurations respecting fusion rules at each vertex

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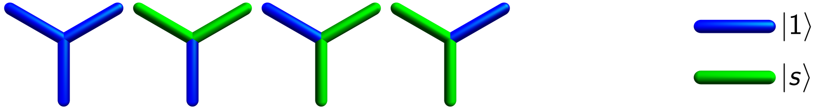
Ex: $\mathbb{Z}_2$ category

- Two labels: $\{1, s\}$
- Fusion rules: $1 \times 1 = 1$, $1 \times s = s$, $s \times s = 1$



Hilbert space for N_v trivalent vertices on any graph

- Dim $\mathcal{H} = 2^{\frac{N_\text{v}}{2}+1}$

The Levin-Wen model in a nutshell

Local constraints and Hilbert space

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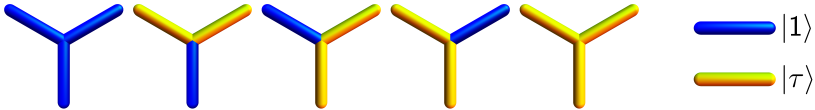
- Hilbert space $\mathcal{H}$: set of configurations respecting fusion rules at each vertex

Ex: Fibonacci category

- Two objects: $\{1, \tau\}$
- Fusion rules: $1 \times 1 = 1$, $1 \times \tau = \tau$, $\tau \times \tau = 1 + \tau$



Hilbert space dimension for any trivalent graph with N_v vertices

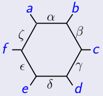
- $\dim \mathcal{H} = (1+\varphi^2)^{\frac{N_\mathrm{v}}{2}} + (1+\varphi^{-2})^{\frac{N_\mathrm{v}}{2}}$, $\quad \varphi = \frac{1+\sqrt{5}}{2}$ (golden ratio)

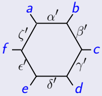
The Levin-Wen model in a nutshell

The string-net Hamiltonian

$$H = -\sum_p P_p$$

- H is a local commuting projector Hamiltonian: $[P_p, P_{p'}] = 0$
- P_p: projector onto the vacuum of $\mathrm{D}(\mathcal{C})$ in the plaquette p

P_p 

$= \sum_s \frac{d_s}{D^2} \sum{}' F^{a\zeta\alpha}_{s\alpha'\zeta'} F^{b\alpha\beta}_{s\beta'\alpha'} F^{c\beta\gamma}_{s\gamma'\beta'} F^{d\gamma\delta}_{s\delta'\gamma'} F^{e\delta\epsilon}_{s\epsilon'\delta'} F^{f\epsilon\zeta}_{s\zeta'\epsilon'}$ 

- d_s: quantum dimension of the string $s \in \mathcal{C}$
- $D = \left(\sum_s d_s^2\right)^{1/2}$: total quantum dimension of $\mathcal{C}$

M. Levin and X.-G. Wen, Phys. Rev. B **71**, 045110 (2005)

Outline

Ground-state degeneracy

On a genus-g orientable compact surface (sphere, torus,...)

- Ground-state degeneracy: $\mathcal{D}_0 = \sum_{i \in \mathrm{D}(\mathcal{C})} \left(\frac{d_i}{D^2} \right)^{\chi}$ =Turaev-Viro invariant
- D^2: total quantum dimension of $\mathrm{D}(\mathcal{C})$
- Euler-Poincaré characteristic: $\chi = 2 - 2g$
- $g = 0$: $\mathcal{D}_0 = 1$
- $g = 1$: $\mathcal{D}_0 =$ Number of objects in $\mathrm{D}(\mathcal{C})$
- $g \geqslant 2$: $\mathcal{D}_0$ depends (non trivially) on $\mathrm{D}(\mathcal{C})$

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F. J. Burnell and S. H. Simon, Ann. Phys. **325**, 2550 (2010)

V. G. Turaev and O. Y. Viro, Topology **31**, 865 (1992)

G. Moore and N. Seiberg, Comm. Math. Phys. **123**, 177 (1989)

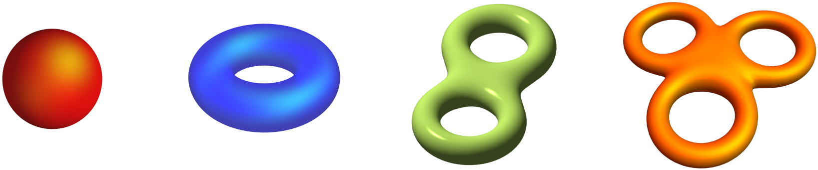
E. Verlinde, Nucl. Phys. B **300**, 360 (1988)

Ground-state degeneracy

Ex: $\mathbb{Z}_2$ category (commutative, braided, non-modular)

- Objects of $\mathcal{C}$: $\{1, s\}$
- $d_1 = 1,\ d_s = 1$
- Objects of $\mathrm{D}(\mathcal{C})$: $\{(1,1),(1,s),(s,1),(s,s)\}$
- $d_{(i,j)} = d_i\, d_j$

$g = 0$	$g = 1$	$g = 2$	$g = 3$



$\mathcal{D}_0 = 1$	$\mathcal{D}_0 = 4$	$\mathcal{D}_0 = 16$	$\mathcal{D}_0 = 64$

Ground-state degeneracy

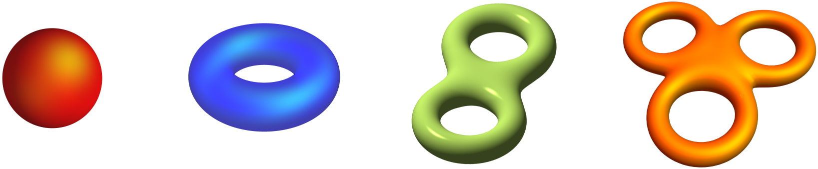
Ex: Fibonacci category (commutative, braided, modular)

- Objects of $\mathcal{C}$: $\{1, \tau\}$
- $d_1 = 1$, $d_\tau = \frac{1+\sqrt{5}}{2}$
- Objects of $\mathrm{D}(\mathcal{C})$: $\{(1,1),(1,\tau),(\tau,1),(\tau,\tau)\}$
- $d_{(i,j)} = d_i\, d_j$

M. D. Schulz, S. Dusuel, K. P. Schmidt, and J. Vidal, Phys. Rev. Lett. **110**, 147203 (2013)

Y. Hu, S. D. Stirling, and Y.-S. Wu, Phys. Rev. B **85**, 075107 (2012)

$g = 0$	$g = 1$	$g = 2$	$g = 3$



$\mathcal{D}_0 = 1$	$\mathcal{D}_0 = 4$	$\mathcal{D}_0 = 25$	$\mathcal{D}_0 = 225$

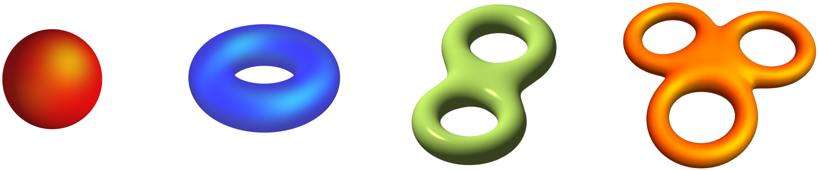
Ground-state degeneracy

Ex: Rep(S_3) category (commutative, braided, non-modular)

- Objects of $\mathcal{C}$: $\{1, 2, 3\}$
- $d_1 = 1,\ d_2 = 1,\ d_3 = 2$
- Objects of $\mathrm{D}(\mathcal{C})$: $\{A, B, C, D, E, F, G, H\}$
- $d_A = 1,\ d_B = 1,\ d_C = 2,\ d_D = 3,\ d_E = 3,\ d_F = 2,\ d_G = 2,\ d_H = 2$

J. Preskill, Lecture Notes for Physics 219, Chapter 9 (2004)
A. Kitaev, Ann. Phys. **303**, 2 (2003)

$g = 0$	$g = 1$	$g = 2$	$g = 3$



$\mathcal{D}_0 = 1$	$\mathcal{D}_0 = 8$	$\mathcal{D}_0 = 116$	$\mathcal{D}_0 = 2948$

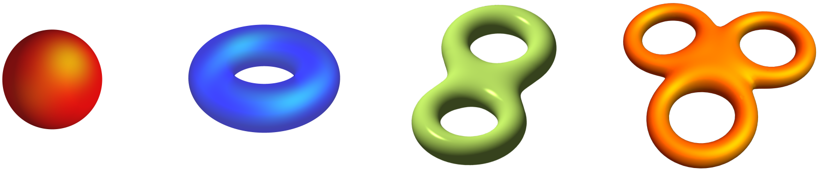
Ground-state degeneracy

Ex: Haagerup subfactor category $\mathcal{H}_3$ (non-commutative, non-braided)

- Objects of $\mathcal{C}$: $\{1, \alpha, \alpha^*, \rho, \rho\alpha, \rho\alpha^*\}$
- $d_1 = d_\alpha = d_{\alpha^*} = 1,\ d_\rho = d_{\rho\alpha} = d_{\rho\alpha^*} = \frac{3+\sqrt{13}}{2}$
- Objects of $\mathrm{D}(\mathcal{C})$: $\{0, \mu^1, \mu^2, \mu^3, \mu^4, \mu^5, \mu^6, \pi^1, \pi^2, \sigma^1, \sigma^2\, \sigma^3\}$
- $d_0 = 1,\ d_\mu = 3d_\rho,\ d_\pi = 3d_\rho + 1,\ d_\sigma = 3d_\rho + 2$

M. Asaeda and U. Haagerup, Comm. Math. Phys. **202**, 1 (1999)
R. Vanhove, L. Lootens, M. Van Damme, R. Wolf, T. J. Osborne, J. Haegeman, and F. Verstraete, arXiv:2110.03532

$g = 0$ | $g = 1$ | $g = 2$ | $g = 3$



$\mathcal{D}_0 = 1$ | $\mathcal{D}_0 = 12$ | $\mathcal{D}_0 = 1401$ | $\mathcal{D}_0 = 1\,603\,329$

Outline

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Excitations of the Levin-Wen model (2005)

- Excitations are the nontrivial objects of $\mathrm{D}(\mathcal{C})$
- General construction via the Ocneanu tube algebra
- Some objects only violate the plaquette constraints (fluxons)
- Some objects violate the vertex/plaquette constraints (not considered here !)

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Old results from CFT (1989)

- Number of states with q fluxons on a genus-g surface with N_p plaquettes:

$$\mathcal{D}_q = \binom{N_\mathrm{p}}{q} \sum_{i \in \mathrm{D}(\mathcal{C})} \mathbf{S}_{1,i}^{\chi - q} \left(\sum_{j \in \mathrm{fluxons}} \mathbf{S}_{i,j} \right)^q$$

- $\mathbf{S}-$matrix: symmetric unitary matrix diagonalizing fusion matrices of $\mathrm{D}(\mathcal{C})$

G. Moore and N. Seiberg, Comm. Math. Phys. **123**, 177 (1989)

Partition function of the Levin-Wen model

New results for commutative $\mathcal{C}$

- Number of states with q fluxons on a genus-g surface with N_p plaquettes:

$$\mathcal{D}_q = \binom{N_\mathrm{p}}{q} (-1)^q \left\{ \mathcal{D}_0 + \sum_{j \in \mathcal{C}} S_{1,j}^{2\chi} \left[\left(1 - S_{1,j}^{-2} \right)^q - 1 \right] \right\},$$

- $S-$matrix: unitary matrix diagonalizing fusion matrices of $\mathcal{C}$

J. Vidal, Phys. Rev. B **105**, L041110 (2022) / arXiv:2108.13425

A. Ritz-Zwilling *et al.*, in preparation

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Energy spectrum of fluxons

- Levin-Wen Hamiltonian: $H = -\sum_p P_p$
- Ground-state energy: $E_0 = -N_\mathrm{p}$
- $q-$fluxon state energy: $E_q = -N_\mathrm{p} + q$

Partition function of the Levin-Wen model

Exact finite-size and finite-temperature partition function

- Partition function: $\mathcal{Z} = \mathrm{Tr}(\mathrm{e}^{-\beta H}) = \sum_{q=0}^{N_\mathrm{p}} \mathcal{D}_q \, \mathrm{e}^{-\beta E_q}$ $\quad (\beta = 1/T)$

$$\boxed{\mathcal{Z} = \left(\mathrm{e}^{\beta} - 1\right)^{N_\mathrm{p}} \left\{ \mathcal{D}_0 + \sum_{j \in \mathcal{C}} S_{1,j}^{2\chi} \left[\left(1 + \frac{S_{1,j}^{-2}}{\mathrm{e}^{\beta} - 1} \right)^{N_\mathrm{p}} - 1 \right] \right\}}$$

- $\mathcal{Z}$ depends on input fusion rules ($S_{1,j} = d_j / D$)
- $\mathcal{Z}$ depends on the surface topology ($\chi = 2 - 2g$)
- $\mathcal{Z}$ depends on the number of plaquettes (N_p)
- Infinite-temperature limit: $\mathcal{Z}(\beta = 0) = \dim \mathcal{H} = \sum_{j \in \mathcal{C}} S_{1,j}^{2(\chi - N_\mathrm{p})} = \sum_{j \in \mathcal{C}} S_{1,j}^{-N_\mathrm{v}}$

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Specific heat and absence of thermal phase transition

- Specific heat per plaquette : $c = \frac{\beta^2}{N_\mathrm{p}} \frac{\partial^2 \ln \mathcal{Z}}{\partial \beta^2}$
- Thermodynamical limit ($N_\mathrm{p} \to \infty$): $c = \frac{\mathrm{e}^\beta \, \beta^2 (D^2 - 1)}{(D^2 - 1 + \mathrm{e}^\beta)^2}$

No finite-temperature phase transition in the Levin-Wen model !

J. Vidal, Phys. Rev. B **105**, L041110 (2022) / arXiv:2108.13425