Hilbert Space Fragmentation and Commutant Algebras

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SM, Olexei I. Motrunich, arXiv: 2108.10824 (2021) [To appear in PRX]
SM, Olexei I. Motrunich, (in preparation)

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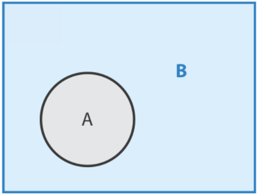
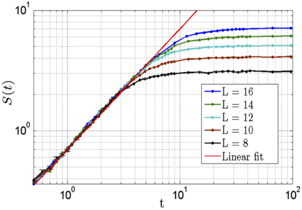
Outline

Ergodicity in Isolated Quantum Systems

- A quantum Hamiltonian is said to be ergodic if *any* initial state $|\psi(0)\rangle$ evolves into a "thermal" state $|\psi(t)\rangle = e^{-iHt}\,|\psi(0)\rangle$
- Reduced density matrix of a thermal state is the Gibbs density matrix of the subsystem

$$\rho = |\psi\rangle\langle\psi|\,, \quad \rho_A = \mathrm{Tr}_B\left(\rho\right), \quad \rho_A \sim e^{-\beta H|_A}$$

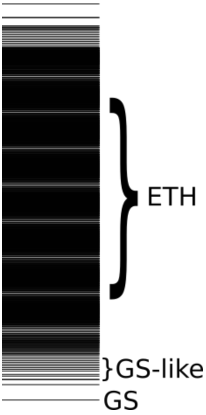




- Entanglement quantified by the von Neumann entropy $S = -\mathrm{Tr}_A\left(\rho_A \log \rho_A\right)$
- Local information gets scrambled throughout the system

Eigenstate Thermalization Hypothesis (ETH)

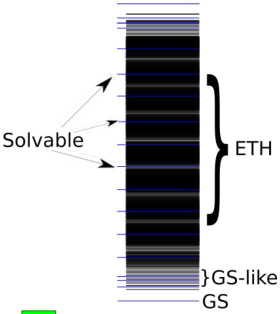
- A fundamental principle governing the thermalization of initial states in a quantum system
- Eigenstate Thermalization:[1] Eigenstates $|E_n\rangle$ in the middle of the spectrum are thermal, entanglement entropy obeys a **volume law** $S \sim \log D \sim L$
- Strong ETH: *ALL* eigenstates at finite energy density satisfy ETH *after resolving symmetries*
- Hamiltonians without an extensive number of conserved quantities believed to satisfy strong ETH
- Ergodicity breaking (violation of ETH) was believed to *only* occur in two types of systems
  - Integrable
  - Many-Body Localized

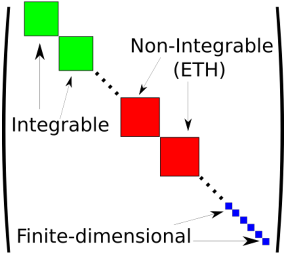


[1] M. Srednicki Phys. Rev. E 50, 888 (1994)

Outstanding Questions for Eigenstates in Non-Integrable Systems

- Can ETH be violated in *some* states in the absence of an extensive number of conserved quantities?
- A paradigm of ergodicity breaking beyond integrability and MBL?
- Issues: no good numerical methods to address this problem
- Recent analytical progress has identified two new types of "weak" ergodicity breaking[2]
    - Quantum Many-Body Scars
    - Hilbert Space Fragmentation





[2] M.Serbyn, D.A.Abanin, Z.Papic (2020); **SM**, B.A.Bernevig, N.Regnault (2021)

Weak Ergodicity Breaking: Quantum Many Body Scars

Quantum Many-Body Scars

- Non-integrable models with quasiparticle towers of eigenstates deep in the spectrum have been discovered[3]
- AKLT spin chain:[4] $\mathcal{P} = \sum_j (-1)^j (S_j^+)^2$, states with N quasiparticles dispersing with $k = \pi$ are **exact eigenstates** for finite system sizes L!

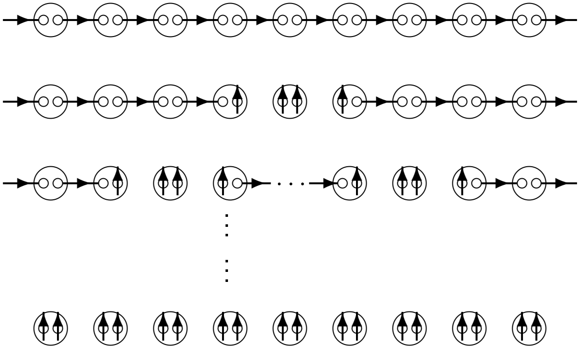
$|G\rangle$

$|S_2\rangle = \mathcal{P}\,|G\rangle$

$|S_4\rangle = \mathcal{P}^2\,|G\rangle$

$\vdots$

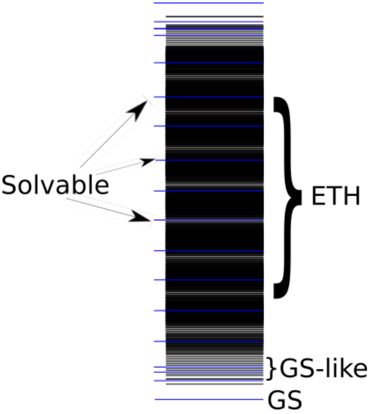
$|S_L\rangle = \mathcal{P}^{\frac{L}{2}}\,|G\rangle = |F\rangle$

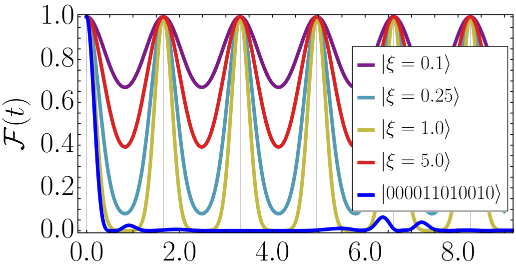


[3] **SM**, B.A.Bernevig, N.Regnault (2021)

[4] **SM**, S. Rachel, B. A. Bernevig, N. Regnault (2017)

- States have entanglement entropy $S \sim \log L \implies$ Violation of Strong ETH!
- Equally spaced tower: leads to exact revivals from simple initial states[5]



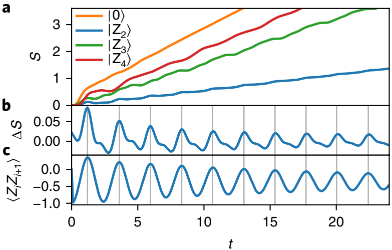


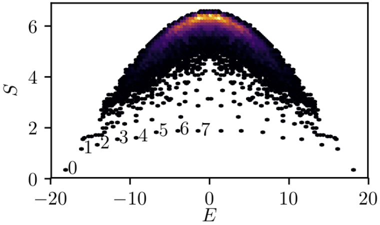
[5] T. Iadecola, M. Schecter (2019)

Connections to Recent Experiments: PXP Model

- Rydberg experiment[6] modelled by the constrained Hamiltonian

$$H_{PXP} = \sum_{n=1}^{L} P^{\circ}_{n-1} \; X_n \; P^{\circ}_{n+1} = |\circ \;\bullet\; \circ\rangle \langle \circ \;\circ\; \circ| + h.c.$$





- Initial charge density wave configuration $|\mathbb{Z}_2\rangle = |\circ \;\bullet\; \circ \;\cdots\; \bullet \;\circ\; \bullet\rangle$ shows anomalous dynamics[7]
- QMBS understood as a consequence of *approximately* disconnected low-entanglement subspace $\mathrm{span}_t \left\{ e^{-iH_{PXP}t} |\mathbb{Z}_2\rangle \right\}$[8]

[6] Bernien *et al.* Nature 551, 579-584 (2017)

[7] C.J. Turner *et al.* Nature Physics 14, 745-749 (2018)

[8] M. Serbyn, D.A.Abanin, Z. Papic (2020)

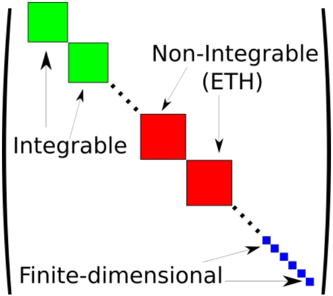
Weak Ergodicity Breaking: Hilbert Space Fragmentation

Hilbert Space Fragmentation

- What happens to ETH in constrained systems? Hard constraints typically arise in effective Hamiltonians
- Hilbert space fractures into *exponentially many* dynamically disconnected Krylov subspaces, $|R_i\rangle$ being product states

$$\mathcal{H} = \bigoplus_{i=1}^{K} \mathcal{K}\left(H, |R_i\rangle\right), \quad \mathcal{K}\left(H, |R\rangle\right) = \mathrm{span}_t \left\{ e^{-iHt} |R\rangle \right\}$$

- Different subspaces are *not distinguished* by obvious symmetry quantum numbers, can show vastly different properties![9]
- Violation of conventional ETH due to block-diagonal structure *after resolving known symmetries*



[9] **SM**, A. Prem, R. Nandkishore, N. Regnault, B.A. Bernevig (2019)

Dipole-Moment Conserving Models

- Fragmentation *generically* occurs in one dimensional systems conserving dipole moment ($\sum_j jS_j^z$ with OBC)[10,11]
- Example: spin-1 dipole conserving Hamiltonian that implements the following rules ($H = \sum_j (S_{j-1}^-(S_j^+)^2 S_{j+1}^- + h.c.)$)

$$|+-0\rangle \leftrightarrow |0+-\rangle\,, \quad |0-+\rangle \leftrightarrow |-+0\rangle$$
$$|+-+\rangle \leftrightarrow |0+0\rangle\,, \quad |-+-\rangle \leftrightarrow |0-0\rangle$$

- Exponentially many one-dimensional subspaces ("frozen" eigenstates)

$$|++--\cdots++--\rangle\,, \quad |0++0++\cdots 0++\rangle$$

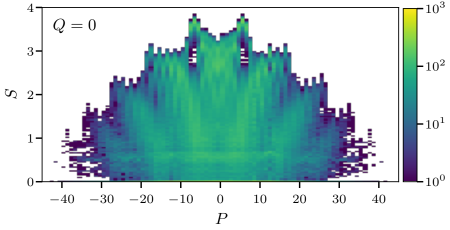
- Subspaces with non-local conserved quantities, e.g. a product state $|0\cdots 0+0\cdots 0\rangle$ can only evolve to states with "string-order" $|0\cdots 0+0\cdots 0-0\cdots 0+\cdots 0\rangle$

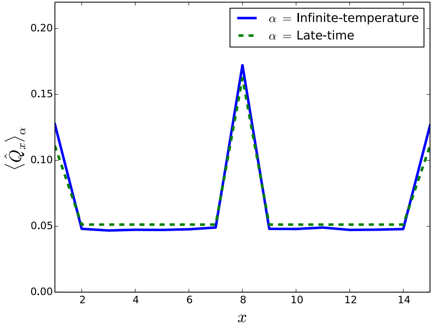
[10] P. Sala, T. Rakovszky, R. Verresen, M. Knap, F. Pollmann (2019)

[11] V. Khemani, M. Hermele, R. Nandkishore (2019)

Violation of conventional ETH

- Initial product states never thermalize w.r.t. the full Hilbert space[12,13]
- Eigenstate entanglement entropy within in the blocks satisfy $S \sim \log D$ ($S \sim L$ if $D \sim \exp(L)$, $S \sim \log L$ if $D \sim L^{\alpha}$)
- Krylov-restricted ETH principle: **ETH or its absence holds only within each subspace $\mathcal{K}(H, |R_i\rangle)$**[14]





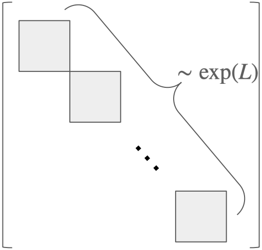
[12] P.Sala, T.Rakovszky, R.Verresen, M.Knap, F.Pollmann (2019)

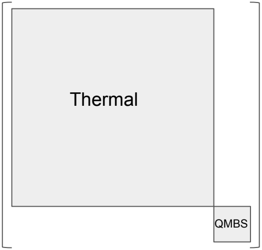
[13] V.Khemani, M.Hermele, R.Nandkishore (2019)

[14] **SM**, A.Prem, R.Nandkishore, N.Regnault, B.A.Bernevig (2019)

Dynamically Disconnected Subspaces

- These phenomena of weak ergodicity breaking are essentially the existence of unexpected "dynamically disconnected subspaces" in the Hilbert space





- Basis? Product state basis $\implies$ "classical" phenomenon (most of the fragmentation literature)
- Dynamically disconnected subspaces always exist in the presence of symmetries (usual quantum number sectors)

How do these sectors differ from symmetry sectors?

Symmetries in Quantum Many-Body Systems

- Conventional symmetries: usually on-site unitary representations of a group G

$$\hat{U}(g) = \hat{u}(g) \otimes \hat{u}(g) \otimes \cdots \otimes \hat{u}(g), \quad \text{e.g. } \hat{u}(g) = \begin{cases} e^{i\alpha Z} & \text{if } G = U(1) \\ e^{i\vec{\alpha}\cdot\vec{\sigma}} & \text{if } G = SU(2) \end{cases}.$$

- Conserved quantities are typically sums of local operators, e.g. total charge, number of domain walls, etc.
- Issue: These conserved quantities do not explain dynamically disconnected subspaces in QMBS or fragmentation
- Allow arbitrary commuting operators to be conserved quantities $\Longrightarrow$ every finite-dimensional Hamiltonian is fragmented?!

$$[H, |E_n\rangle\langle E_n|] = 0 \quad \Longrightarrow \text{ exponentially many conserved quantities}$$

What is an appropriate definition of a conserved quantity?[15]

[15] Similar problems exist in defining integrability in finite-dimensional systems: E.A.Yuzbashyan, B.S.Shastry (2013)

Commutant Algebras

Commutant algebras

- Key observation: Same fragmentation structure appears for entire classes of Hamiltonians $\{\sum_j J_j h_{j,j+1}\}$
- Natural to look for operators that commute with this entire family.

$$[\widehat{O}, \sum_j J_j h_{j,j+1}] = 0 \ \ \forall \{J_j\}.$$

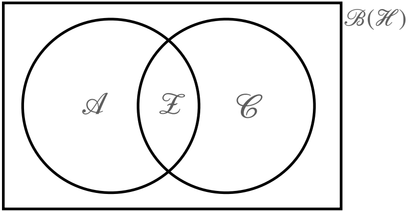
- Commutant Algebra $\mathcal{C}$: algebra of operators $\widehat{O}$ (not necessarily local) such that $[h_{j,j+1}, \widehat{O}] = 0 \ \ \forall j$

$$\widehat{O}_1 \in \mathcal{C}, \ \ \widehat{O}_2 \in \mathcal{C} \quad \implies \quad \begin{cases} \alpha_1 \widehat{O}_1 + \alpha_2 \widehat{O}_2 \in \mathcal{C} \ \text{ for any } \ \alpha_1, \alpha_2 \in \mathbb{C} \\ \widehat{O}_1 \widehat{O}_2, \widehat{O}_2 \widehat{O}_1 \in \mathcal{C} \end{cases}$$

- $\mathcal{C}$ commutes with the full "bond algebra" $\mathcal{A}$ generated by $\{h_{j,j+1}\}$ ($\mathcal{A} = \langle\langle \{h_{j,j+1}\} \rangle\rangle$).

Commutant Algebras

- $\mathcal{A}$ and $\mathcal{C}$ are unital †-closed (von Neumann) algebras
- They are centralizers of each other in the algebra of all operators on $\mathcal{H}$ (Double commutant theorem)



- Representation theory: There exists a basis in which operators $\widehat{h}_{\mathcal{A}} \in \mathcal{A}$ and $\widehat{h}_{\mathcal{C}} \in \mathcal{C}$ have the matrix representations

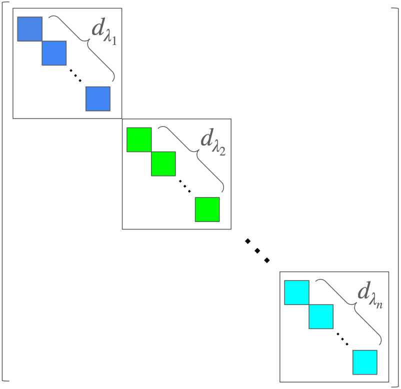
$$\widehat{h}_{\mathcal{A}} = \bigoplus_{\lambda} (M_{D_\lambda} \otimes \mathbb{1}_{d_\lambda}), \quad \widehat{h}_{\mathcal{C}} = \bigoplus_{\lambda} (\mathbb{1}_{D_\lambda} \otimes N_{d_\lambda})$$

- $\{D_\lambda\}$ and $\{d_\lambda\}$: dimensions of irreducible representations of $\mathcal{A}$ and $\mathcal{C}$.
- Alternately: Basis in which *all* elements of $\mathcal{A}$ are maximally block diagonal

Dynamically Disconnected Subspaces

- Hamiltonian H in $\mathcal{A}$, block diagonal form defines dynamically disconnected subspaces.
- For each λ: d_λ number of degenerate D_λ-dimensional Krylov subspaces.
- Number of Krylov subspaces $K = \sum_\lambda d_\lambda$, bounded using $\dim(\mathcal{C}) = \sum_\lambda d_\lambda^2$

$$\frac{1}{2}\log(\dim(\mathcal{C})) \leq \log K \leq \log(\dim(\mathcal{C}))$$



$\log(\dim(\mathcal{C}))$	**Example**
$\sim \mathcal{O}(1)$	Discrete Global Symmetry
$\sim \log L$	Continuous Global Symmetry
$\sim L$	Fragmentation

Simple Examples: Abelian $\mathcal{C}$

- Abelian $\mathcal{C} \implies d_\lambda = 1, \;\; K = \dim(\mathcal{C})$
- Generic Hamiltonians $\sum_j J_j h_{j,j+1}$ with no symmetries

$$[h_{j,j+1}, \widehat{O}] = 0 \quad \implies \quad \mathcal{C} = \{\mathbb{1}\}, \; K = \dim(\mathcal{C}) = 1.$$

- Example: Ising models $H = \sum_{j=1}^{L} [J_j X_j X_{j+1} + h_j Z_j]$, solve for $[X_j X_{j+1}, \widehat{O}] = 0$ and $[Z_j, \widehat{O}] = 0$

$$\mathcal{C} = \text{span}\{\mathbb{1}, \prod_j Z_j\} = \mathbb{C}[\mathbb{Z}_2], \quad K = \dim(\mathcal{C}) = 2.$$

- Example: Spin-$\frac{1}{2}$ XX models $H = \sum_{j=1}^{L} [J_j (X_j X_{j+1} + Y_j Y_{j+1}) + h_j Z_j]$, solve for $[X_j X_{j+1} + Y_j Y_{j+1}, \widehat{O}] = 0$ and $[Z_j, \widehat{O}] = 0$

$$\mathcal{C} = \langle\!\langle \widehat{Z} \rangle\!\rangle = \text{span}\{\mathbb{1}, \widehat{Z}, (\widehat{Z})^2, \cdots, (\widehat{Z})^L\}, \;\; K = \dim(\mathcal{C}) = L + 1$$
$$\widehat{Z} = \textstyle\sum_j Z_j$$

Simple Examples: Non-Abelian $\mathcal{C}$

- Example: spin-$\frac{1}{2}$ Heisenberg model
  $H = \sum_j J_j \; \vec{S}_j \cdot \vec{S}_{j+1}$, $\mathcal{A} = \langle\langle \vec{S}_j \cdot \vec{S}_{j+1} \rangle\rangle = \mathbb{C}[S_L]$

$$[\vec{S}_j \cdot \vec{S}_{j+1}, \widehat{X}] = 0, \;\; [\vec{S}_j \cdot \vec{S}_{j+1}, \widehat{Y}] = 0, \;\; [\vec{S}_j \cdot \vec{S}_{j+1}, \widehat{Z}] = 0 \; \forall \; j$$

$$\mathcal{C} = \langle\langle \widehat{X}, \widehat{Y}, \widehat{Z} \rangle\rangle = \mathrm{span}_{\alpha,\beta,\gamma}\{(\widehat{X})^{\alpha}(\widehat{Y})^{\beta}(\widehat{Z})^{\gamma}\} = U(\mathfrak{su}(2))$$

- Block-diagonal form (Schur-Weyl duality):
  $0 \leq \lambda \leq L/2$: S^2 eigenvalues, $d_\lambda = 2\lambda + 1$: irreps of $\mathfrak{su}(2)$
  D_λ: irreps of S_L
- Double Commutant Theorem: Any $SU(2)$-symmetric operator is within the algebra $\mathcal{A} = \langle\langle \{\vec{S}_j \cdot \vec{S}_{j+1}\} \rangle\rangle$.

In these simple cases, the full commutant is generated by "conventional" conserved quantities, but not always the case

[15] **SM**, O. I. Motrunich (2021)

Hilbert space fragmentation

"Classical" fragmentation: $t - J_z$ model

- Consider the $t - J_z$ Hamiltonian: hopping with two species of particles $|\uparrow 0\rangle \leftrightarrow |0 \uparrow\rangle$, $|\downarrow 0\rangle \leftrightarrow |0 \downarrow\rangle$

$$H_{t-J_z} \equiv \sum_j (-t_{j,j+1} \sum_{\sigma\in\{\uparrow,\downarrow\}} \left(\tilde{c}_{i,\sigma}\tilde{c}^\dagger_{j,\sigma} + h.c.\right) + J^z_{j,j+1} S^z_i S^z_j)$$

$$\tilde{c}_{j,\sigma} \equiv c_{j,\sigma}\left(1 - c^\dagger_{j,-\sigma} c_{j,-\sigma}\right)$$

- Has two $U(1)$ symmetries $N^\uparrow \equiv \sum_j N^\uparrow_j$ and $N^\downarrow \equiv \sum_j N^\downarrow_j$
- Full pattern of spins ($\uparrow$ or $\downarrow$) preserved in one dimension with OBC

$$|0 \uparrow\downarrow 0 \downarrow\uparrow 0\rangle \not\leftrightarrow |0 \uparrow\uparrow 0 \downarrow\downarrow 0\rangle$$

- Fragmentation in the product state basis, number of Krylov subspaces $K = \sum_{j=0}^{L} 2^j = 2^{L+1} - 1$.

"Classical" fragmentation: $t - J_z$ model

- Local operators $N_j^\uparrow$ and $N_j^\downarrow$ satisfy the relations

$$[h_{j,j+1}, N_j^\alpha + N_{j+1}^\alpha] = 0, \quad [h_{j,j+1}, N_j^\alpha N_{j+1}^\beta] = 0, \quad \alpha, \beta \in \{\uparrow, \downarrow\}$$

- The full commutant algebra $\mathcal{C}$ can be explicitly constructed, $\dim(\mathcal{C}) = 2^{L+1} - 1 \sim \exp(L)$

$$N^{\sigma_1 \sigma_2 \cdots \sigma_k} = \sum_{j_1 < j_2 < \cdots < j_k} N_{j_1}^{\sigma_1} N_{j_2}^{\sigma_2} \cdots N_{j_k}^{\sigma_k}, \quad \sigma_j \in \{\uparrow, \downarrow\}$$

- Most of these are functionally independent from the conventional conserved quantities $N^\uparrow$ and $N^\downarrow$ $\implies$ new dynamically disconnected subspaces
- Similar construction works for dipole-conserving models, exact results in some cases (e.g. $\dim(\mathcal{C}) \sim (1 + \sqrt{2})^L$ for range-3 spin-1 model)
- Classical fragmentation: All operators in $\mathcal{C}$ are diagonal in the product state basis

"Quantum" fragmentation: Spin-1 biquadratic model

- Disordered $SU(3)$-symmetric spin-1 "ferromagnetic" biquadratic model $H = \sum_{j=1}^{L} J_j(\vec{S}_j \cdot \vec{S}_{j+1})^2$, ground state degeneracy grows exponentially with $L \implies$ hidden symmetries
- Bond algebra $\mathcal{A} = \langle\!\langle \vec{S}_j \cdot \vec{S}_{j+1} \rangle\!\rangle$ is the Temperley-Lieb Algebra $TL_L(q = \frac{3+\sqrt{5}}{2})$
- Commutant $\mathcal{C}$ can be explicitly constructed,[16] not generated by local operators, $\dim(\mathcal{C}) \sim \exp(L)$

$$[(\vec{S}_j \cdot \vec{S}_{j+1})^2, (M^{\alpha}_{\beta})_j + (M^{\alpha}_{\beta})_{j+1}] = 0, \quad [(\vec{S}_j \cdot \vec{S}_{j+1})^2, (M^{\alpha}_{\beta})_j (M^{\gamma}_{\delta})_{j+1}] = 0,$$

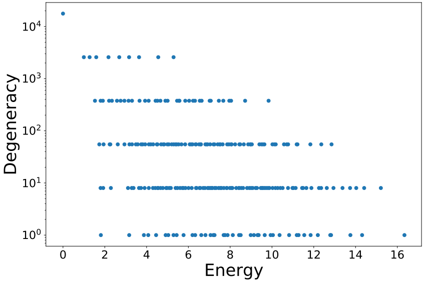
$$M^{\alpha_1 \alpha_2 \cdots \alpha_k}_{\beta_1 \beta_2 \cdots \beta_k} = \sum_{j_1 < j_2 < \cdots < j_k} (M^{\alpha_1}_{\beta_1})_{j_1} (M^{\alpha_2}_{\beta_2})_{j_2} \cdots (M^{\alpha_k}_{\beta_k})_{j_k}.$$

- Quantum fragmentation: Block-diagonal structure of the Hamiltonian understood in the spin-1 singlet basis, not the product state basis

[16] N. Read, H. Saleur (2007)

"Quantum" fragmentation: Spin-1 biquadratic model

- Non-abelian $\mathcal{C}$ leads to large degeneracies in the spectrum
- Violates conventional ETH if only the $SU(3)$ symmetry is resolved



- Hamiltonian restricted to a Krylov subspace is the XXZ model with $SU(2)_q$ symmetry

$$\sum_{j=1}^{L-1} J_j \left[X_j X_{j+1} + Y_j Y_{j+1} + \frac{q+q^{-1}}{2} Z_j Z_{j+1} + \frac{q-q^{-1}}{4} (Z_j - Z_{j+1}) \right]$$

- Satisfies Krylov-Restricted ETH[17] (equivalent to resolving non-local conserved quantities)

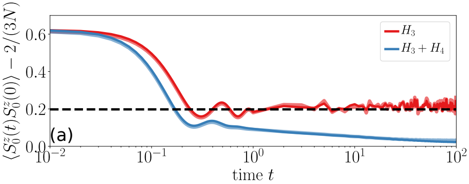
[17] **SM**, A. Prem, R. Nandkishore, N. Regnault, B. A. Bernevig (2019)

Application: Mazur bounds

- Autocorrelation functions of local operators can be bounded using the "conserved quantities" $\{Q_\alpha\}$ of the system

$$\lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau dt \, \langle A(t)A(0)\rangle \geq \sum_\alpha \frac{(A|Q_\alpha)\,(Q_\alpha|A)}{(Q_\alpha|Q_\alpha)}, \quad (A|B) := \frac{1}{D}\mathrm{Tr}(A^\dagger B)$$

- "Measure" of operator spreading, expected to decay to 0 as $L \to \infty$ (e.g., as $\sim 1/L$ for $U(1)$-symmetric systems)
- Fragmentation associated with anomalous saturation of autocorrelation functions[18]
- Puzzles resolved if all the operators in the commutant are incorporated in the Mazur bound[19]

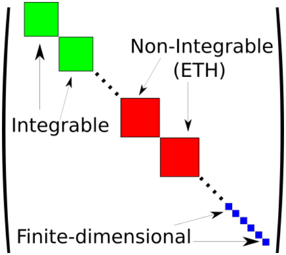


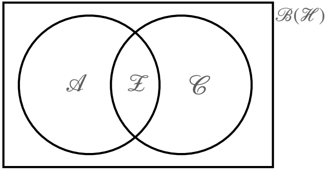
[18] P.Sala, T.Rakovszky, R.Verresen, M.Knap, F.Pollmann (2019)

[19] **SM**, O. I. Motrunich (2021)

Summary

- Commutant algebras natural language for dynamically disconnected subspaces, gives a concrete definition for fragmentation $\dim(\mathcal{C}) \sim \exp(L)$
- Conventional symmetries: $\mathcal{C}$ generated by conventional conserved quantities, $\dim(\mathcal{C}) \sim \mathcal{O}(1)$ or $\dim(\mathcal{C}) \sim \mathrm{poly}(L)$
- "Classical" fragmentation in the product state basis v/s "Quantum" fragmentation in an entangled basis
- Systematic consideration explains numerical observations of autocorrelation functions/operator spreading, leads to new Mazur bounds





Details in:

SM, O. I. Motrunich, arXiv: 2108.10324