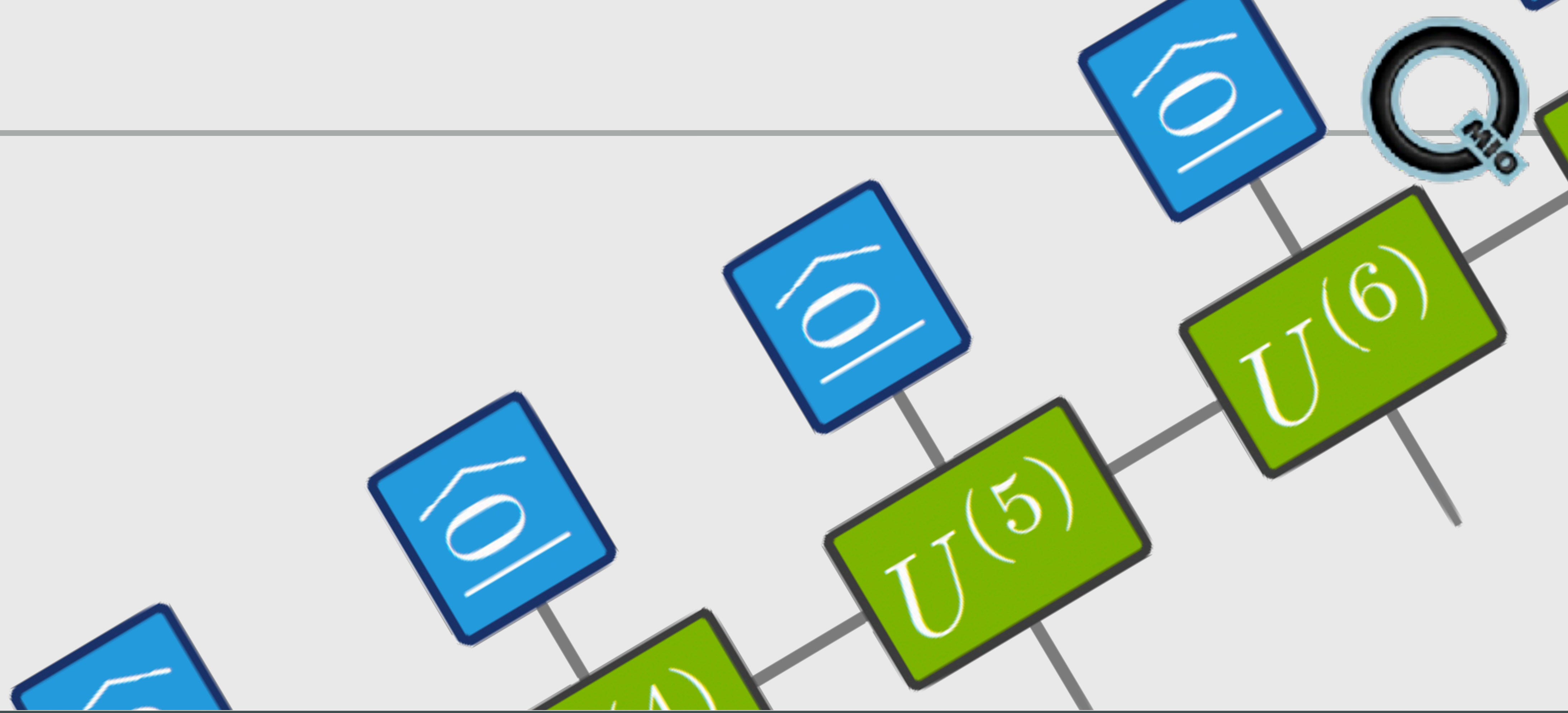


SOME FUN NEW IDEAS ON

TENSOR NETWORKS

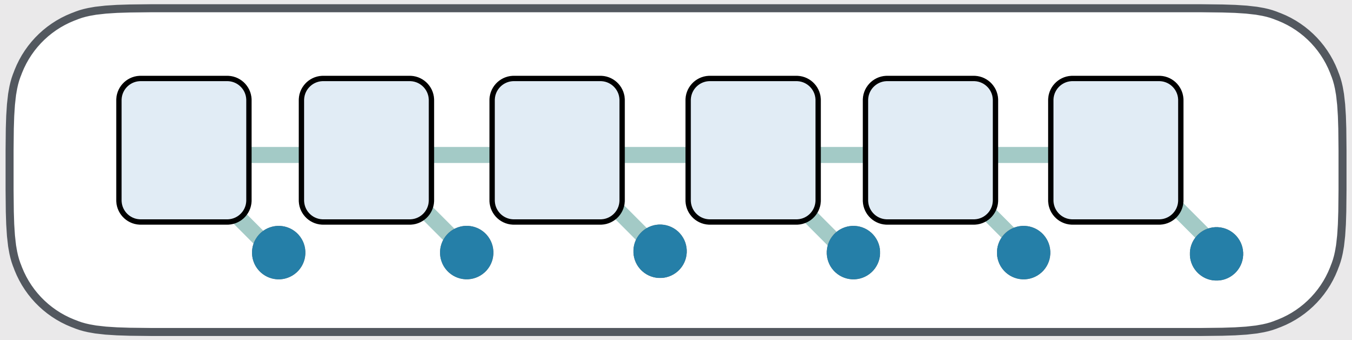
TO CAPTURE ENTANGLEMENT IN QUANTUM MANY-BODY SYSTEMS

JENS EISERT, FU BERLIN

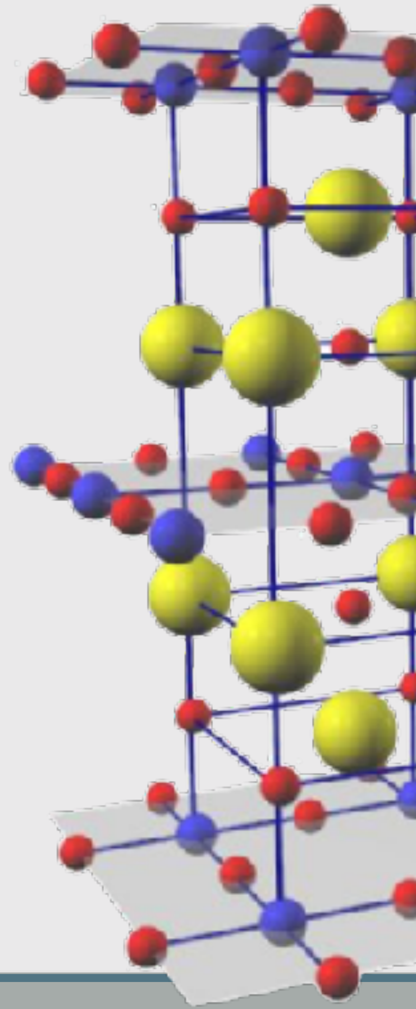
ENTANGLEMENT IN QUANTUM MANY-BODY SYSTEMS

TENSOR NETWORKS

- Tensor networks capture common **entanglement** patterns in **quantum many-body systems**

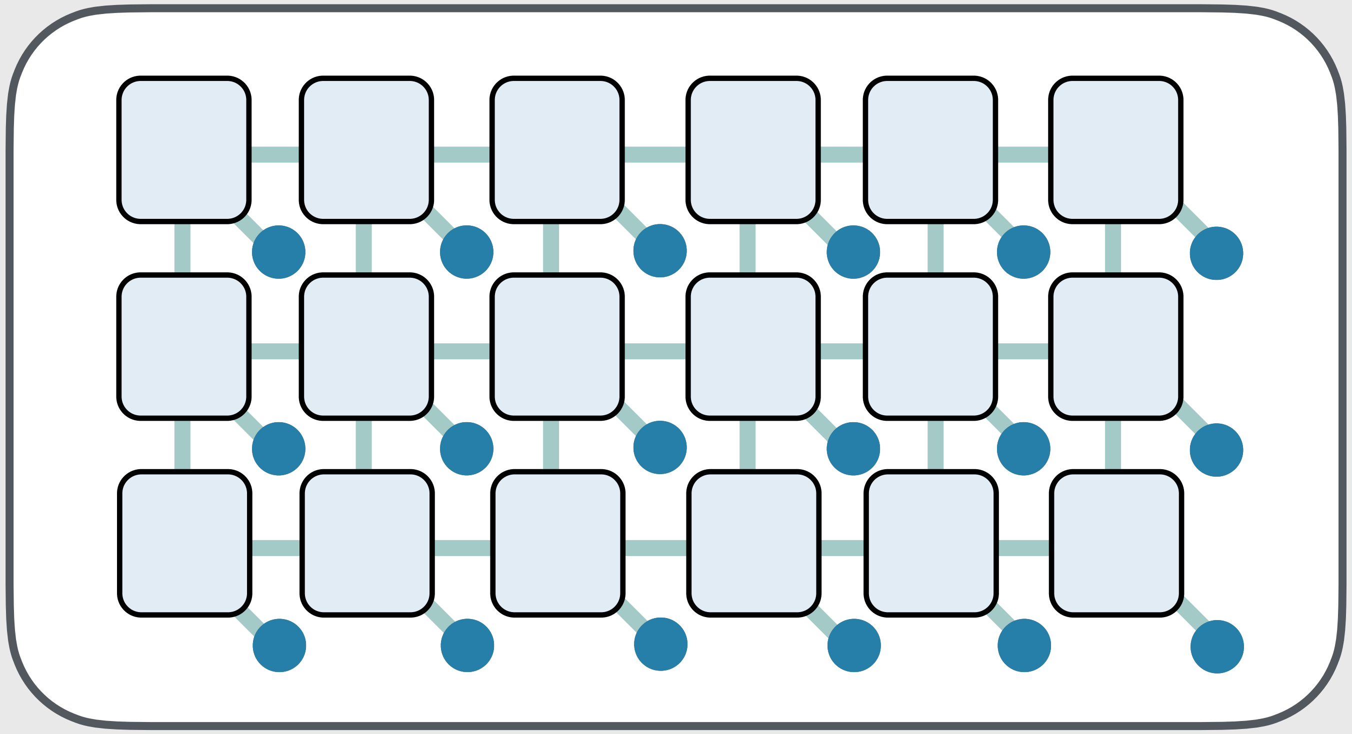


White, Phys Rev Lett 69, 2863 (1992)
Fannes, Nachtergaele, Werner, Commun Math Phys 3, 443 (1992)
Schollwoeck, Ann Phys 326, 96-192 (2011)

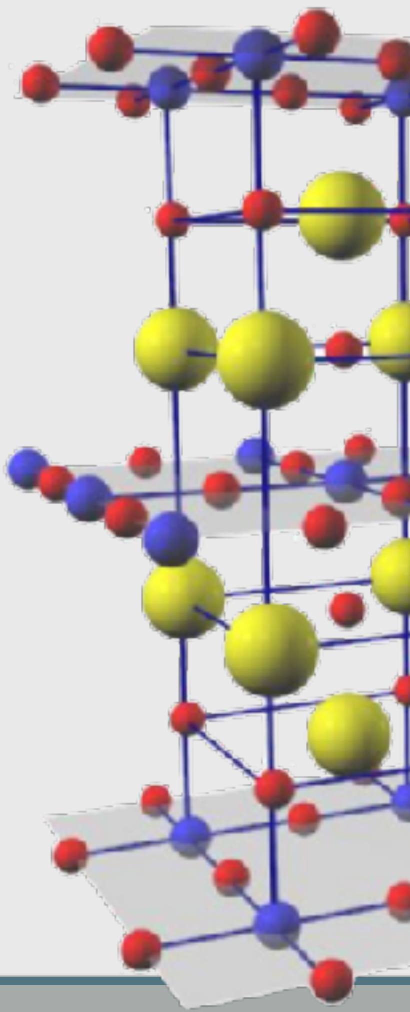


TENSOR NETWORKS

- Tensor networks capture common **entanglement** patterns in **quantum many-body systems**

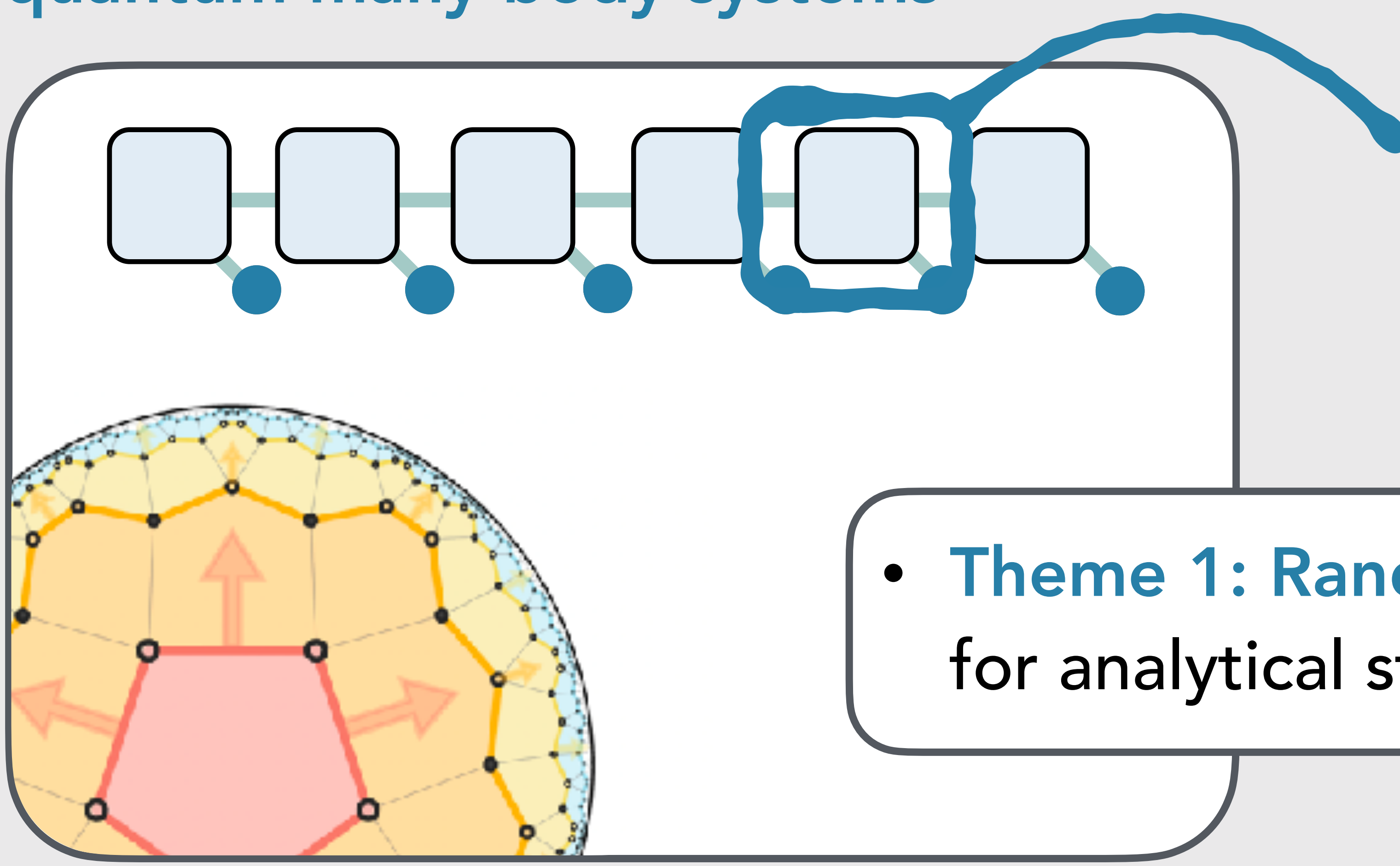


Verstraete, Cirac, Murg, Adv Phys 57, 143 (2008)
Eisert, Cramer, Plenio, Rev Mod Phys 82, 277 (2010)
Orús, Ann Phys 349, 117–158 (2014)
Cirac, Perez-Garcia, Schuch, Verstraete, Rev Mod Phys 93, 045003 (2021)



RANDOM TENSOR NETWORKS

- Tensor networks capture common **entanglement** patterns in **quantum many-body systems**



Random tensors

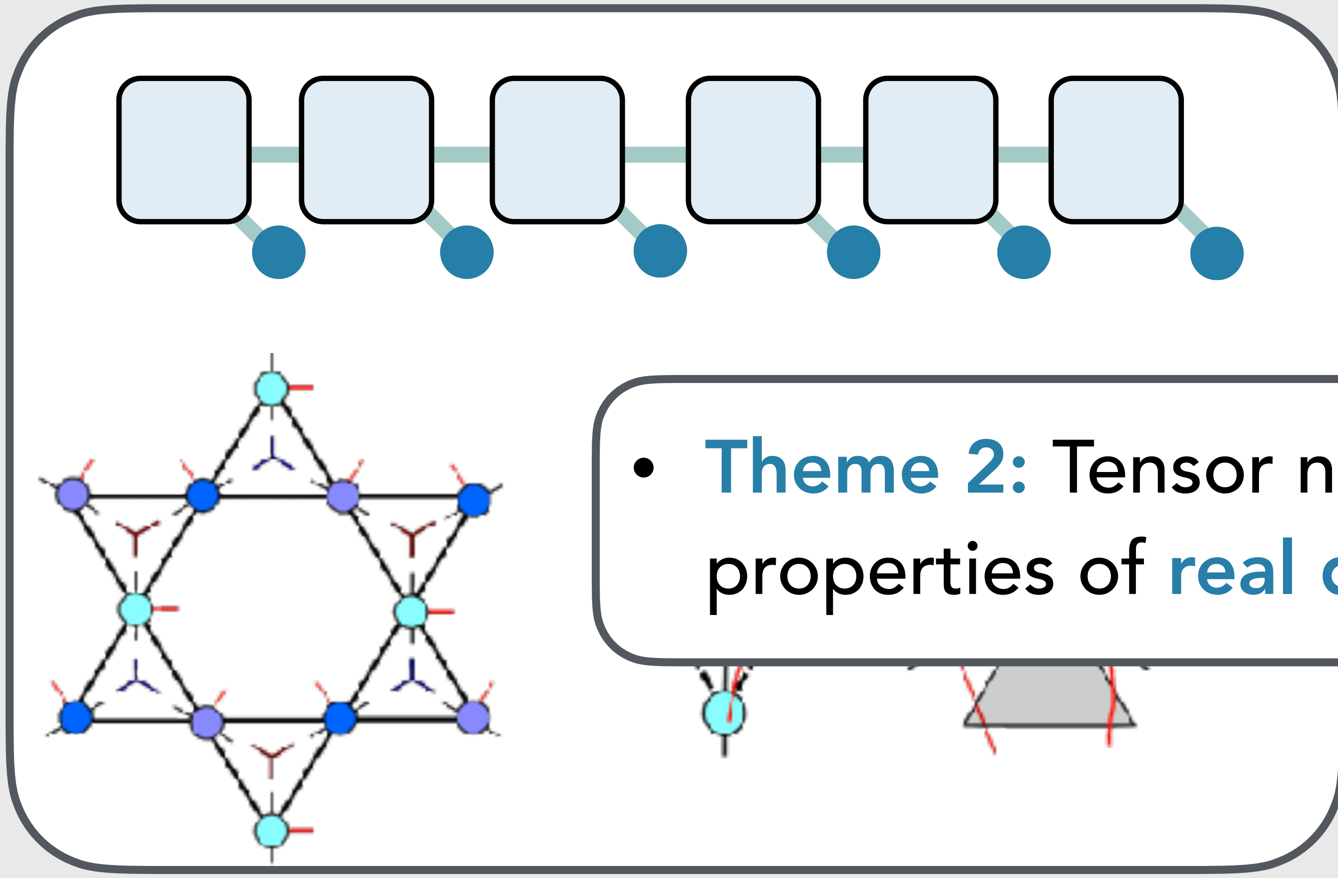
- **Theme 1: Random tensor networks** for analytical studies

Haferkamp, Bertoni, Roth, Eisert, PRX Quantum 2, 040308 (2021)
Jahn, Zimboras, Eisert, Jahn, Quantum 6, 643 (2022)
Haferkamp, Faist, Kothakonda, Eisert, Halpern, Nature Physics, in press (2022)
Wille, Altland, Jahn, Eisert, in preparation (2022)
Feldman, Kshetrimayum, Eisert, Goldstein, arXiv:2202.04089 (2022)

(ANALYTICAL)

TENSOR NETWORKS FOR QUANTUM MATERIALS

- Tensor networks capture common **entanglement** patterns in **quantum many-body systems**



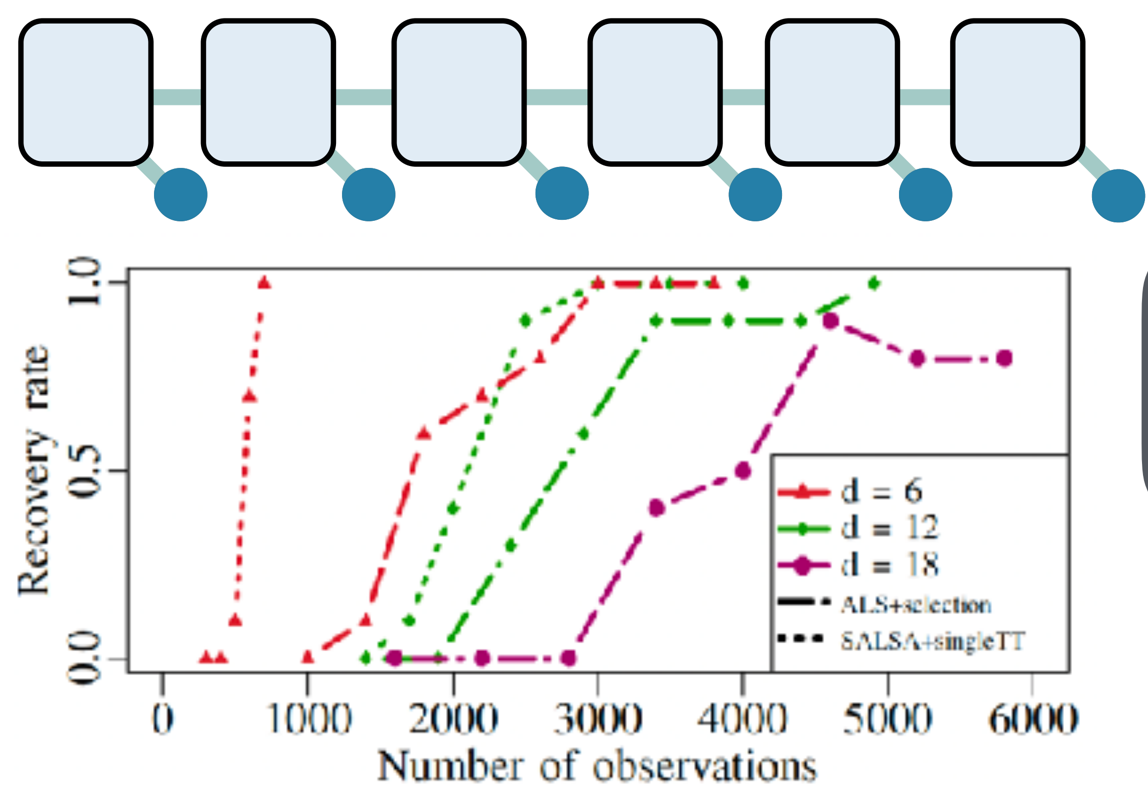
- **Theme 2:** Tensor networks to assess properties of **real quantum materials**

Nietner, Kshetrimayum, Eisert, Lake, in preparation (2022)
Kshetrimayum, Balz, Lake, Eisert, Ann Phys 421, 168292 (2020)
Krumnow, Veis, Eisert, Legeza, Phys Rev B 104, 075137 (2021)
Schmoll, Nietner, Kshetrimayum, Eisert, Chen, in preparation (2022)

(NUMERICAL)

TENSOR NETWORKS IN MACHINE LEARNING

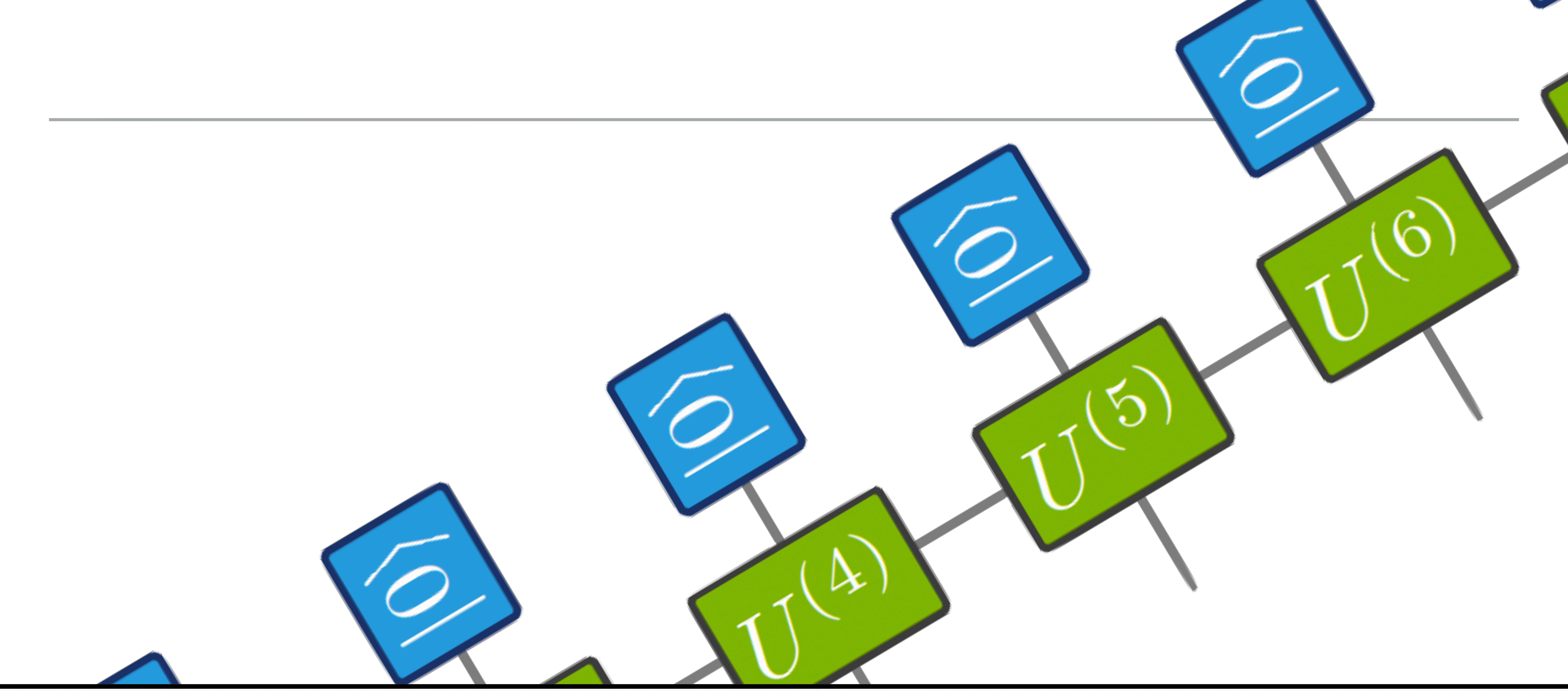
- Tensor networks capture common **entanglement** patterns in **quantum many-body systems**



Goeßmann, Götte, Roth, Sweke, Kutyniok, Eisert, NeurIPS (2021)
Wilde, Kshetrimayum, Roth, Eisert, in preparation (2022)

- **Theme 3:** Tensor networks in **machine learning** tasks

(CONCEPTUAL)

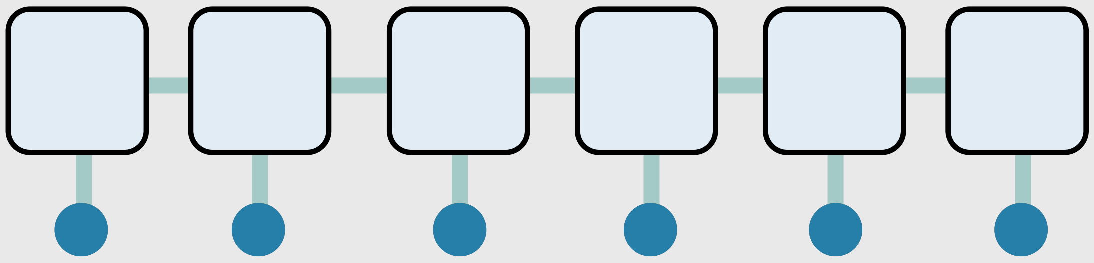


RANDOM TENSOR NETWORKS

- How can **randomness** be harnessed to explore **typical** properties of tensor networks?

RANDOM MATRIX PRODUCT STATES

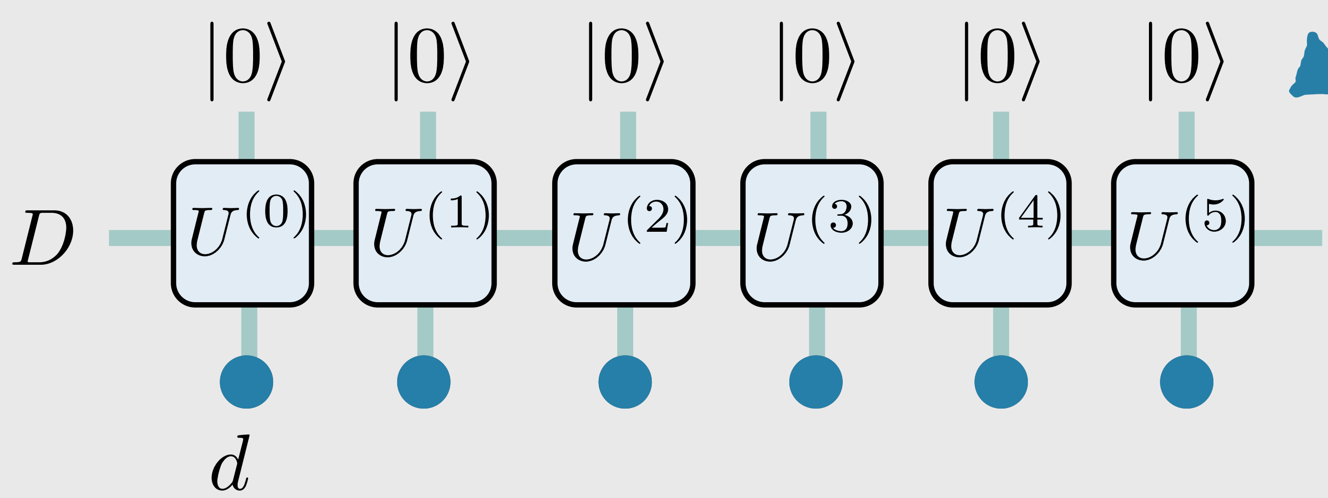
- Random **matrix product states**



Haferkamp, Bertoni, Roth, Eisert, PRX Quantum 2, 040308 (2021)
Lancien, García, arXiv:1906.11682 (2019)
Garnerone, de Oliveira, Haas, Zanardi, Phys Rev A 82, 052312 (2010)

RANDOM MATRIX PRODUCT STATES

- Random **matrix product states**



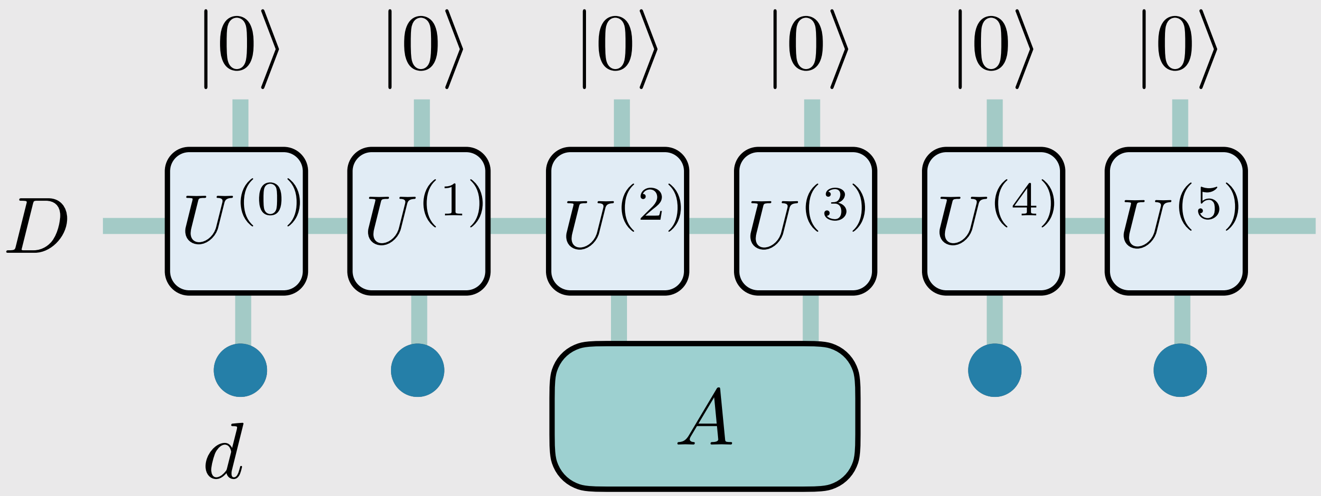
Haar random tensors
$U^{(i)} \in U(dD)$

- Several **interesting properties** can be shown

Haferkamp, Bertoni, Roth, Eisert, PRX Quantum 2, 040308 (2021)
Lancien, García, arXiv:1906.11682 (2019)
Garnerone, de Oliveira, Haas, Zanardi, Phys Rev A 82, 052312 (2010)

STATISTICAL MECHANICS OF RANDOM STATES

- Random **matrix product states**



- E.g., seeing them as generic ground states, do they **equilibrate** in time under generic local Hamiltonians?

- Observables A evolving under H have **time averages**

$$A_\psi^\infty := \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle\psi|A(t')|\psi\rangle \mathrm{d}t'$$

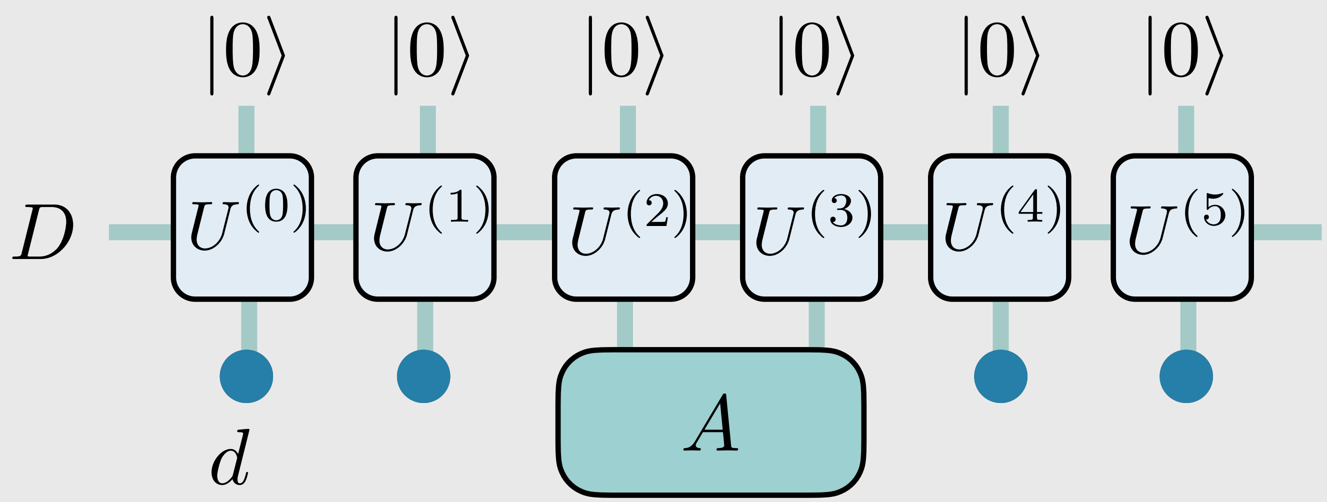
- **Fluctuations**

$$\Delta A_\psi^\infty := \lim_{t\to\infty} \frac{1}{t}\int_0^t |\langle\psi|A(t')|\psi\rangle - A_\psi^\infty|^2 \mathrm{d}t'$$

Haferkamp, Bertoni, Roth, Eisert, PRX Quantum 2, 040308 (2021)

STATISTICAL MECHANICS OF RANDOM STATES

- Random **matrix product states**



- They **equilibrate exponentially** well:

$$\Pr\left(\Delta A_\psi^\infty \leq e^{-c_1 \alpha(d,D) n}\right) \geq 1 - e^{-c_2 \alpha(d,D) n}$$

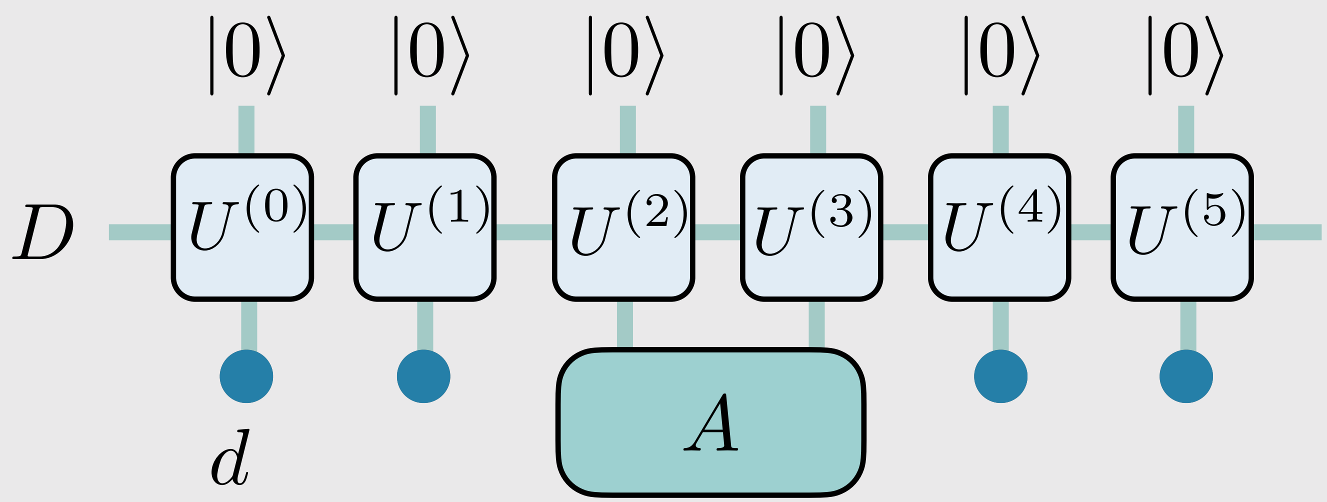
for

$$\alpha(d,D) = \log\left(\frac{d - \frac{1}{dD^2}}{(1+\frac{1}{D})(1+\frac{1}{dD})}\right)$$

Haferkamp, Bertoni, Roth, Eisert, PRX Quantum 2, 040308 (2021)

STATISTICAL MECHANICS OF RANDOM STATES

- Random **matrix product states**



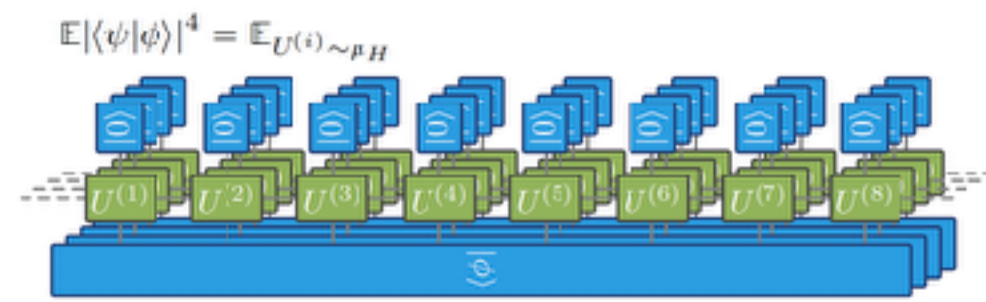
- Bound "effective dimension"

$$\Delta A_\psi^\infty = O(1/D_{\text{eff}})$$
$$1/D_{\text{eff}} := \sum_j |\langle \psi | j \rangle|^4$$

- Use Weingarten calculus

$$\mathbb{E}_{U\sim\mu_H} U^{\otimes t} \otimes \overline{U}^{\otimes t} = \sum_{\sigma,\pi \in S_t} \mathrm{Wg}(\sigma^{-1}\pi, q) |\sigma\rangle\langle\pi|$$

- Map to statistical mech model



- Tensor calculus

- They **equilibrate exponentially** well:

$$\Pr\left(\Delta A_\psi^\infty \leq e^{-c_1 \alpha(d,D) n}\right) \geq 1 - e^{-c_2 \alpha(d,D) n}$$

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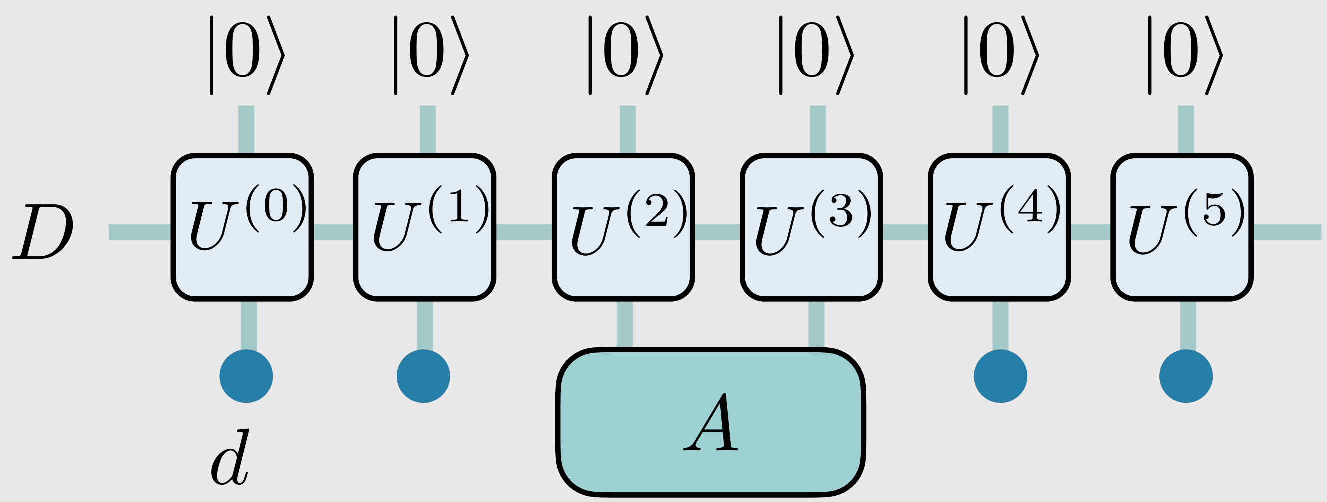
Haferkamp, Bertoni, Roth, Eisert, PRX Quantum 2, 040308 (2021)
Hunter-Jones, arXiv:1905.12053 (2019)

STATISTICAL MECHANICS OF RANDOM STATES

- Random **matrix product states**



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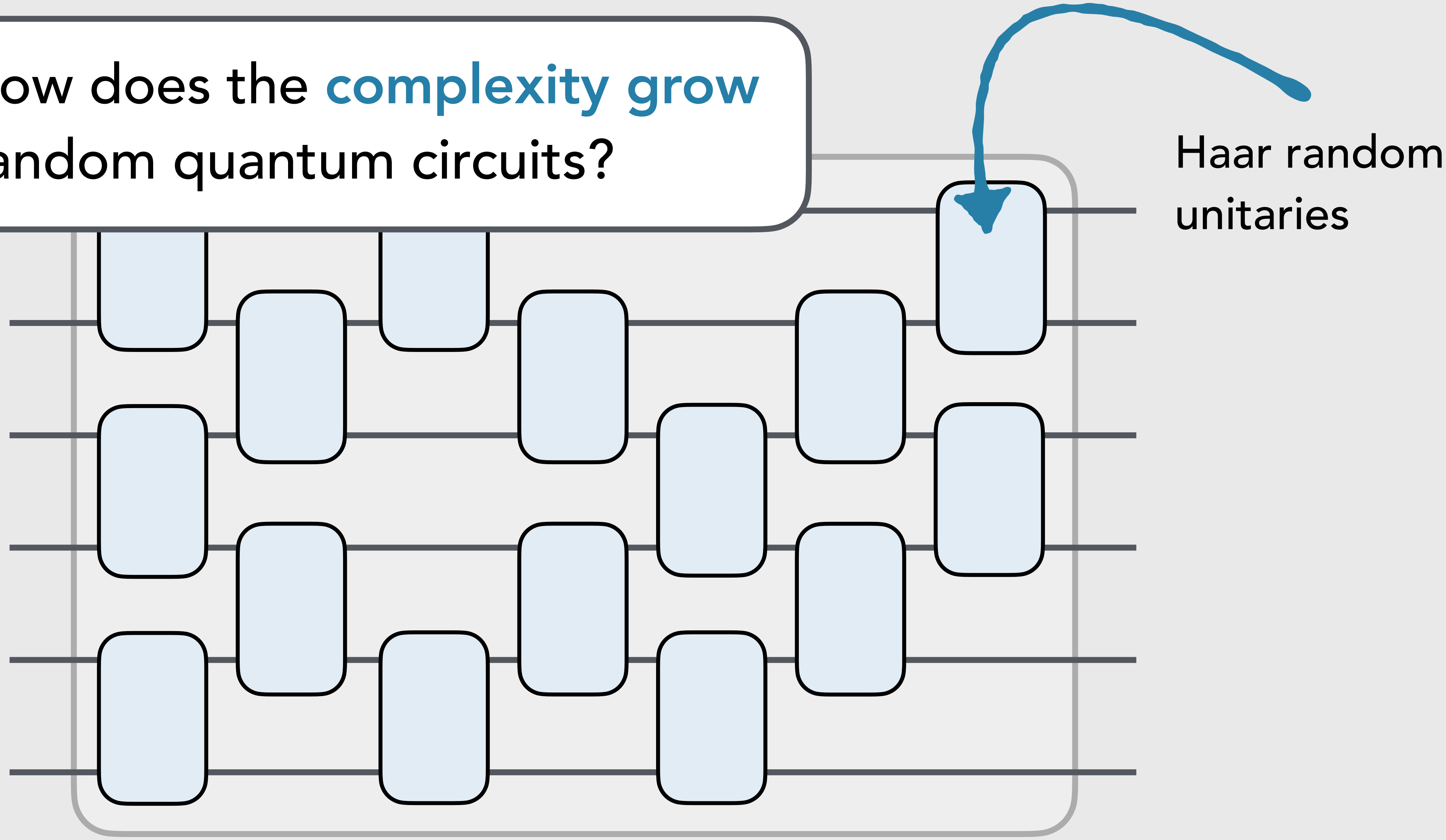
$$\alpha(d,D) = \log\left(\frac{d - \frac{1}{dD^2}}{(1+\frac{1}{D})(1+\frac{1}{dD})}\right)$$

- Extensive **entropies,** insights into generic **phases of matter**, etc

Haferkamp, Bertoni, Roth, Eisert, PRX Quantum 2, 040308 (2021)

- Or, how does the **complexity grow** for random quantum circuits?

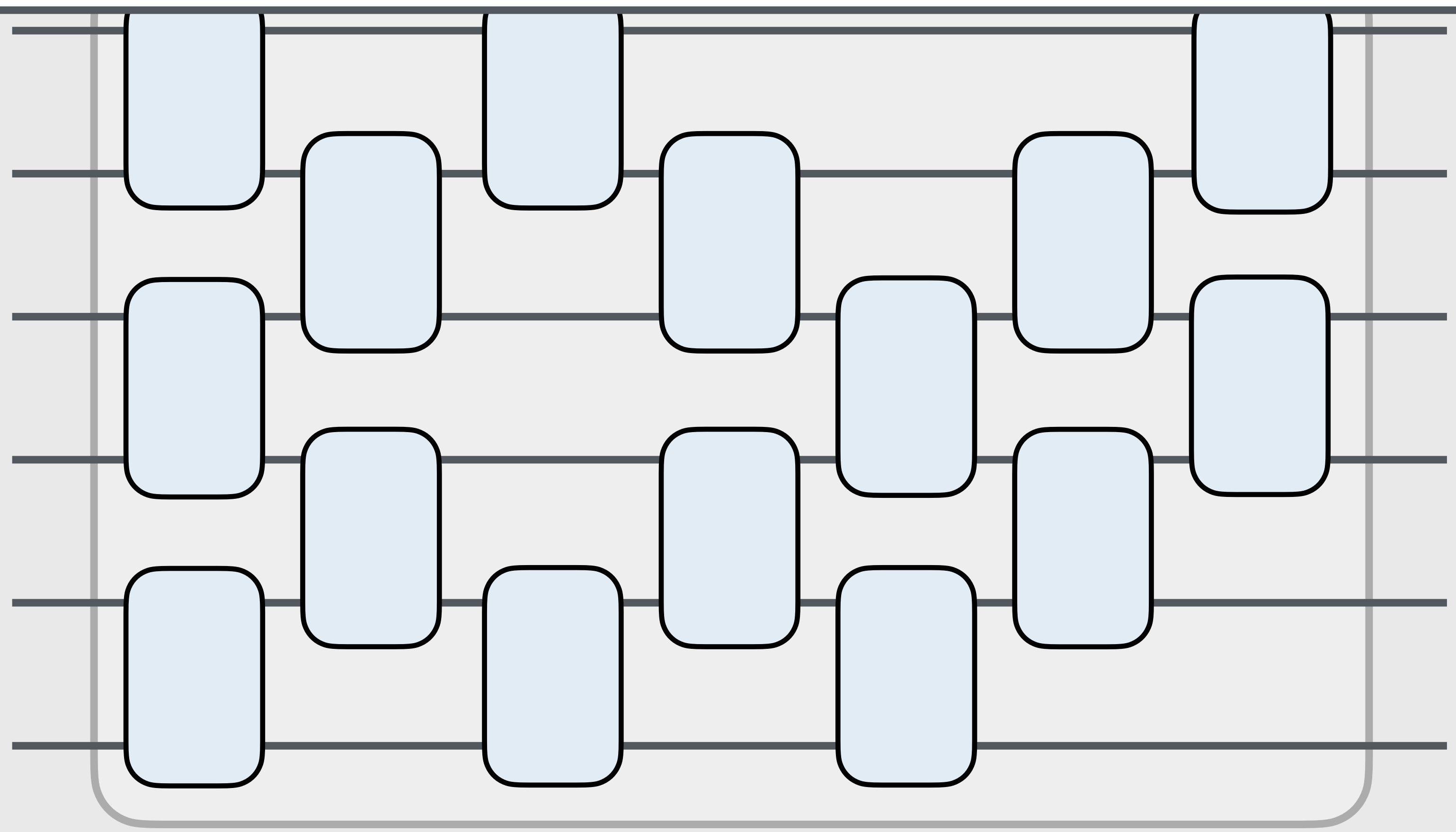


- **Circuit complexity**: Smallest number of quantum gates from gate set to generate a **given unitary**
- **Computationally hard:** Notorious cancellations

Gosset, Gosset, Kliuchnikov, Mosca, Russo, Quant Inf Comp 14, 1277 (2014)
Aaronson, Gottesman, Phys Rev A 70, 02328 (2004)

COMPLEXITY GROWTH IN RANDOM CIRCUITS

- Has risen to prominence as **Brown-Susskind** conjecture

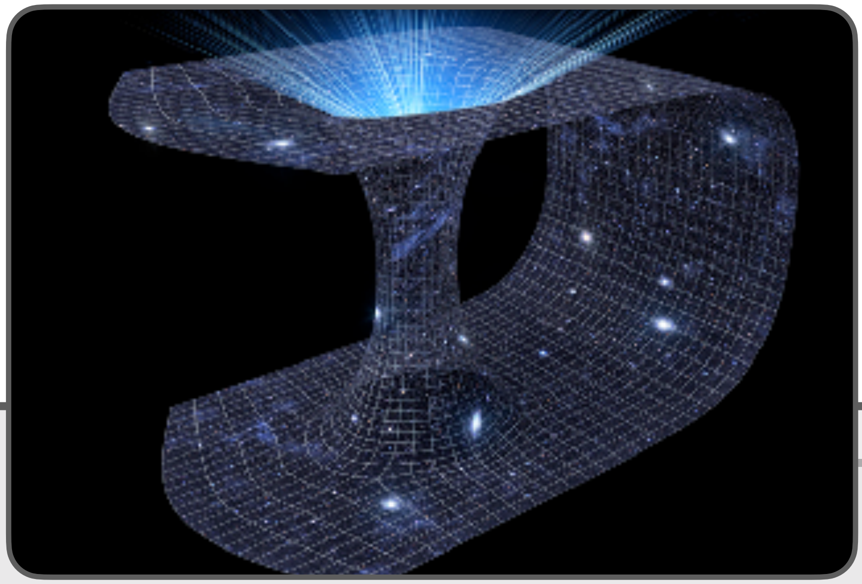


Brown, Roberts, Susskind, Swingle, Zhao, Phys Rev Lett 116, 191301 (2016)
Brown, Roberts, Susskind, Swingle, Zhao, Phys Rev D 93, 086006 (2016)
Chapman, Eisert, Hackl, Heller, Jefferson, Marrochio, Myers, SciPost Phys 6, 034 (2019)
Brown, Susskind, Phys Rev D 97, 086015 (2018)

BROWN-SUSSKIND CONJECTURE

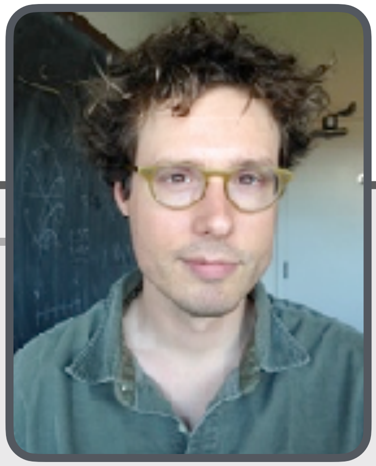
- Has risen to prominence as Brown-Susskind conjecture

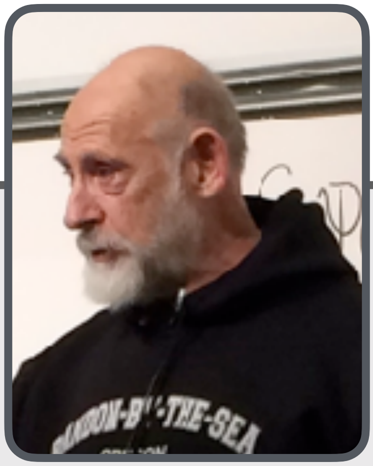
- **AdS:** Volume grows for exponentially long time



- **CFT:** Local observables equilibrating?

$|\psi\rangle$





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Brown, Roberts, Susskind, Swingle, Zhao, Phys Rev D 93, 086006 (2016)
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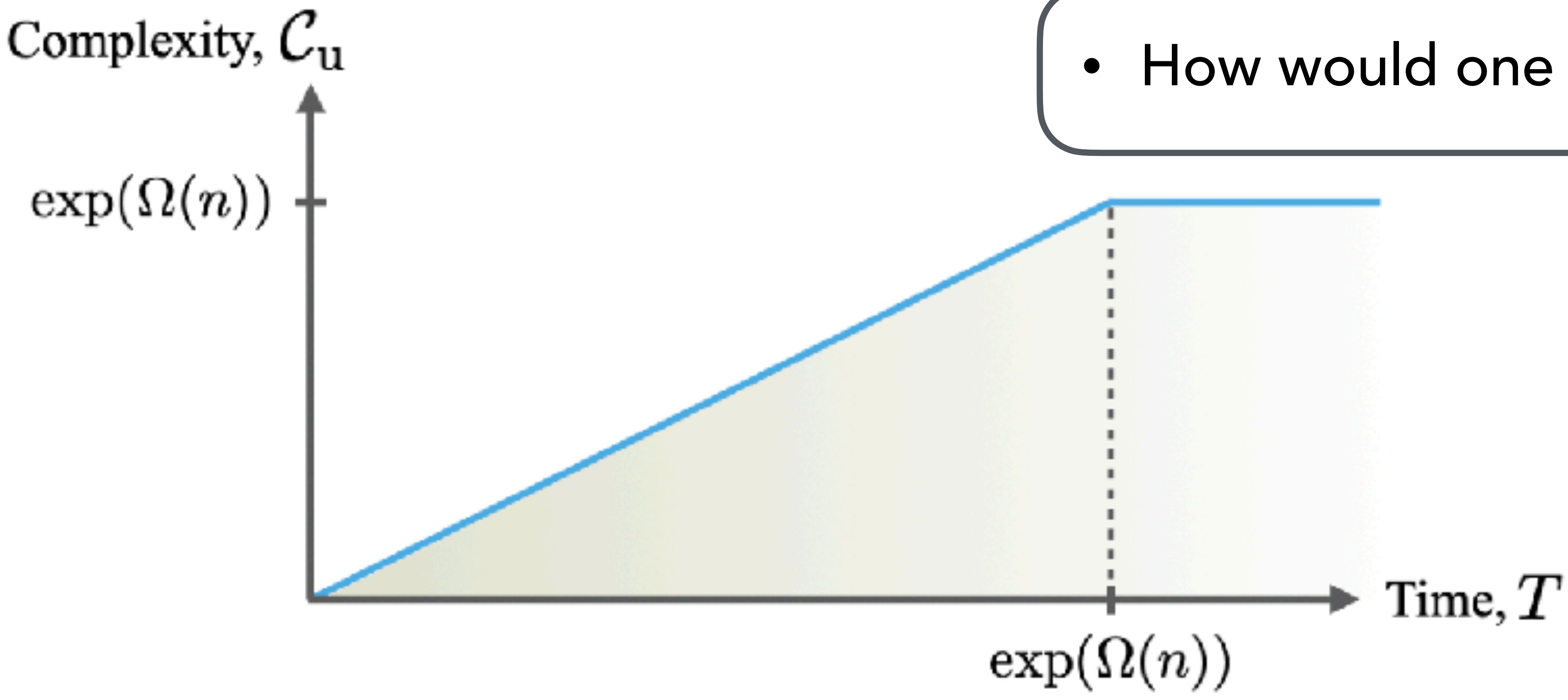


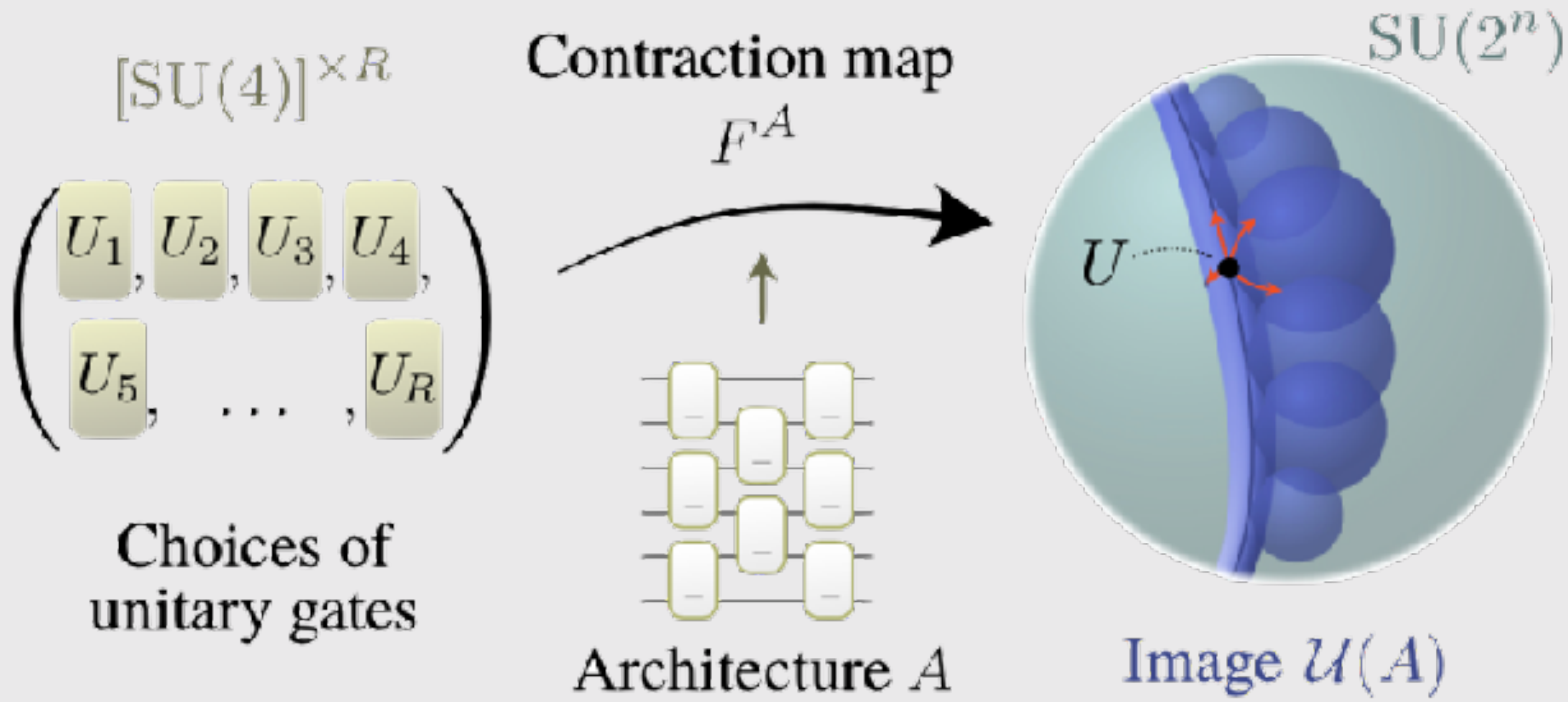
- How would one judge?

Brown, Susskind, Phys Rev D 97, 086015 (2018)

LINEAR COMPLEXITY GROWTH

- Indeed, the linear growth conjecture (until exponential times) is provably **true**!

- **Random Clifford** circuits attain bound



- "Almost all circuits have **maximum dimension**"

- **Tarski-Seidenberg** principle

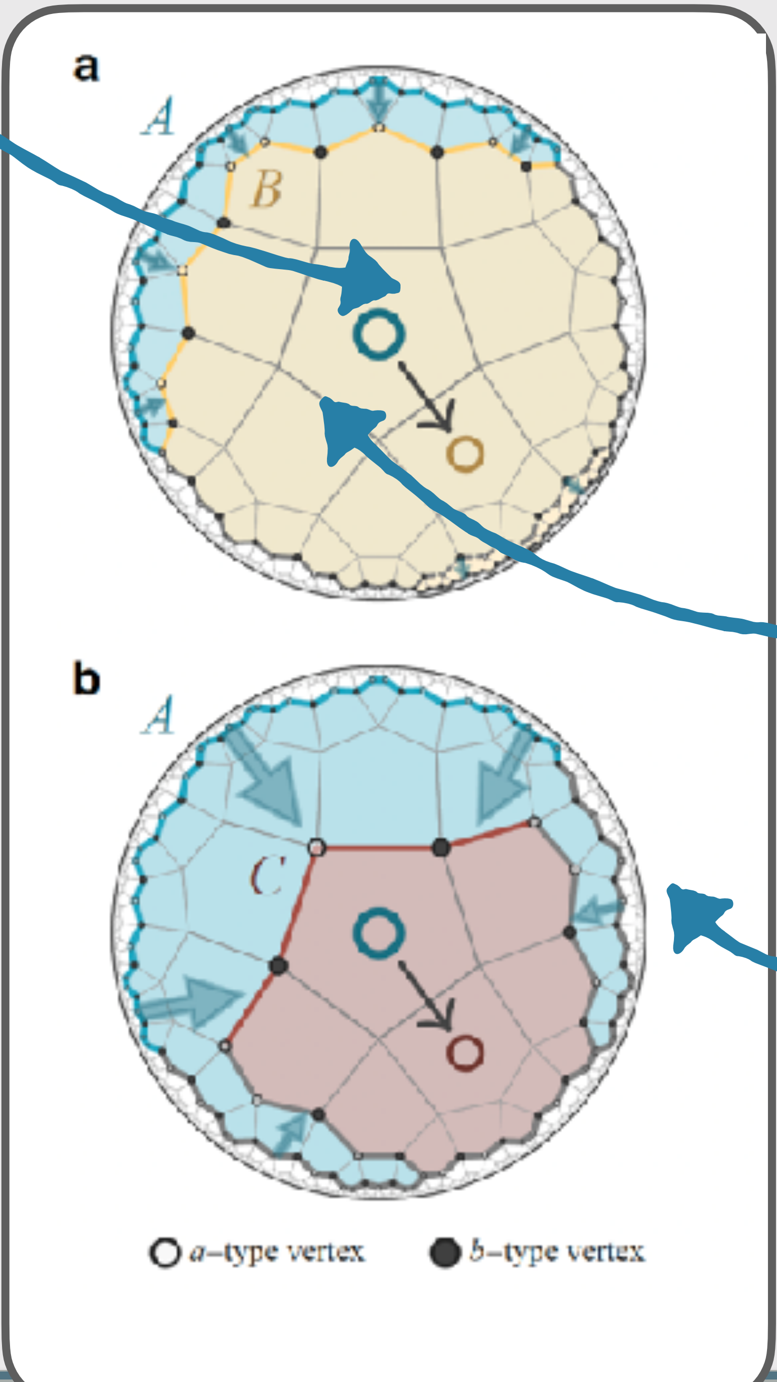
- Is "**quasialgebraic set**": Polynomial equalities and inequalities

Haferkamp, Faist, Kothakonda, Eisert, Yunger-Halpern, Nature Physics, in press (2022)

HOLOGRAPHY AND CRITICAL MODELS

- Toy models of AdS-CFT can be formulated as **matchgate tensor networks**

Matchgate "free fermionic" **tensor networks** on hyperbolic tiling of plane



Inflation rules to go from one layer to the next

Critical theory on boundary with effective central charges depending on tiling, e.g.

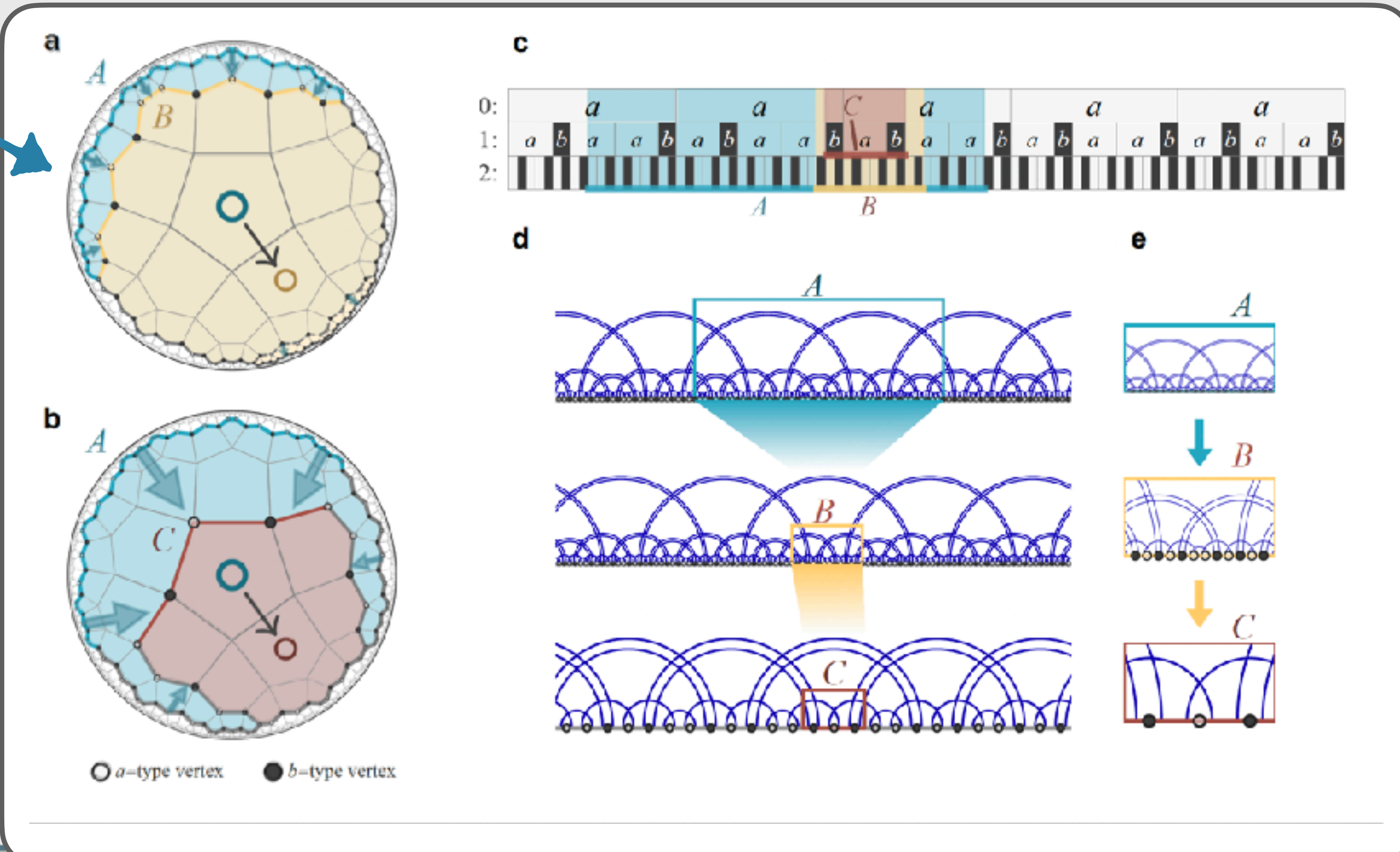
$$c_{\{5,4\}} \approx 4.74$$

Jahn, Zimboras, Eisert, Quantum 6, 643 (2022)
Jahn, Eisert, Quant. Sc. Tech. 6, 033002 (2021)
Wille, Altland, Jahn, Eisert, in preparation (2022)

HOLOGRAPHY AND CRITICAL MODELS

- Toy models of AdS-CFT can be formulated as **matchgate tensor networks**

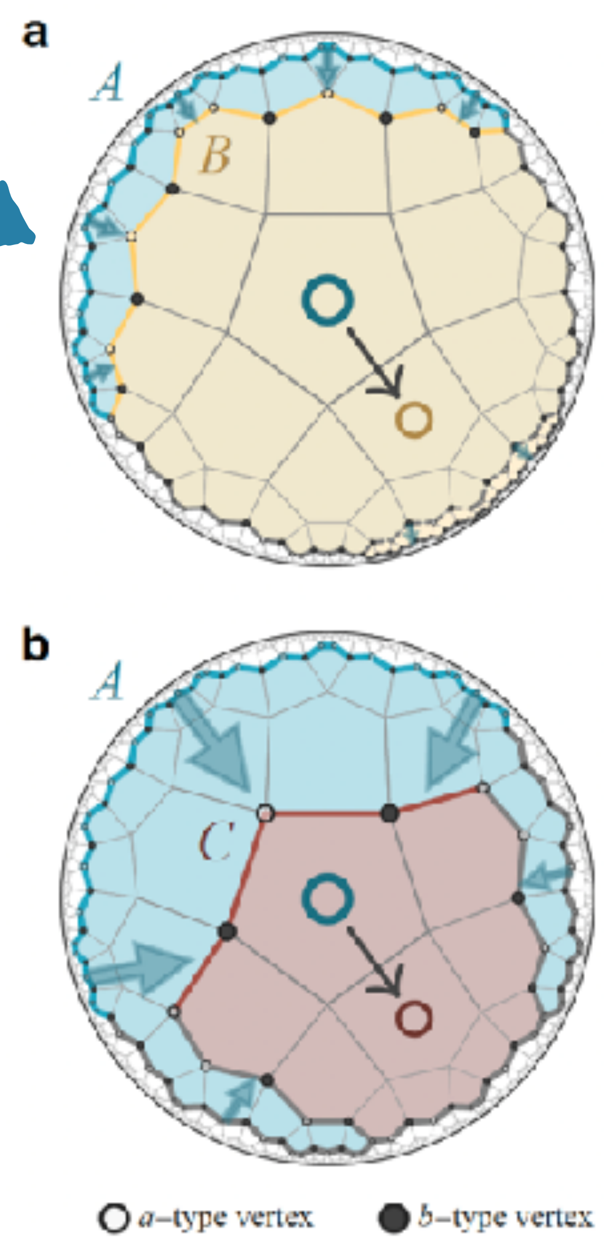
Get actual CFT (up to quasi-crystalline symmetry)



HOLOGRAPHY AND CRITICAL MODELS

Get actual CFT (up to quasi-crystalline symmetry)

- Using random matchgate tensors, can go to the continuum (in prep)



Jahn, Zimboras, Eisert, Quantum 6, 643 (2022)
Jahn, Eisert, Quant. Sc. Tech. 6, 033002 (2021)
Wille, Altland, Jahn, Eisert, in preparation (2022)

ENTANGLEMENT IN MANY-BODY SYSTEMS

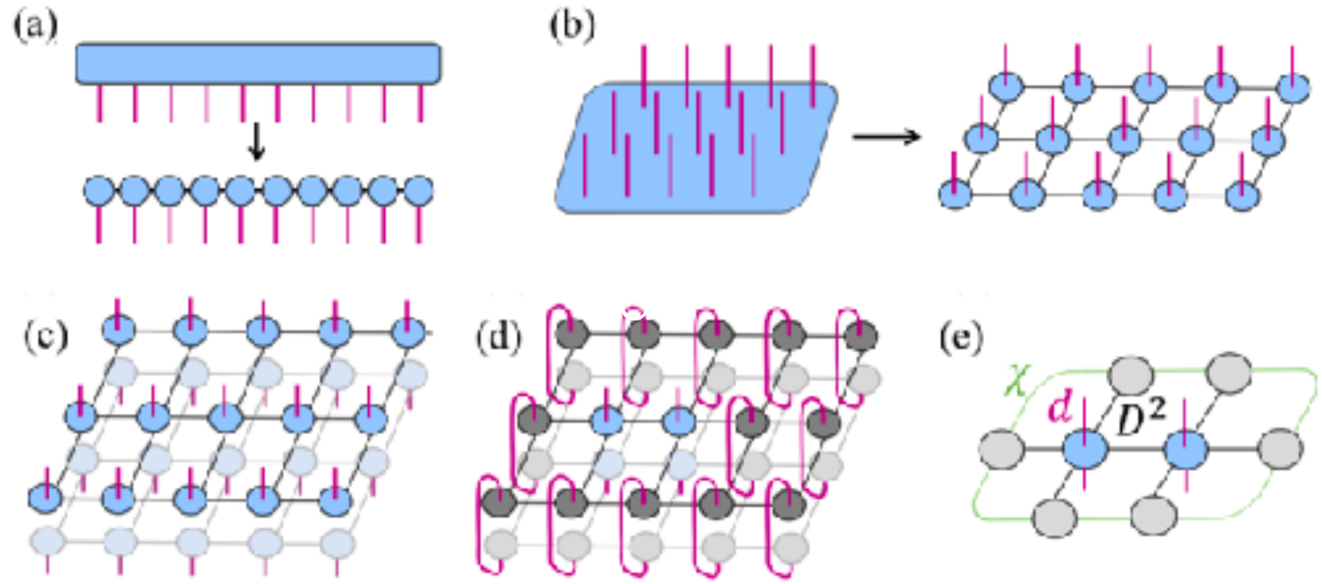
- Last insight: Use random sampling to literally **estimate entanglement** in quantum many-body systems

- Resource-economically estimate **Renyi entanglement entropies**

$$E_n(A) = \frac{1}{1-n} \log \mathrm{tr}(\rho_A^n)$$

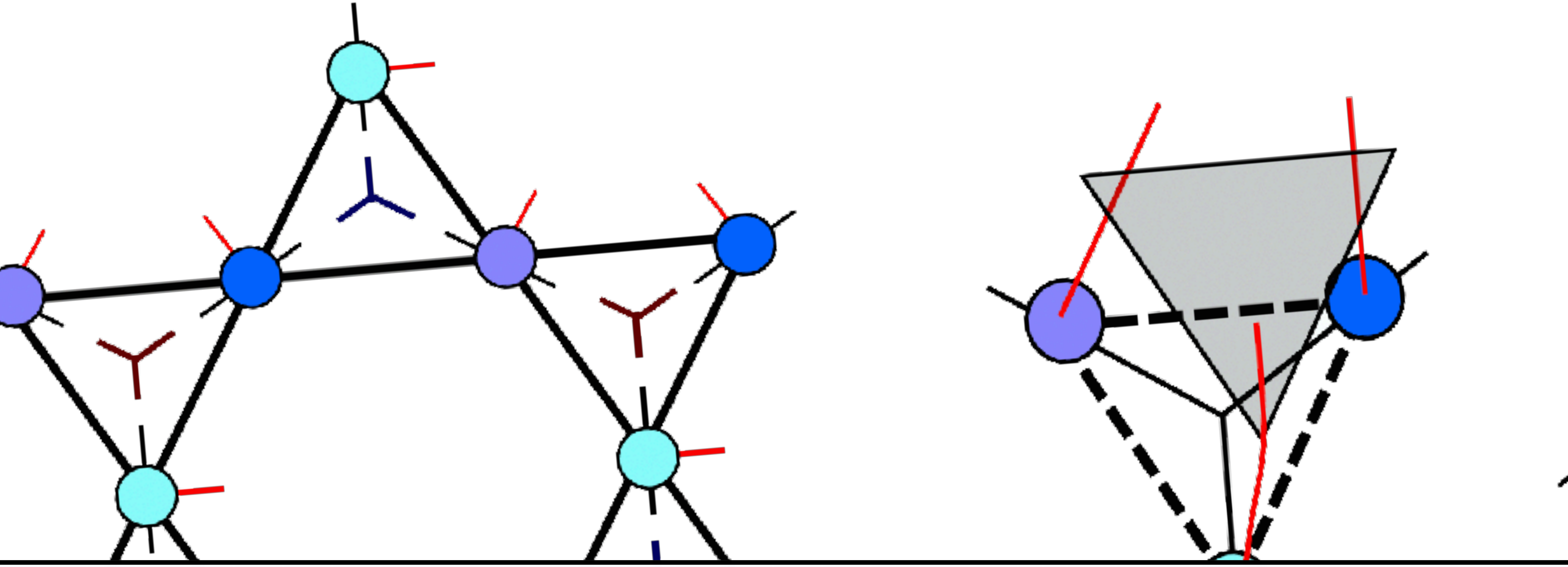
and negativity moments using **frames**, random vectors $|v\rangle \in \mathbb{C}^d$ with

$$\mathbb{E}(|v\rangle\langle v|) = \mathbb{I}$$



Feldman, Kshetrimayum, Eisert, Goldstein, arXiv:2202.04089 (2022)

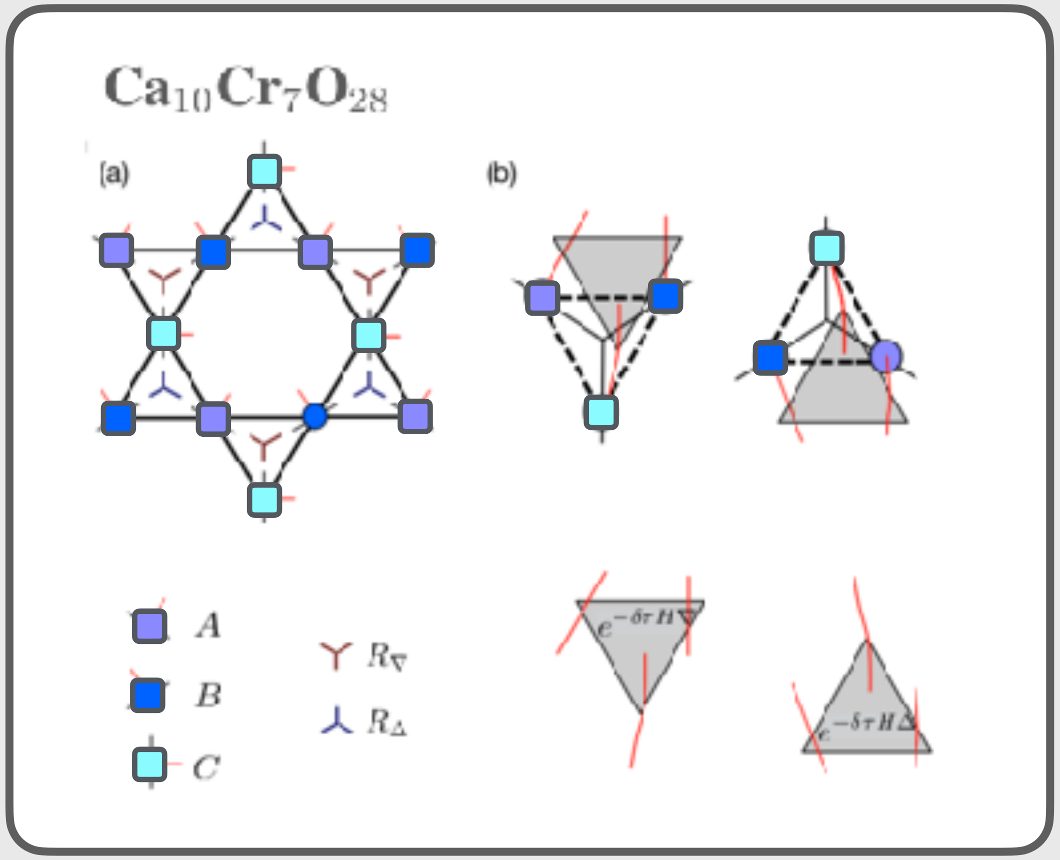
- **Lesson:** Random tensor networks are a fun playground for analytical studies



TENSOR NETWORKS FOR QUANTUM MATERIALS

- Tensor networks to explore **experimental quantum materials**?

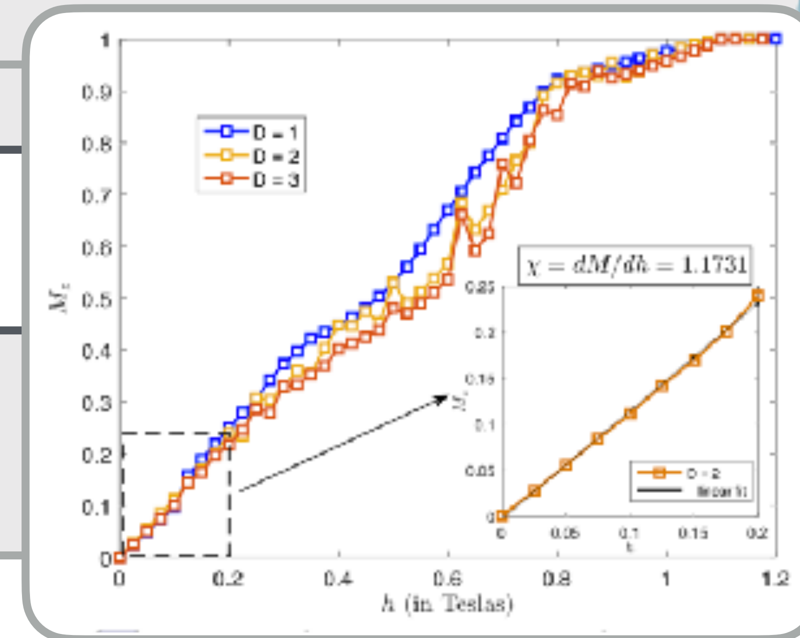
- Collaboration with Bella Lake



- Tensor network study of **double layer Kagome** compound (PESS algorithm)

Kshetrimayum, Balz, Lake, Eisert, Ann Phys 421, 168292 (2020)

- Tensor networks to explore **experimental**



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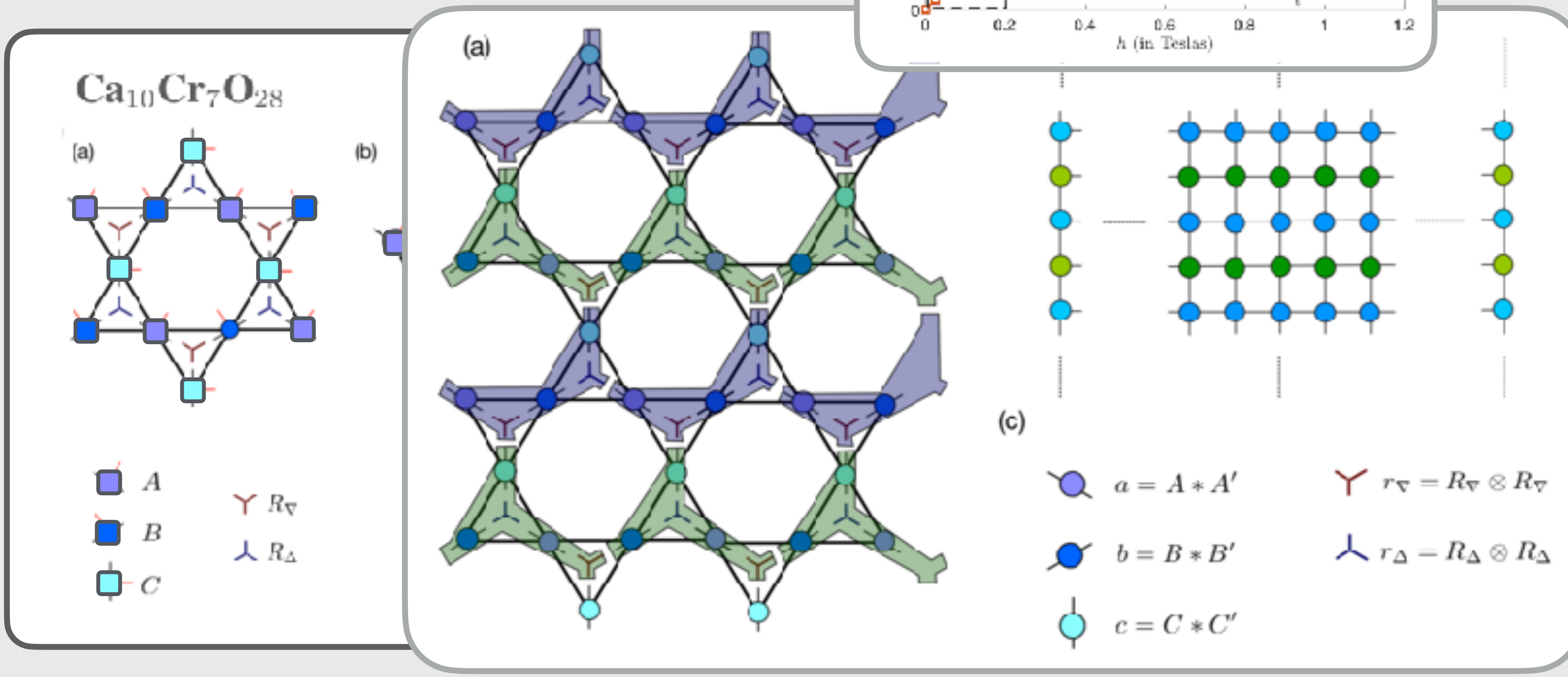


- Tensor network study of **double layer Kagome** compound (PESS algorithm)

Kshetrimayum, Balz, Lake, Eisert, Ann Phys 421, 168292 (2020)

MANY BODY LOCALIZATION IN QUANTUM MATERIALS?

- Can one think of quantum materials that feature **many-body localization**?

Nietner, Kshetrimayum, Eisert, Lake, in preparation (2022)
Ovadia, Scientific Rep 5, 1 (2015)

MANY BODY LOCALIZATION IN QUANTUM MATERIALS?

- Exploit the inherent randomness in a **doping process** of two quantum materials as source of randomness

$$H_{\text{doping}} = \sum_i \frac{J_\perp^{(i)}}{2}(s_i^+ s_{i+1}^- + s_i^- s_{i+1}^+) + \sum_i J_z^{(i)} s_i^z s_{i+1}^z$$

  - **Discrete** disorder
  - Disorder in **hoppings** and on-site
  - Reflect actual **doping** process

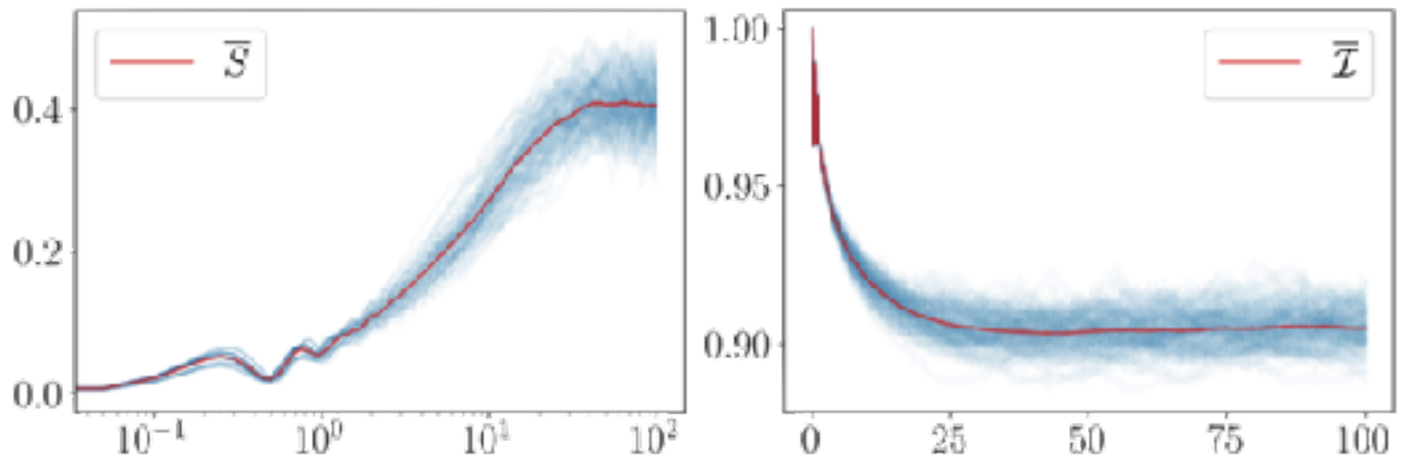
- **Doping strength** $\delta \in [0, 1]$ reflects relative weight of $J_\perp^{(i)}$ and $J_z^{(i)}$, $i = 0, 1$

- **Candidate materials**: $CsCoBr_3$ and $CsCoCl_3$

Nietner, Kshetrimayum, Eisert, Lake, in preparation (2022)

MANY BODY LOCALIZATION IN QUANTUM MATERIALS?

- Simulate with **matrix product states**



Entanglement entropy

Imbalance

$$\mathcal{I}(t) := \frac{1}{L}\sum_{i=1}^{L}(-1)^i \langle\psi(t)|s_i^z|\psi(t)\rangle$$

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Nietner, Kshetrimayum, Eisert, Lake, in preparation (2022)

MANY BODY LOCALIZATION IN QUANTUM MATERIALS?

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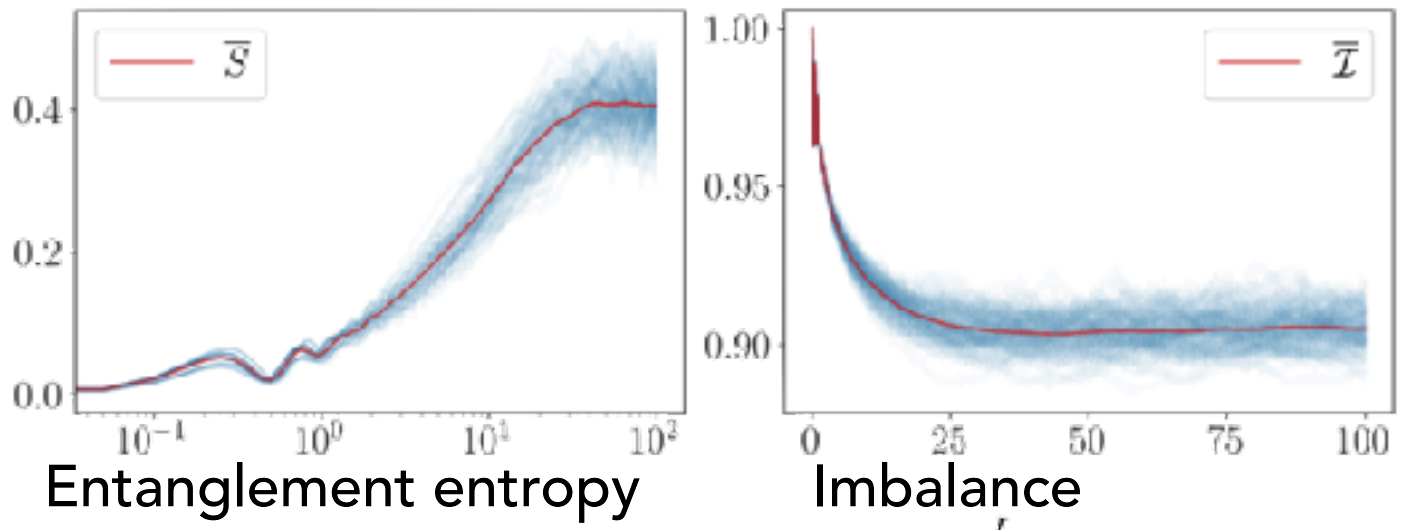


$$\mathcal{I}(t) := \frac{1}{L}\sum_{i=1}^{L}(-1)^i \langle\psi(t)|s_i^z|\psi(t)\rangle$$

- **Heart of matter**: Robustness to phonons

$$H = H_{\text{spin}} + H_{\text{phonon}} + H_{\text{spin-phonon}}$$

$$H_{\text{phonon}} = \sum_{\langle i,j\rangle}\omega_{i,j}a^\dagger_{i,j}a_{i,j} + \sum_{\langle\langle i,j\rangle,\langle k,l\rangle\rangle}\kappa_{i,j,k,l}a^\dagger_{i,j}a_{k,l}$$

$$H_{\text{spin-phonon}} = \sum_{\langle i,j\rangle}g_{i,j}\left(a^\dagger_{i,j}+a_{i,j}\right)\left(s_i^+s_j^- + s_i^-s_j^+\right)$$

- Evidence that they remain **localized** even in presence of interactions

Nietner, Kshetrimayum, Eisert, Lake, in preparation (2022)

MANY BODY LOCALIZATION IN QUANTUM MATERIALS?

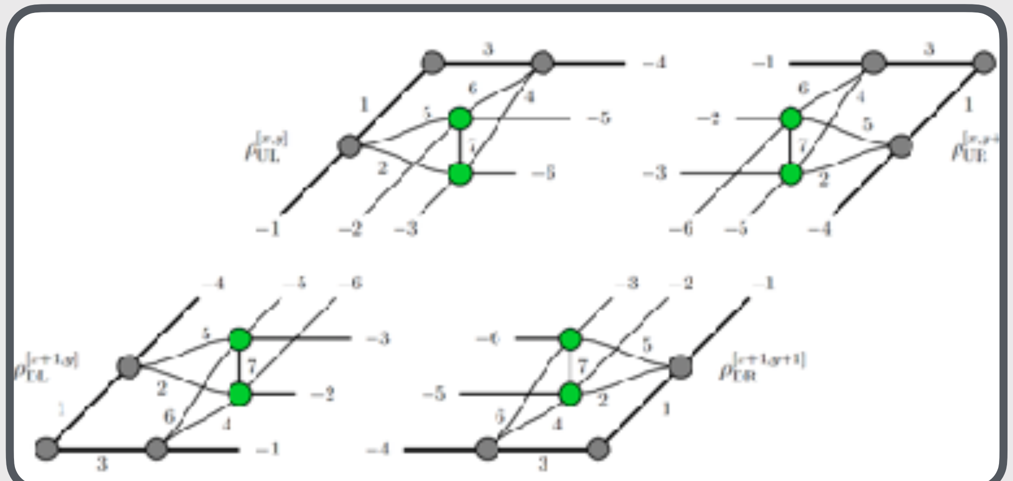
- There is substantiated hope that many-body localized material can be synthesized

Nietner, Kshetrimayum, Eisert, Lake, in preparation (2022)

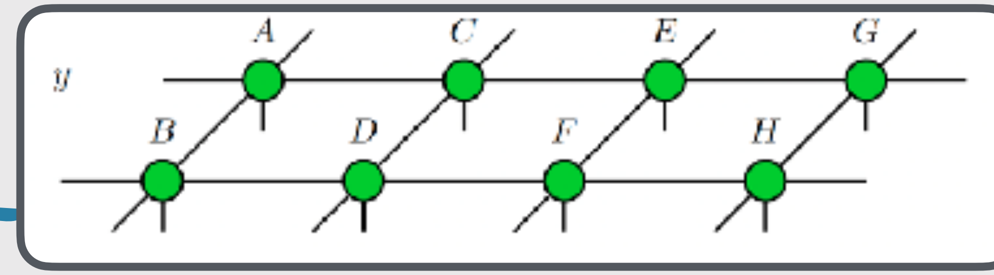
AUTOMATIC DIFFERENTIATION IPEPS

- Implement a library of corner transfer matrix renormalization group (CTMRG) optimization for iPEPS based on **automatic differentiation**?

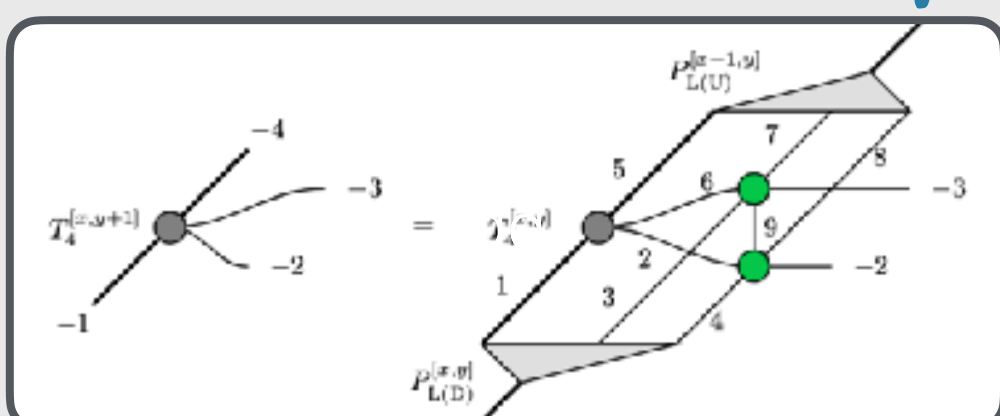
- CTMRG tensors used to compute **truncation projectors**



- Arbitrary **unit cells**



- Update **transfer matrices**



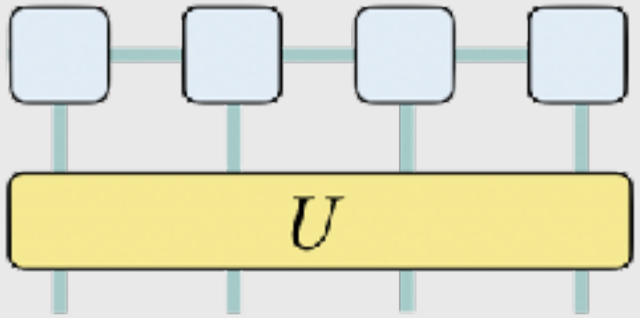
- Exploit **automatic differentiation** in tensor parameters t for environment e^*

$$L : \mathbb{C}^d \longrightarrow \mathbb{R}$$
$$t \longmapsto E(t, e^*, H)$$
$$\nabla_t L = \nabla_0 E(t, e^*, H) + \nabla_1 E(t, e^*, H) \nabla_t e^*$$

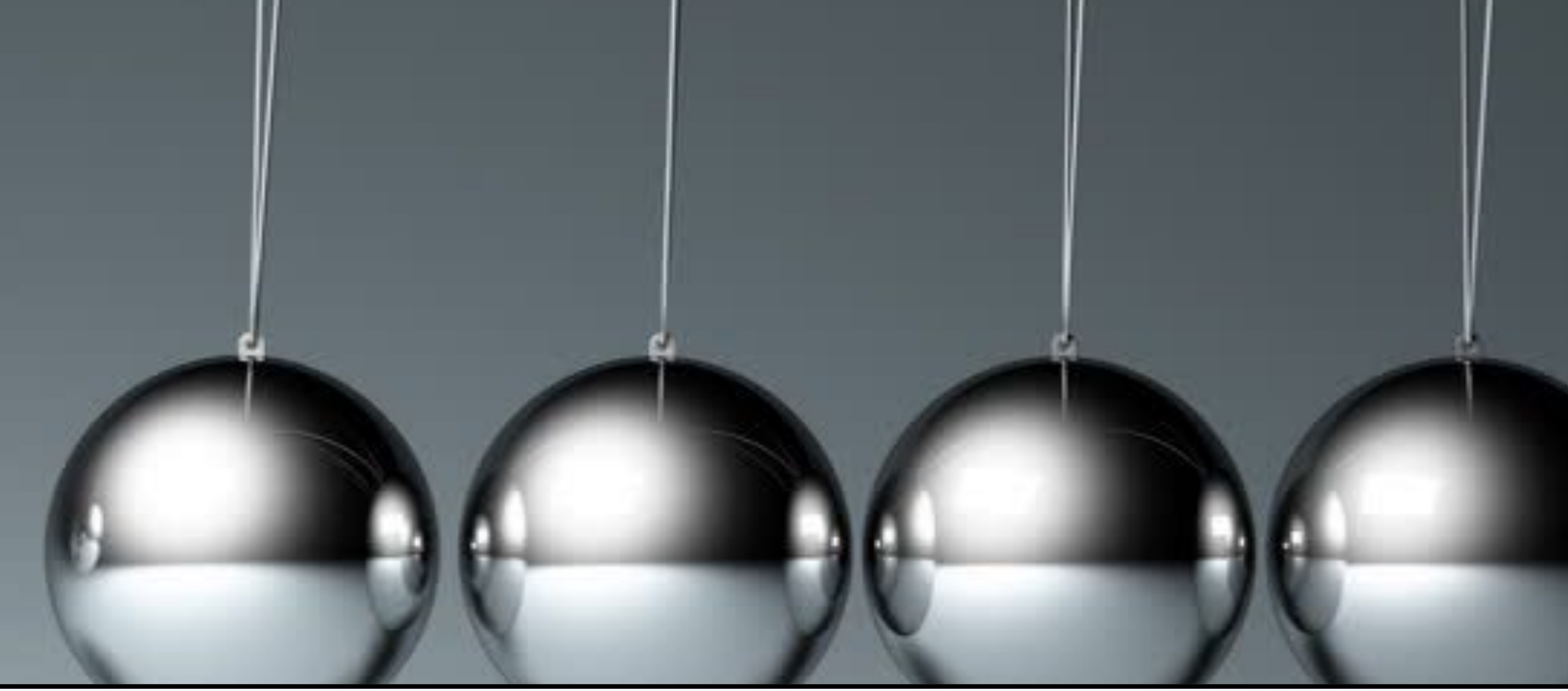
Naumann, Schmoll, Nietner, Kshetrimayum, Eisert, Chen, in preparation (2022)
Liao, Liu, Wang, Xiang, Phys Rev X 9, 031041 (2019)
Ponsioen, Assaad, Corboz, arXiv:2107.03399 (2021)
Tu, Wu, Schuch, Kawashima, Chen, arXiv:2101.03935 (2021)

- **Lesson**: It is fun to see ideas on tensor networks related to features of **quantum materials**

- Tensor networks with **mode transformations** (coffee break)



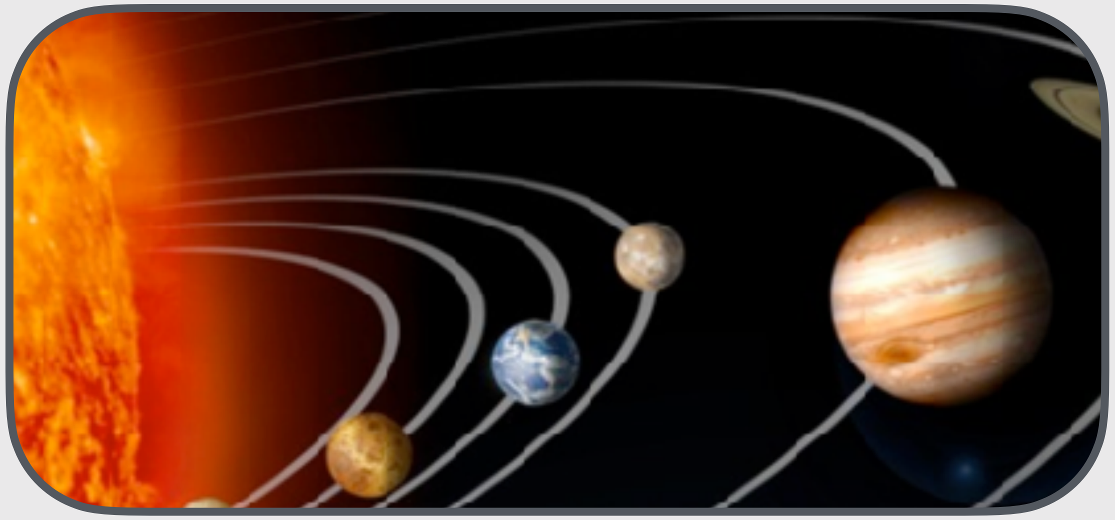
Krumnow, Veis, Eisert, Legeza, Phys Rev B 104, 075137 (2021)
Krumnow, Veis, Legeza, Eisert, Phys Rev Lett 117, 210402 (2016)

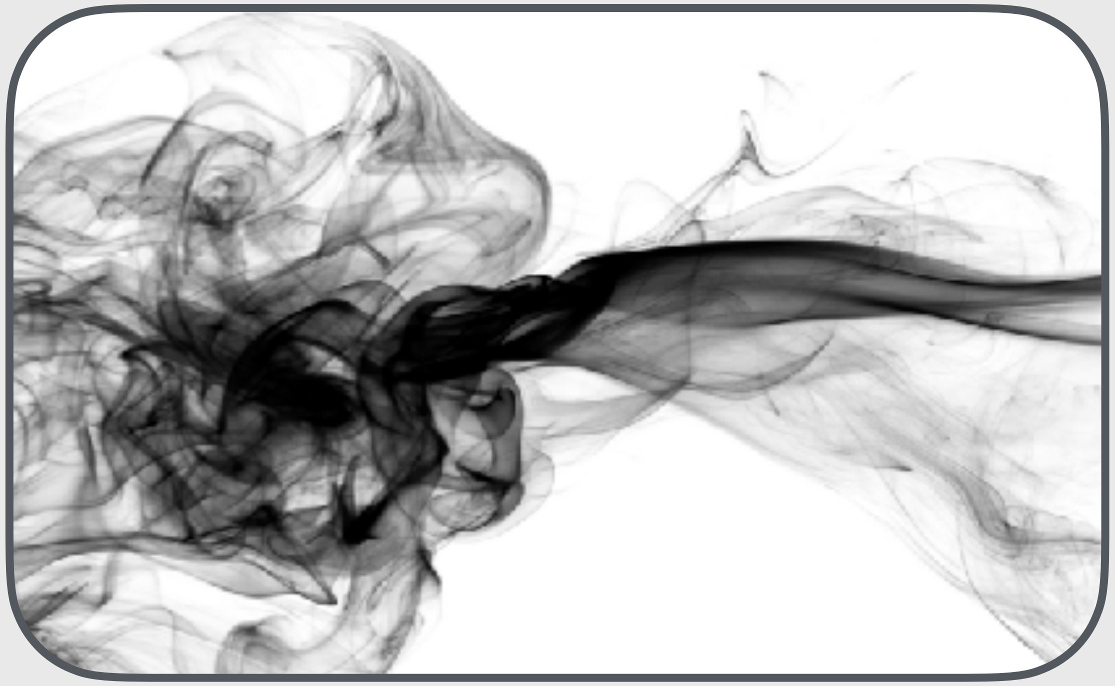


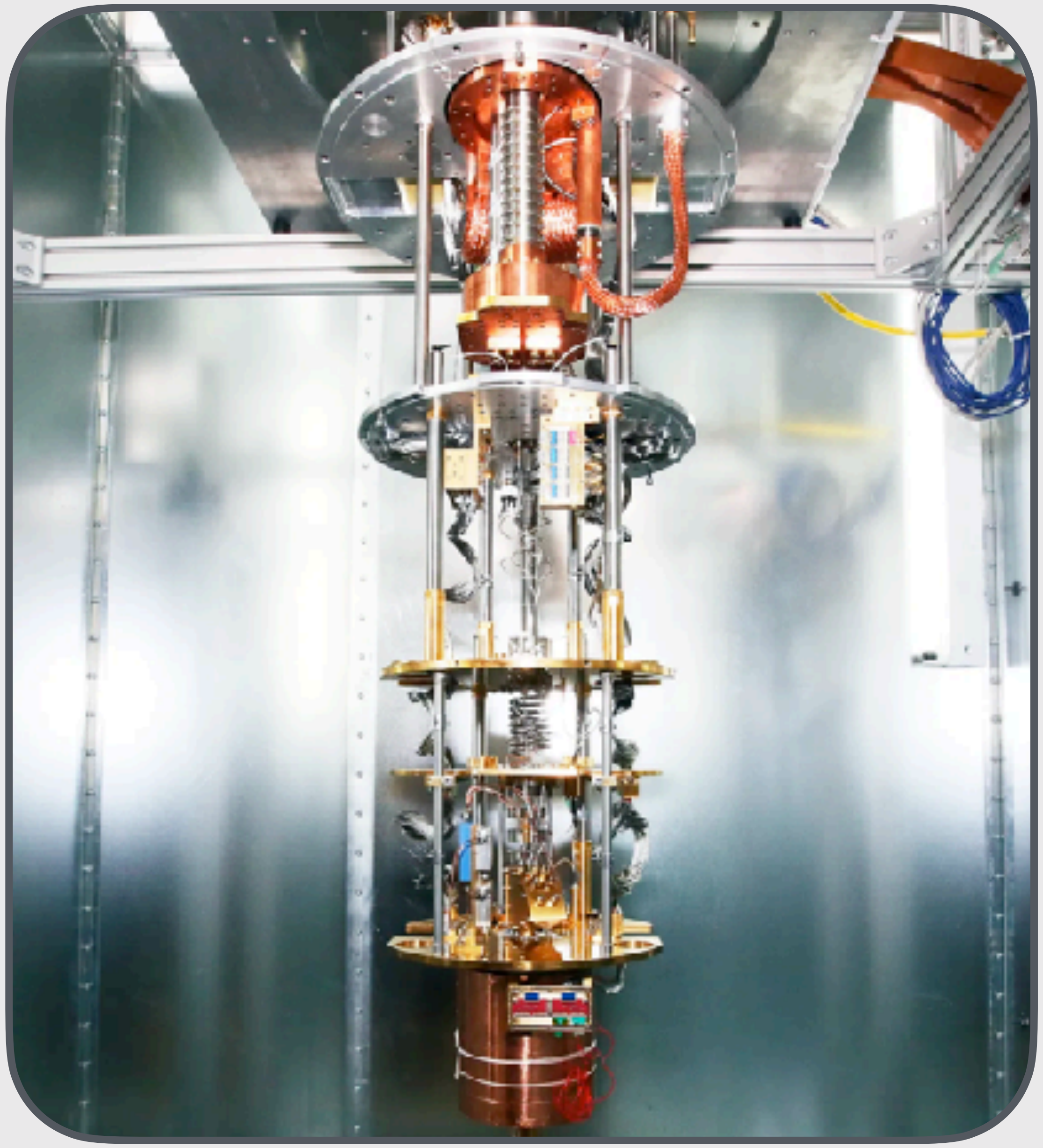
TENSOR NETWORKS IN (MACHINE) LEARNING TASKS

LEARNING DYNAMICAL LAWS

- How can one scalably learn **dynamical laws** from data?

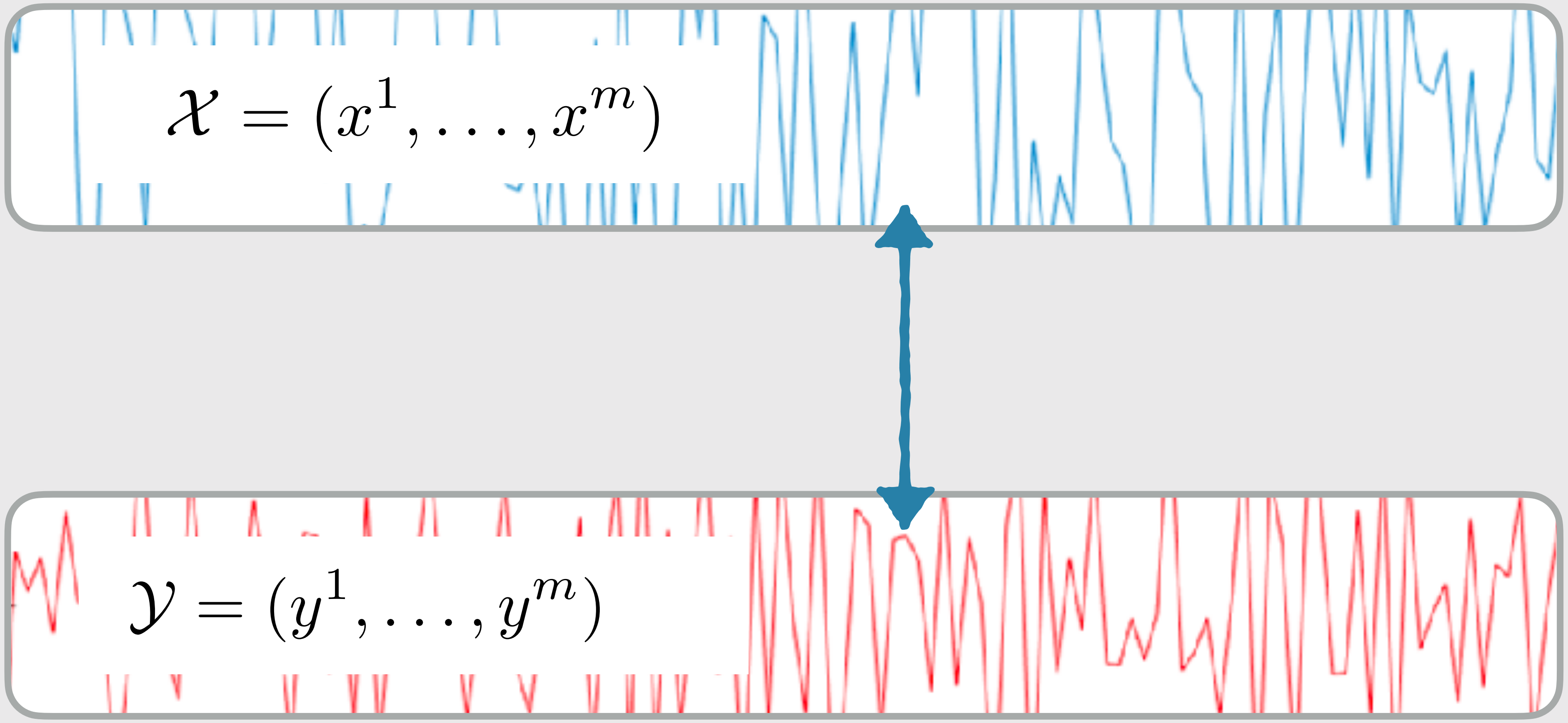






LEARNING CLASSICAL LAWS WITH TENSOR NETWORKS

- Learn dynamical laws from data



- Task: Identify the governing equations $f = (f_1, \ldots, f_d)$ from observations

$$\{x^j, y^j := f(x^j)\}_{j=1}^m$$

Goeßmann, Götte, Roth, Sweke, Kutyniok, Eisert, NeurIPS (2021)
Gelss, Klus, Eisert, Schuette,
J Comput Nonlinear Dynam 14, 061006 (2019)

- Define a **dictionary** $\{\psi_1, \ldots, \psi_p\}$ of basis functions
- Construct transformed **data matrix**

$$\Psi(\mathcal{X}) = \begin{bmatrix} \psi_1(x_1) & \cdots & \psi_1(x_m) \\ \vdots & \ddots & \vdots \\ \psi_p(x_1) & \ldots & \psi_p(x_m) \end{bmatrix}$$

- Determine the **coefficient matrix**

$$\Xi = [\xi_1 \ldots \xi_d]$$

such that the cost function is minimized

$$\|\mathcal{Y} - \Xi^T \Psi(\mathcal{X})\|_2 + \lambda \|\Xi\|_1$$

Brunton, Proctor, Kutz, Proc Natl Ac Sc 113, 3932 (2016)

- Define a **dictiona**
- Construct transf

$$\Psi(\mathcal{X}) = \begin{bmatrix} \psi \\ \\ \psi \end{bmatrix}$$

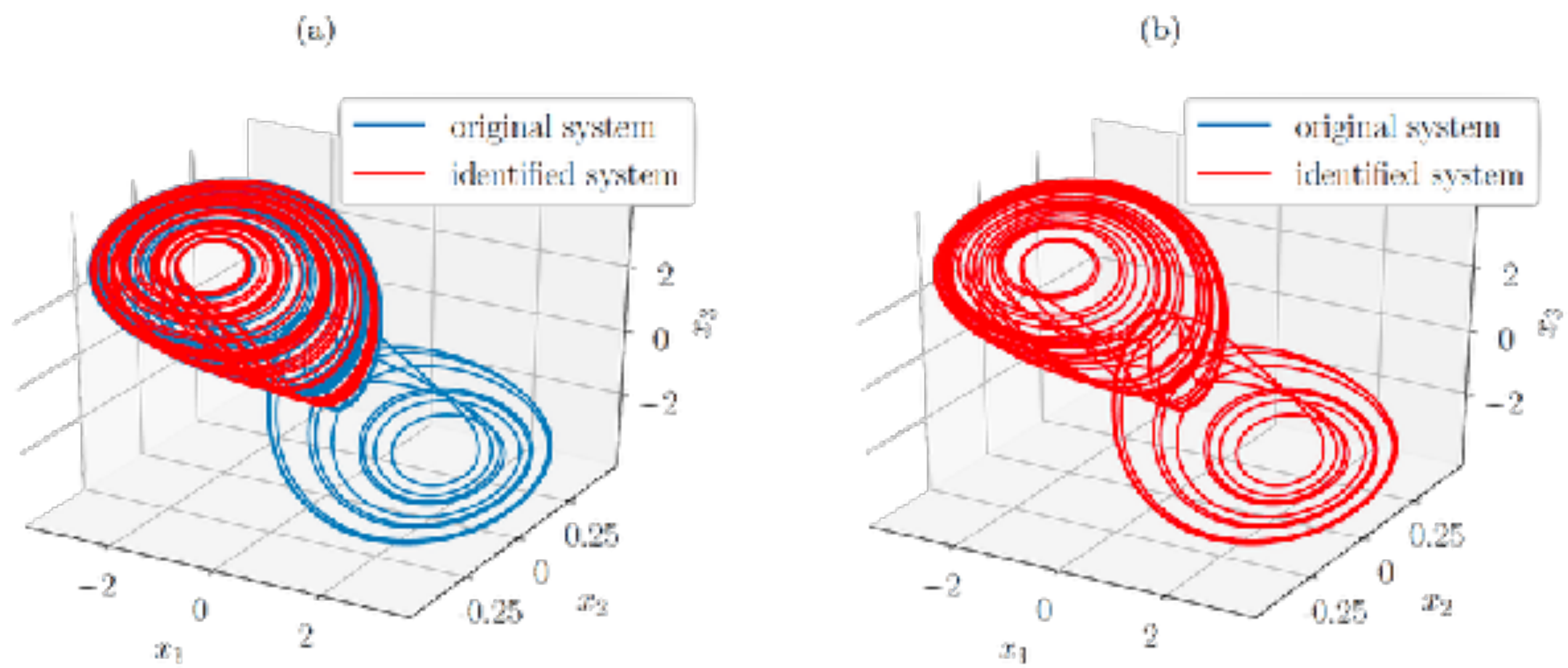
- Determine the **c**

such that the co

$$\|\mathcal{Y} - \Xi^T$$

- **SINDy** finds a **sparse** coefficient matrix
- E.g., recovers Chua's circuit well

$$\dot{x}_1 = \alpha(x_2 - x_1 - g(x_1))$$
$$\dot{x}_2 = x_1 - x_2 + x_3$$
$$\dot{x}_3 = -\beta x_2$$

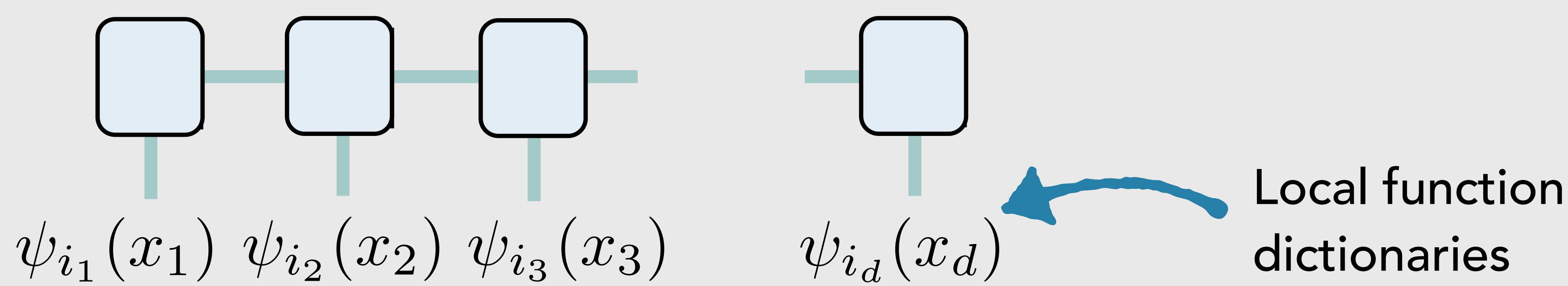


Brunton, Proctor, Kutz, Proc Natl Ac Sc 113, 3932 (2016)

LEARNING CLASSICAL LAWS

- For large systems, need **hypothesis class** for basis dictionaries

    - Use **matrix product ansatzes**



Goeßmann, Götte, Roth, Sweke, Kutyniok, Eisert, arXiv:2002.12388, NeurIPS (2021)

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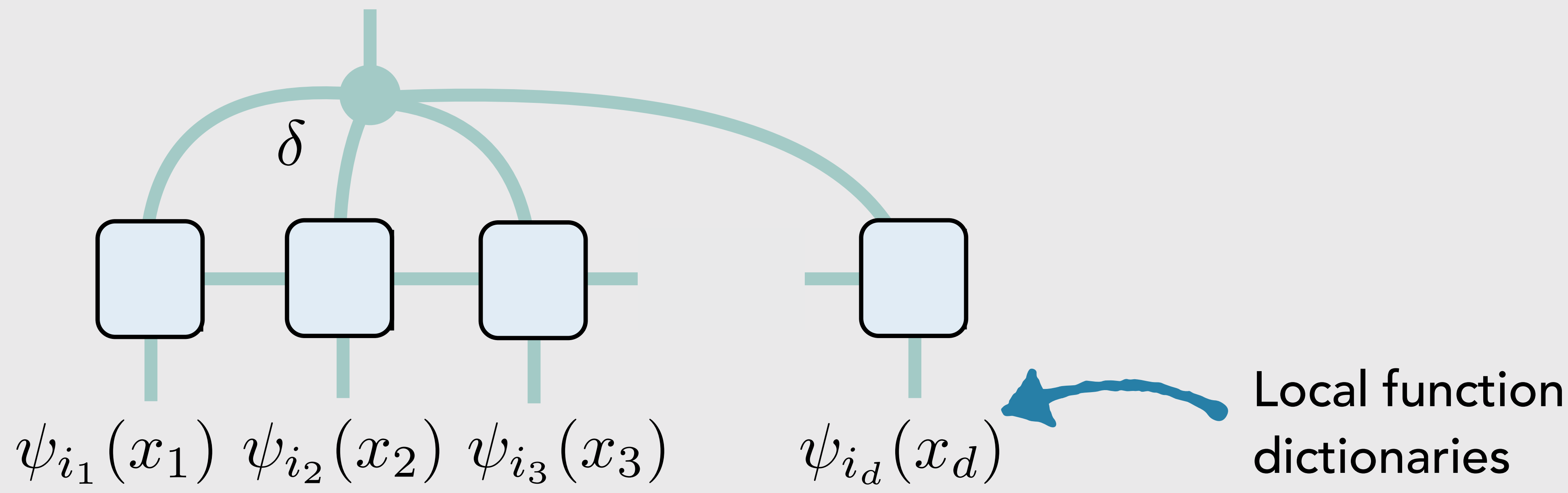

Goeßmann, Götte, Roth, Sweke, Kutyniok, Eisert, arXiv:2002.12388, NeurIPS (2021)

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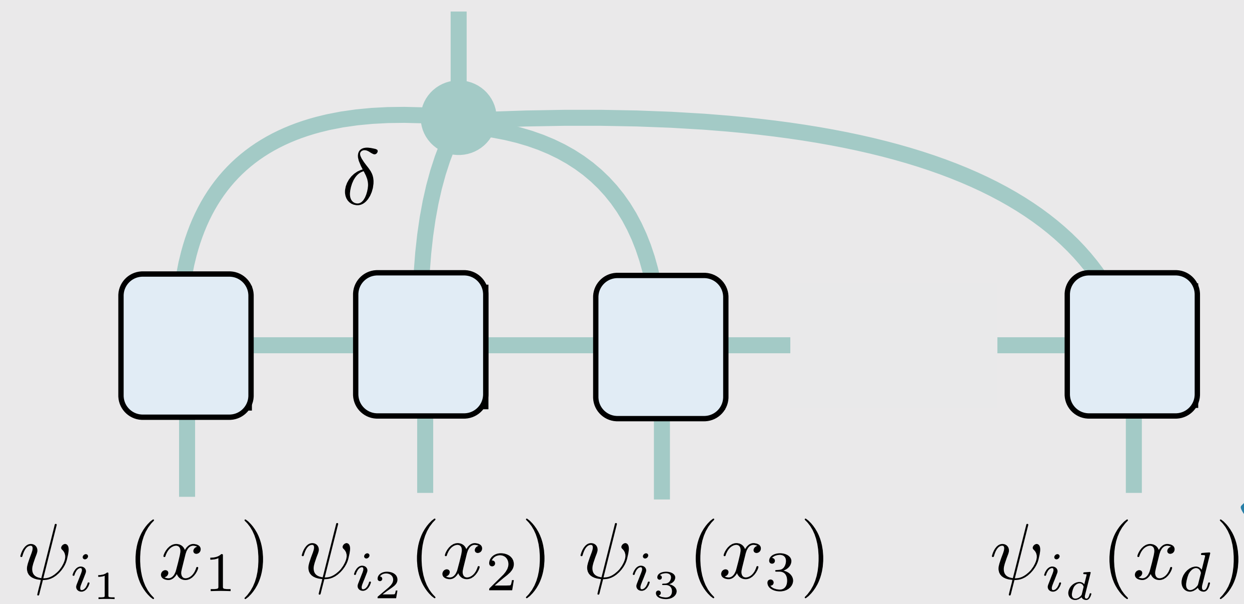


- **Stabilized alternating least squares** (SALSA)

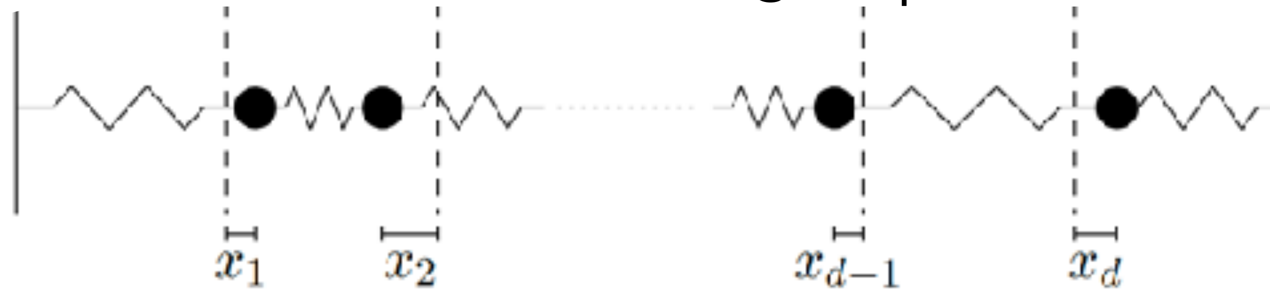
$$\underset{\mathcal{N}^k \in \mathbb{R}^{r_{k-1} \times \tilde{n} \times r_k}}{\text{minimize}} \|f(\mathcal{L}, \mathcal{N}^k, \mathcal{R}) - y\|_F^2 + \omega^2 \left(\|\Sigma_{\mathcal{L},c}^{-1} \mathcal{N}^k\|_F^2 + \|\mathcal{N}^k \Sigma_{\mathcal{R},c}^{-1}\|_F^2 \right). \qquad (P_{k-s})$$

- Get **good recovery**

Goeßmann, Götte, Roth, Sweke, Kutyniok, Eisert, arXiv:2002.12388, NeurIPS (2021)

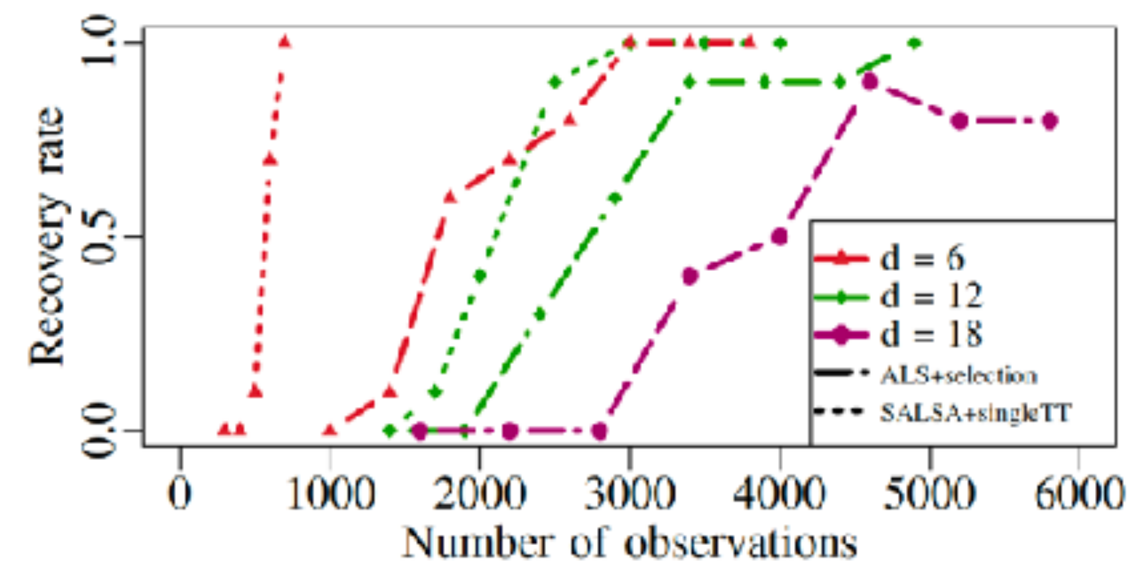
LEARNING CLA

- E.g., Fermi-Pasta-Ulam-Tsingou problem



with dictionary of the first $p = 4$ Legendre polys per site

$$\ddot{x}_i = (x_{i+1} - 2x_i + x_{i-1}) + \beta[(x_{i+1} - x_i)^3 - (x_i - x_{i-1})^3]$$



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$$+ \omega^2 \left(\|\Sigma_{\mathcal{L},c}^{-1} \mathcal{N}^k\|_F^2 + \|\mathcal{N}^k \Sigma_{\mathcal{R},c}^{-1}\|_F^2 \right).$$

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Goeßmann, Götte, Roth, Sweke, Kutyniok, Eisert, arXiv:2002.12388, NeurIPS (2021)
In preparation (2022)

LEARNING CLASSICAL LAWS

- Using tensor networks, one can learn **classical dynamical laws**
- **Now**: Expressivity, “entanglement” in function spaces

Goeßmann, Götte, Roth, Sweke, Kutyniok, Eisert, arXiv:2002.12388, NeurIPS (2021)
In preparation (2022)

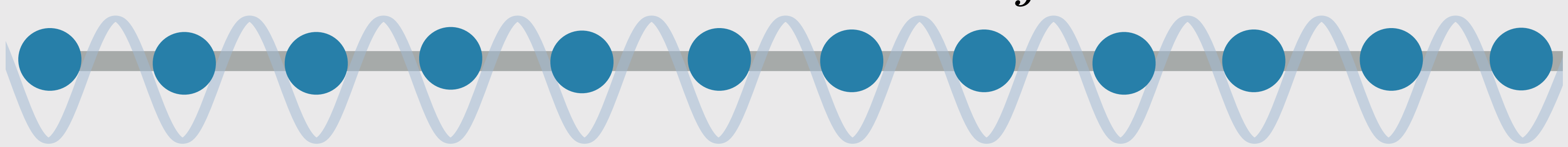
- How can **quantum Hamiltonians** be recovered?
- Important in the **quantum technologies**

Goeßmann, Götte, Roth, Sweke, Kutyniok, Eisert, arXiv:2002.12388, NeurIPS (2021)
In preparation (2022)

HAMILTONIAN LEARNING

To be learned parameters

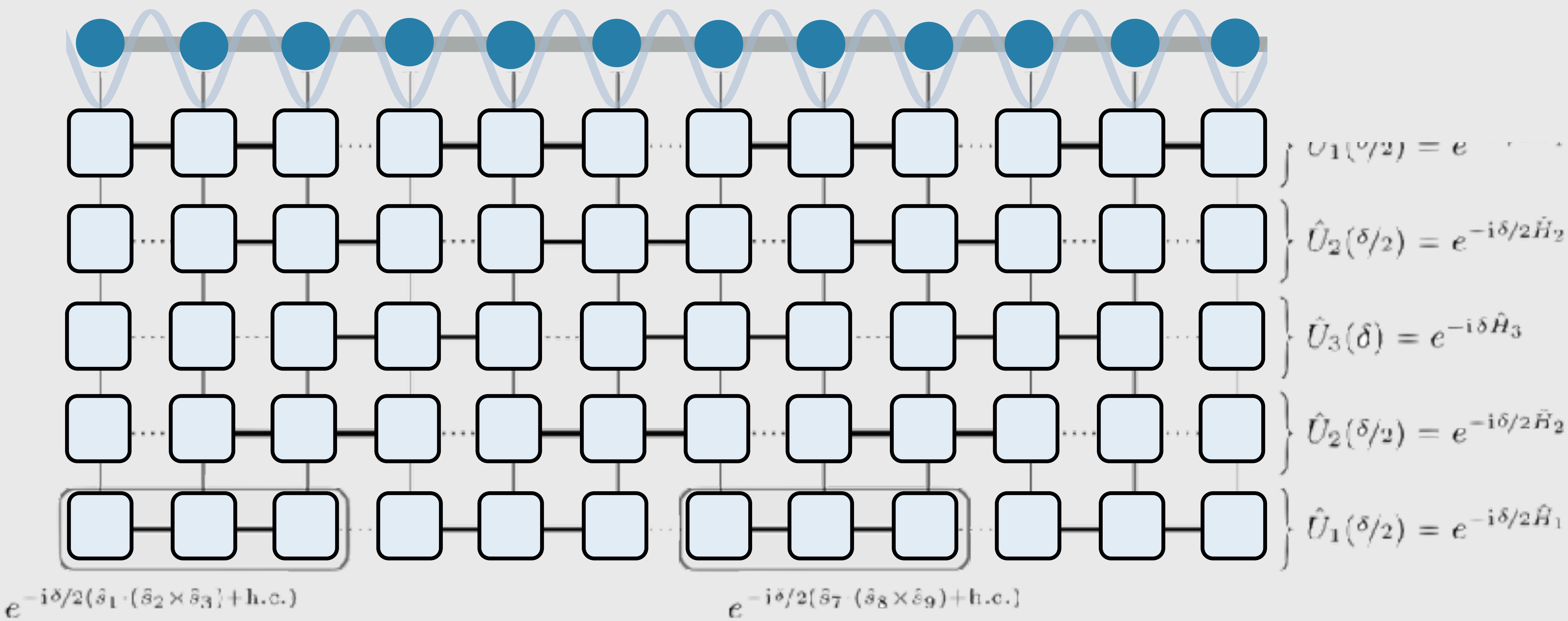
$$H = \sum_j \alpha_j H_j$$



- Recover $\{\alpha_j\}$ from time series data

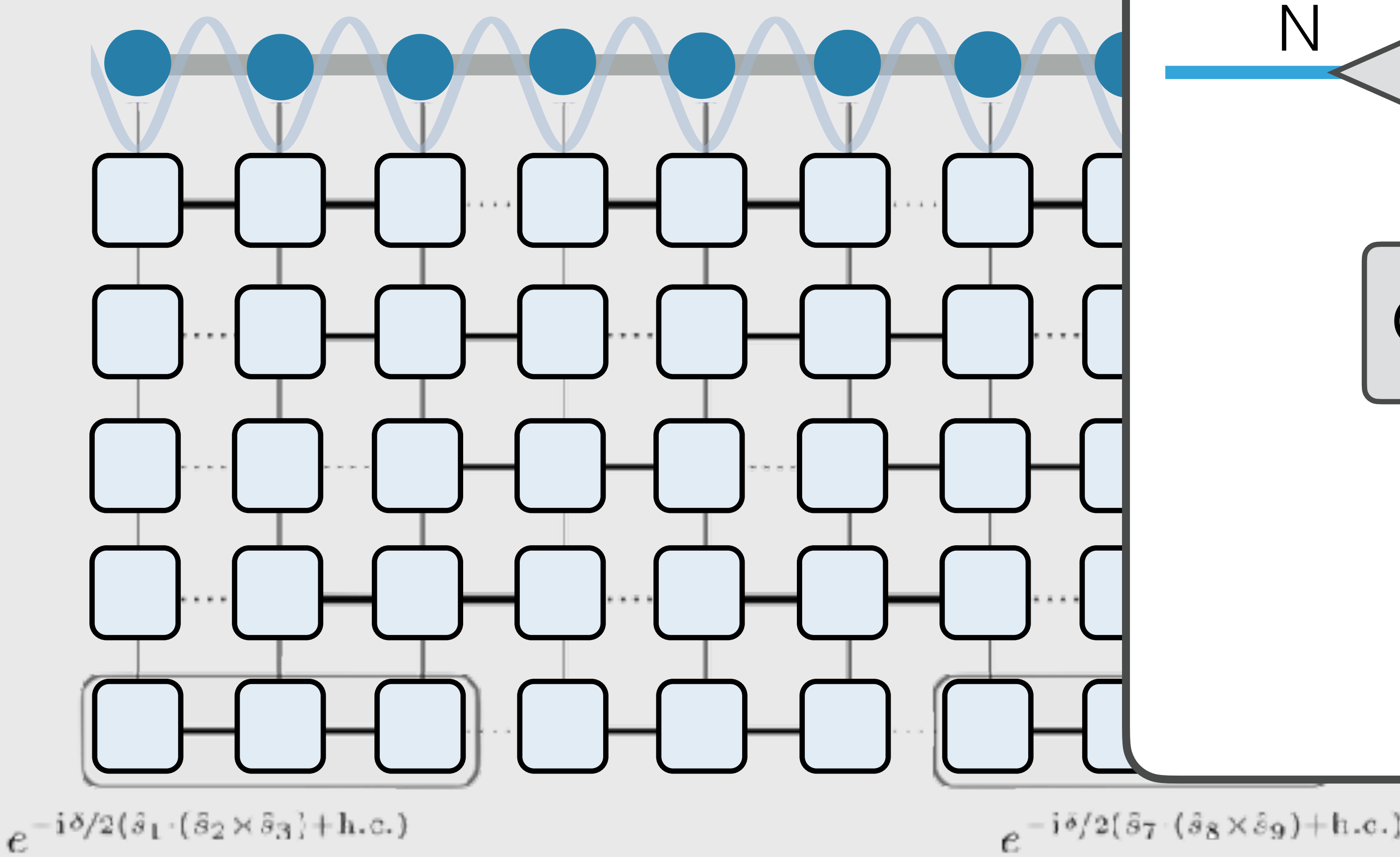
$$y_{m,n}[l] = \mathrm{tr}(e^{-it_l H} \rho_n e^{-it_l H} A_m)$$

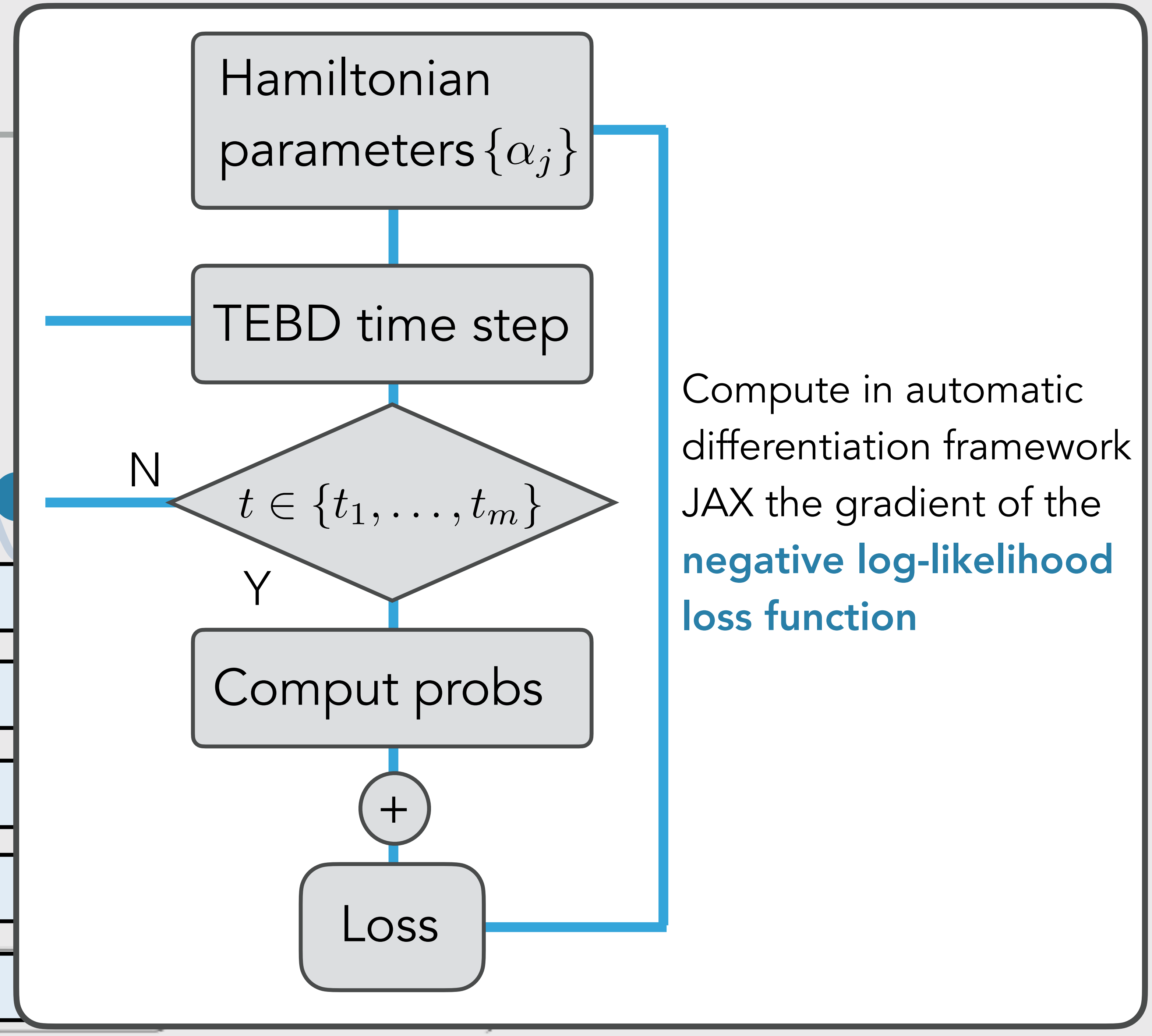
at times $t_1, t_2, \ldots, t_L$ up to tolerance $\epsilon > 0$



Wilde, Sweke, Kshetrimayum, Roth, Eisert, in preparation (2022)

HAMILTONIAN LEARNING





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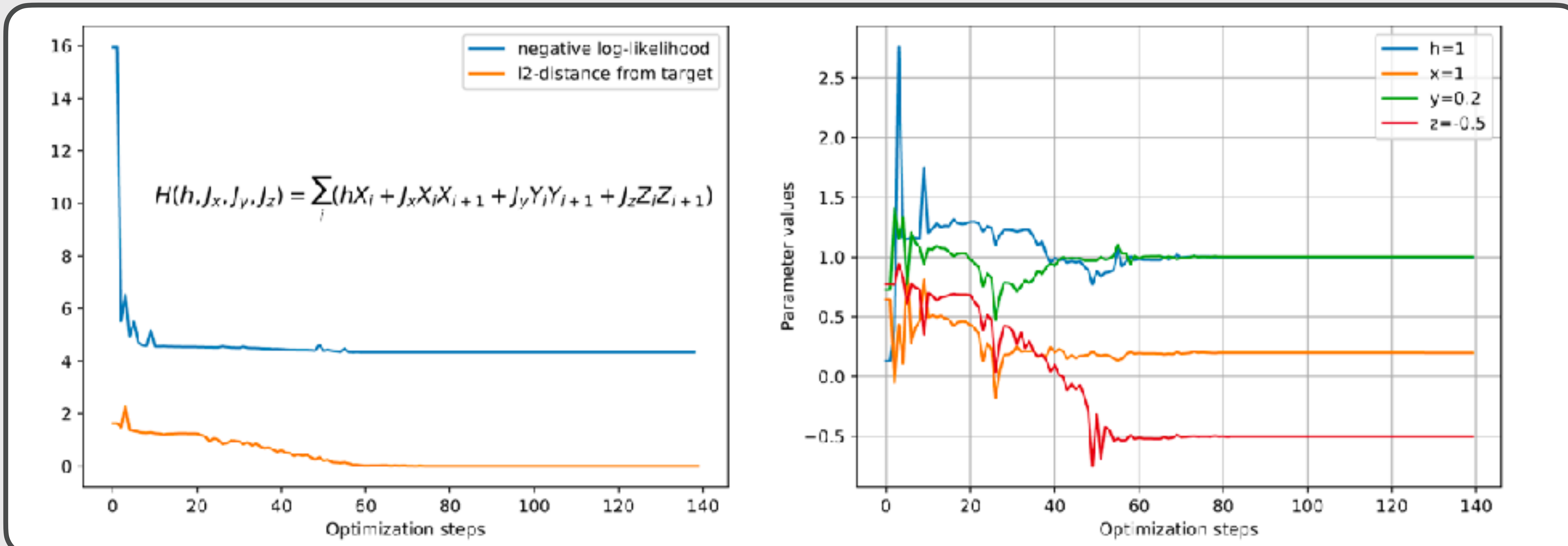
HAMILTONIAN LEARNING



$e^{-i\delta/2(\hat{s}_1\cdot(\hat{s}_2\times\hat{s}_3)+\mathrm{h.c.})}$ $e^{-i\delta/2(\hat{s}_7\cdot(\hat{s}_8\times\hat{s}_9)+\mathrm{h.c.})}$

- Works well to e.g. learn **Heisenberg-type Hamiltonians** from concept class

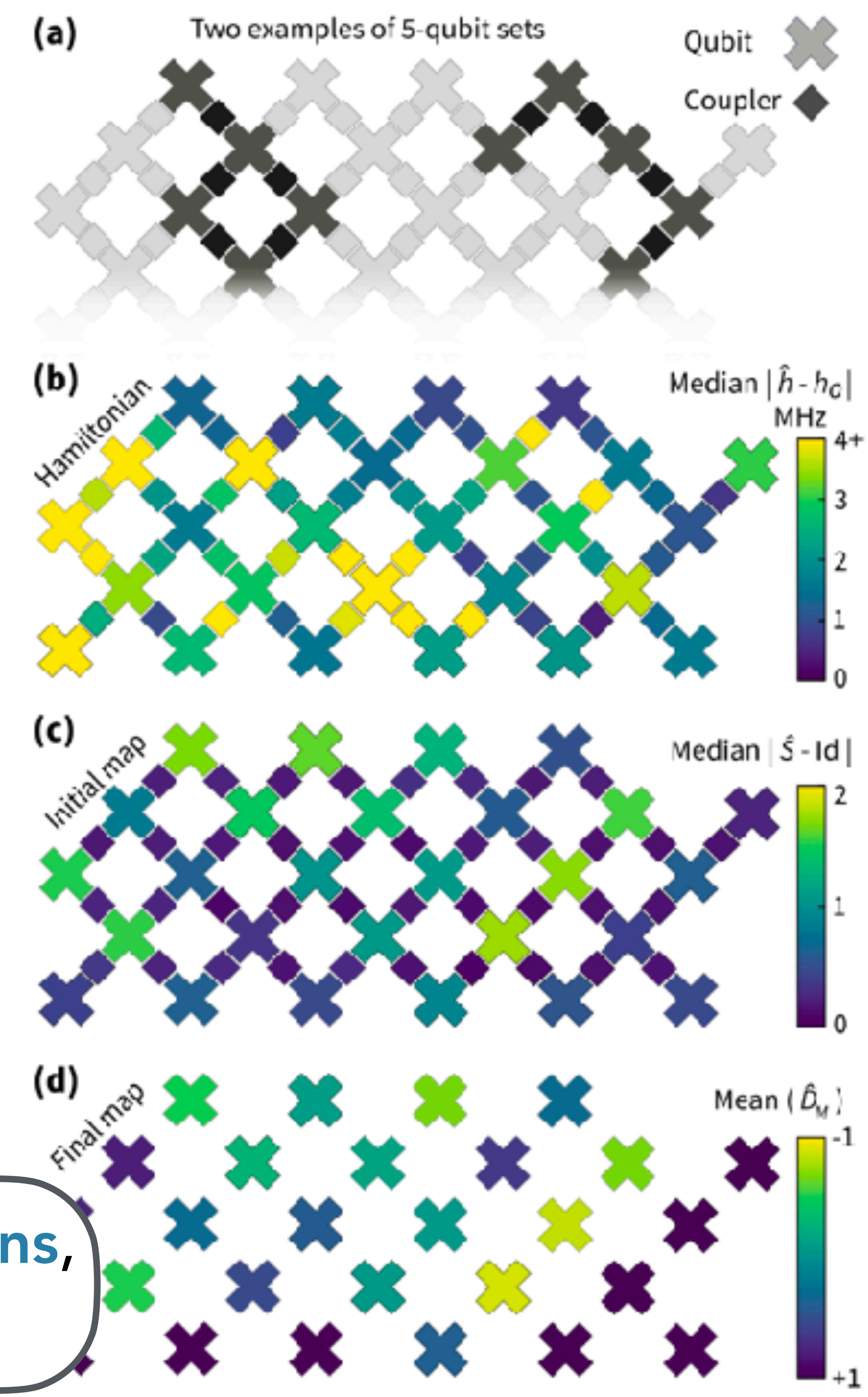
$$H(h, J_x, J_y, J_z) = \sum_j (hX_j + J_xX_jX_{j+1} + J_yY_jY_{j+1} + J_zZ_jZ_{j+1})$$

- **Errors** of 10^{-2} in l_1-distance for system sizes of $n = 100$

Wilde, Sweke, Kshetrimayum, Roth, Eisert, in preparation (2022)

HAMILTONIAN LEARNING

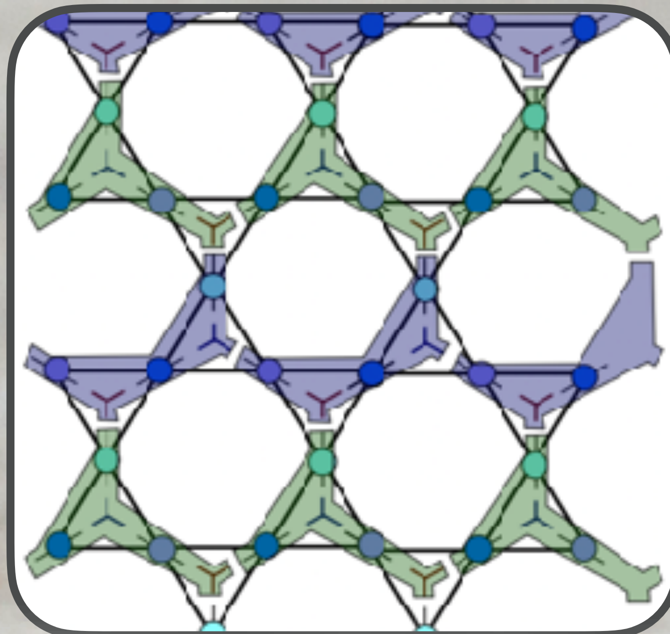
- Similar techniques to learn Google AI's Sycamore chip



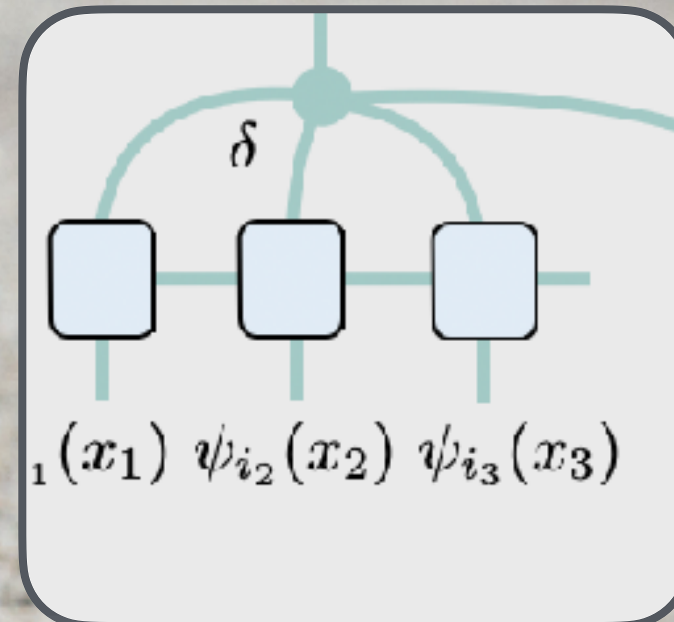
- Can very precisely learn Hamiltonians, then make predictions

Hangleiter, Roth, Eisert, Roushan, arXiv:2108.08319 (2021)

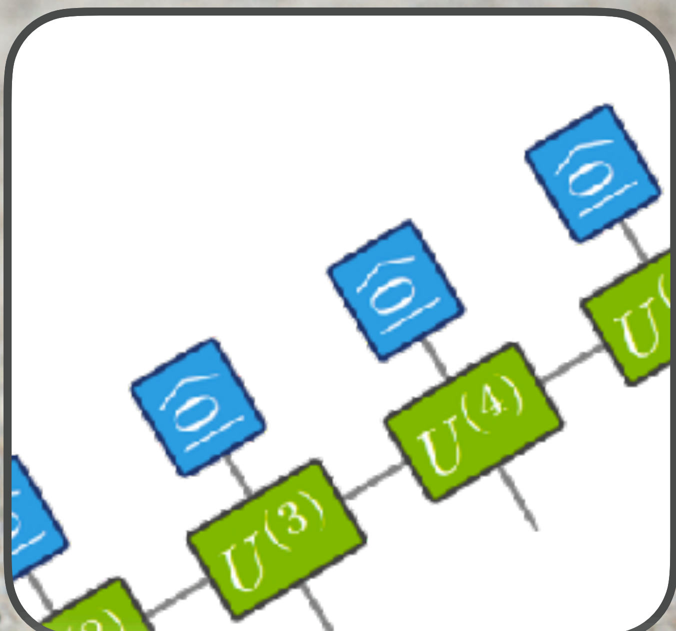
OUTLOOK



- **Tensor networks** can be use to capture properties of quantum materials



- **Tensor networks** in (machine) learning tasks



- **Random tensor networks** allow for analytical insights out of reach otherwise

THANKS FOR YOUR ATTENTION