TAE 2015
Centro de Ciencias de Benasque

Cosmology - Theory

Professor: Joan Solà
(Dept. ECM, UB, sola@ecm.ub.edu)

Tutorial: i) Choose to do any 2 of the following exercises.

1. Consider the system of equations formed by the Friedmann-Lemaître equation with non-vanishing cosmological constant ($\Lambda = 8\pi\, G_N\, \rho_\Lambda$),

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi\, G_N}{3}\,(\rho_m + \rho_\Lambda) - \frac{k}{a^2}\,, \tag{1}$$

and the dynamical evolution equation for the scale factor,

$$\ddot{a} = -\frac{4\pi\, G_N}{3}\,(\rho_m + 3\, p_m - 2\,\rho_\Lambda)\, a\,. \tag{2}$$

i) Show that $H^2 + \dot{H} = \ddot{a}/a$ and use the previous equations to obtain $\dot{H}$ in terms of the pressure and density. What is its qualitative time evolution (increases or decreases with the expansion)?

ii) Consider now the possibility that $\Lambda = \Lambda(t)$. Why this is allowed by the Cosmological Principle? Show that the equation

$$\dot{\rho}_\Lambda + \dot{\rho}_m + 3\, H\,(\rho_m + p_m) = 0 \tag{3}$$

is a first integral of the previous system; that is, a differential equation of smaller order that can replace Eq. (2). For $\dot{\rho}_\Lambda \neq 0$ there is an exchange of energy between matter and vacuum. Show in detail that any of these three equations (1),(2) and (3) can be derived from the other two. What happens for strictly constant Λ? Integrate Eq. (3) in this case and provide the explicit form of $\rho_m(a)$ as a function of the scale factor, assuming that matter is non-relativistic. **Hint**: You can trade the derivative with respect the cosmic time for the derivative with respect to the scale factor with the help of $d/dt = aHd/da$ (why?).

iii) If the matter energy-momentum tensor $T_{\mu\nu}$ is non-conserved in the presence of a variable ρ_Λ, what is the conserved energy-momentum tensor now? Write it down explicitly: $\tilde{T}_{\mu\nu} = T_{\mu\nu} + ?$.

iv) Using the FLRW metric explicitly, re-derive equation (3) from the local covariant conservation law of $\tilde{T}_{\mu\nu}$, i.e. $\nabla^\mu \tilde{T}_{\mu\nu} = 0$.

v) Starting from equation (2), prove that the deceleration parameter of the universe ($q \equiv -\ddot{a}\, a/\dot{a}^2$) in the presence of several fluids with *constant* equation of state parameters ω_n (for radiation, matter, vacuum energy etc) can be written, at a given cosmological redshift z, as

$$q(z) = \sum_n (1 + 3\omega_n)\, \frac{\Omega_n(z)}{2}\,, \tag{4}$$

where $\Omega_n(z) = \rho_n(z)/\rho_c(z)$ are the various cosmological parameters, which are normalized to satisfy the sum rule $\sum_n \Omega_n(z) = 1$ at any value of the redshift z. Evaluate this formula for $z = 0$ in the ΛCDM model and show that the current value of the acceleration parameter reads

$$q_0 = \frac{\Omega_m^0}{2} - \Omega_\Lambda = \frac{3\Omega_m - 2}{2}\,, \tag{5}$$

where the second equality is valid *only* for a spatially flat universe (why?). What is the value of q_0 according to the present data? Explain the meaning of this numerical value and its sign.

2. Let us make some study on the time evolution and the age of the universe.
i) Show that the formulae giving the cosmic time as a function of the scale factor is

$$t - t_1 = \int_{a_1}^{a(t)} \frac{da}{a\, H(a)} \tag{6}$$

where $H = \dot{a}/a$ is the expansion rate. What is the formula that gives the cosmic time in terms of the cosmological redshift, $t = t(z)$?

ii) In general these integrals cannot be solved by quadrature, but there are some interesting cases that can be worked out easily (see the Table below). Compute $a = a(t)$ during the matter and radiation dominated epoch for the Einstein-de Sitter's Universe. Then compute the age of each of the universes in the Table in units of the present Hubble time H_0^{-1} (**Solution**: a) $t_0 = 2/(3H_0)$, b) $t_0 = 1/H_0$, c) $t_0 = \infty$.). Explain physically the result of case c).

(a)	$\Omega_M = 1$	$\Omega_\Lambda = 0$	(Einstein-de Sitter's Universe)
(b)	$\Omega_M \simeq 0$	$\Omega_\Lambda \simeq 0$	(Milne's Universe)
(c)	$\Omega_M = 0$	$\Omega_\Lambda = 1$	(Inflationary Universe)

iii) Compute numerically the realistic age of our universe assuming the standard ΛCDM model of cosmology. To this end, take the latest data released by the PLANCK satellite (February 2015) on the cosmological parameters (assuming zero spatial curvature). Express the result in Gigayears (Gyr), recall that 1 Gyr $= 10^9$ yr. Compare the obtained result with the numerical age of the Einstein-de Sitter's Universe. Does it pay to take into account the radiation epoch in the calculation? Why?

(**Note**: You can use the analytic results of Exercise 3 below to cross-check your numerical computation.)

iv) Check numerically what is the impact of the spatial curvature term Ω_k on the age computation. Use the PLANCK data on Ω_k. (Notice that in this case it is not possible to cross-check your numerical answer with the analytical results of Exercise 3.)

3. Let us now face the age computation of the ΛCDM model analytically, again with vanishing spatial curvature, which is realistic (why?). You will be able to cross-check the numerical results of the previous problem with the analytical results obtained here.

i) Use equations (1) and (3) to show that the Hubble rate $H = H(t)$ in the matter-dominated epoch can be obtained by solving the differential equation

$$\dot{H} + \frac{3}{2}H^2 = \frac{\Lambda}{2}\,, \tag{7}$$

ii) Solve explicitly this equation for the case of the ΛCDM model with $\Lambda \neq 0$, assuming that the universe is spatially flat. Express your result as

$$H(t) = \sqrt{\Omega_\Lambda^0}\; H_0 \coth\left(\frac{3H_0\sqrt{\Omega_\Lambda^0}}{2}\; t\right)\,, \tag{8}$$

where H_0 is the Hubble rate at present. Check that for $\Lambda = 0$ you obtain an expected result. Which one? Derive explicitly $a(t)$ from it.

iii) Consider again the matter-dominated epoch. Verify that the cosmic time t is related with the cosmological redshift z as follows:

$$t(z) = \frac{2}{3\sqrt{\Omega_\Lambda^0}H_0}\sinh^{-1}\left(\sqrt{\frac{\Omega_\Lambda^0}{\Omega_m^0}}\;(1+z)^{-3/2}\right)\,. \tag{9}$$

What is the limit $\Omega_\Lambda^0 \to 0$ of this expression? And what is the limit for $z \to \infty$? Explain your results.

iv) Let us compute the age of the universe in the ΛCDM model for the present values of the cosmological parameters. If $\Omega_k^0 \neq 0$, the age cannot be given as a simple analytical formula. However, for $\Omega_k^0 = 0$ you can easily use the previous results to obtain an analytical expression for the age of the universe. Prove the following beautiful formula:

$$t_0 = \frac{2}{3\sqrt{\Omega_\Lambda^0}H_0}\sinh^{-1}\left(\sqrt{\frac{\Omega_\Lambda^0}{\Omega_m^0}}\right) = \frac{2}{3\,H_0}\;\frac{\tanh^{-1}\sqrt{\Omega_\Lambda^0}}{\sqrt{\Omega_\Lambda^0}}\,. \tag{10}$$

where the second equality is the most convenient one. Check it!

v) Verify in detail that for $\Lambda \to 0$ you recover the age of the Einstein-de Sitter universe given in Exercise 2 above, and also the first correction to this result for small values of Λ. Namely, show that

$$t_0 = \frac{2}{3} H_0^{-1} (1 + \frac{1}{3} \Omega_\Lambda^0 + ...) \tag{11}$$

Take the ratio of this result with the exact result, and check if it departs significantly from one for the current value of Ω_Λ^0.

vi) In contrast to the exact formula (10) (which is quite nice but also quite opaque) the approximate formula (11) is numerically crude, but is qualitatively very useful since it transparently shows that for a non-vanishing and positive Λ the age of the universe is larger than without it. However, can you feel the intuitive meaning of this result beyond the mathematical result? Explain physically why an universe with $\Lambda > 0$ is necessarily older than another one with $\Lambda = 0$ (with the same matter content). Justify your answer, clearly showing that you fully understand the physical reason.

vii) Evaluate Eq. (10) with the PLANCK results for the cosmological parameters and check that your result coincides very approximately with the age of the universe computed by direct numerical integration in Exercise 2 **iii)**. Compare also with the result quoted by the published paper on PLANCK observations.

4. Let us consider scalar fields in cosmology. We have derived on the backboard the explicit expressions for the energy density and pressure, and we found

$$\begin{aligned} \rho_\phi &= \frac{1}{2} \dot{\phi}^2 + V(\phi) \\ p_\phi &= \frac{1}{2} \dot{\phi}^2 - V(\phi) \end{aligned} \tag{12}$$

i) In our derivation we used Minkowskian spacetime, and of course assumed that ϕ does not depend on the space coordinates (why?). Show that these equations also hold for the FLRW metric, which means you have to make explicit use of the homogeneity and isotropy of spacetime.

ii) Use the above equations to determine the equation of motion for ϕ in the FLRW metric.

iii) Recompute the density and pressure in the case when ϕ is *not* homogeneous but is still isotropic.

iv) Show that in the last case the equation of state $\omega_\phi = p_\phi / \rho_\phi$ reads

$$\omega_\phi = -1 + \frac{\dot{\phi}^2 / V(\phi) + \frac{1}{3} \left(\nabla\phi\right)^2 / V(\phi)}{1 + \frac{1}{2} \dot{\phi}^2 / V(\phi) + \frac{1}{2} \left(\nabla\phi\right)^2 / V(\phi)} \tag{13}$$

Explain what is the meaning of $(\nabla\phi)^2$ in this context, and check that the above result boils down to the standard one (which we discussed on the blackboard) for homogeneous scalar fields.

v) Explain how space inhomogeneities could make that scalar field ϕ to appear as being phantom-like, without really being a phantom field. What is the quantitative condition on the time and space variations of the field so as to fulfil this phantom-like behavior? Compare it with the standard quintessence case. Could a time-independent scalar field look quintessence or phantom-like?

5. The idea that the vacuum energy could be a dynamical variable in the expanding Universe has been proposed by some authors, see e.g. th papers arXiv:0907.4555, arXiv:1409.7048, arXiv:1412.3785 and arXiv:1509.03298 and references therein. Consider the following types of dynamical vacuum models in which the dynamics of the vacuum is provided by a truncated power series of the Hubble rate:

$$\begin{aligned} A1: \quad & \Lambda = a_0 + a_2 H^2 \\ A2: \quad & \Lambda = a_0 + a_1 \dot{H} + a_2 H^2 \\ B1: \quad & \Lambda = b_0 + b_1 H \\ B2: \quad & \Lambda = b_0 + b_1 H + b_2 H^2 \\ C1: \quad & \Lambda = c_1 H + c_2 H^2 \\ C2: \quad & \Lambda = c_1 \dot{H} + c_2 H^2 \end{aligned} \tag{14}$$

To solve models B1 and B2 is complicated, but still doable. Models A1 and A2, however, are easier, and I propose you to solve A1 here. Specifically:

i) Explain first why A2 is easier than B1 or B2. You have to understand that there is a relation between $\dot{H}$ and H^2. Which one? And why this helps? Or, put another way, why things become more complicated with models B1 and B2? Write down the corresponding differential equation for the Hubble function and compare it with that of models A1 and A2.

ii) Solve now model A1 explicitly and show that the solutions giving the matter density, the vacuum energy density and the Hubble function, are given by equations (4.2), (4.3) and (4.5) of arXiv:1409.7048 in the special case that $\alpha = 0$ (equivalently, $\xi = 1$, see the text).

iii) Compute the transition redshift (where deceleration flips into acceleration) for model A1. Show that is given by Eq. (4.8). Evaluate it numerically and compare the result with the ΛCDM model. Use the inputs from PLANCK 2015. Notice that the transition redshift of the ΛCDM model can be obtained as a particular case of that of model A1.

iv) Show that model C2 is excluded by the simple reason that it does not have a transition redshift. Explain qualitatively why is so, but prove it rigorously.

Cosmology-Theory (Taller de Altas Energias 2015)

1 Exercises on FLRW cosmologies

■ Cosmological constant in the ΛCDM and beyond

1. Consider the system of equations formed by the Friedmann-Lemaître equation with non-vanishing cosmological constant ($\Lambda = 8\pi G_N \rho_\Lambda$),

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3}\left(\rho_m + \rho_\Lambda\right) - \frac{k}{a^2}, \tag{1.1}$$

and the dynamical evolution equation for the scale factor,

$$\ddot{a} = -\frac{4\pi G_N}{3}\left(\rho_m + 3p_m - 2\rho_\Lambda\right) a. \tag{1.2}$$

a) Show that $H^2 + \dot{H} = \ddot{a}/a$ and provide an explicit expression for $\dot{H}$ in terms of the pressure and density. What is its qualitative time evolution (increases or decreases with the expansion)?

b) Consider now the possibility that $\Lambda = \Lambda(t)$. Show that the equation

$$\dot{\rho}_\Lambda + \dot{\rho}_m + 3H\left(\rho_m + p_m\right) = 0 \tag{1.3}$$

is a first integral of the previous system; that is, a differential equation of smaller order that can replace Eq. (1.2). For $\dot{\rho}_\Lambda \neq 0$ there is an exchange of energy between matter and vacuum. Show in detail that any of these three equations (1.1), (1.2) and (1.3) can be derived from the other two. What happens for strictly constant Λ? Integrate Eq. (1.3) in this case, i.e. provide the explicit form of $\rho_m(t)$, assuming that matter is non-relativistic.

c) If the matter energy-momentum tensor $T_{\mu\nu}$ is non-conserved in the presence of a variable ρ_Λ, what is the conserved energy-momentum tensor now? Write it down explicitly: $\tilde{T}_{\mu\nu} = T_{\mu\nu} + ?$ Is it compatible with the Cosmological Principle to assume that ρ_Λ might not be strictly constant?

d) Using the FLRW metric explicitly, re-derive equation (1.3) from the local covariant conservation law of $\tilde{T}_{\mu\nu}$, i.e. $\nabla^\mu \tilde{T}_{\mu\nu} = 0$.

e) Starting from equation (1.2), prove that the deceleration parameter of the universe ($q \equiv -\ddot{a}a/\dot{a}^2$) in the presence of several fluids with *constant* equation of state parameters ω_n (for radiation, matter, vacuum energy, etc.) can be written, at a given cosmological redshift z, as

$$q(z) = \sum_n (1 + 3\omega_n)\frac{\Omega_n(z)}{2}, \tag{1.4}$$

where $\Omega_n(z) = \rho_n(z)/\rho_c(z)$ are the various cosmological parameters, which are normalized to satisfy the sum rule $\sum_n \Omega_n(z) = 1$ at any value of the redshift z. Evaluate this formula for $z = 0$ in the ΛCDM model and show that the current value of the acceleration parameter reads

$$q_0 = \frac{\Omega_m^0}{2} - \Omega_\Lambda = \frac{3\Omega_m - 2}{2}, \tag{1.5}$$

where the second equality is valid *only* for a spatially flat universe (why?). What is the value of q_0 according to the present data? Explain the meaning of this numerical value and its sign.

2. Let us make some study on the time evolution and the age of the universe.

a) Show that the formulae giving the cosmic time as a function of the scale factor is

$$t - t_1 = \int_{a_1}^{a(t)} \frac{\mathrm{d}a}{aH(a)} \tag{1.6}$$

where $H = \dot{a}/a$ is the expansion rate. What is the formula that gives the cosmic time in terms of the cosmological redshift, $t = t(z)$?

b) In general these integrals cannot be solved by quadrature, but there are some interesting cases that can be worked out easily (see the Table below). Compute $a = a(t)$ during the matter and radiation dominated epoch for the Einstein-de Sitter's Universe. Then compute the age of each of the universes in the Table in units of the present Hubble time H_0^{-1}. (*Solution:* **i)** $t_0 = 2/(3H_0)$, **ii)** $t_0 = 1/H_0$, **iii)** $t_0 = \infty$.) Explain physically the result of case **iii)**.

i)	$\Omega_M = 1$	$\Omega_\Lambda = 0$	Einstein-de Sitter's Universe
ii)	$\Omega_M \simeq 0$	$\Omega_\Lambda \simeq 0$	Milne's Universe
iii)	$\Omega_M = 0$	$\Omega_\Lambda = 1$	Inflationary Universe

c) Compute numerically the realistic age of our universe assuming the standard ΛCDM model of cosmology. To this end, take the latest data released by the PLANCK satellite (February 2015) on the cosmological parameters (assuming zero spatial curvature). Express the result in Gigayears (Gyr), recall that 1 Gyr $= 10^9$ yr. Compare the obtained result with the numerical age of the Einstein-de Sitter's Universe. Does it pay to take into account the radiation epoch in the calculation? Why?
(*Note:* You can use the analytic results of Exercise **3** below to cross-check your numerical computation.)

d) Check numerically what is the impact of the spatial curvature term Ω_k on the age computation. Use the PLANCK data on Ω_k. (Notice that in this case it is not possible to cross-check your numerical answer with the analytical results of Exercise **3**.)

3. Let us now face the age computation of the ΛCDM model analytically, again with vanishing spatial curvature, which is realistic (why?). You will be able to cross-check the numerical results of the previous problem with the analytical results obtained here.

a) Use equations (1.1) and (1.3) to show that the Hubble rate $H = H(t)$ in the matter-dominated epoch can be obtained by solving the differential equation

$$\dot{H} + \frac{3}{2}H^2 = \frac{\Lambda}{2}. \tag{1.7}$$

b) Solve explicitly this equation for the case of the ΛCDM model with $\Lambda \neq 0$, assuming that the universe is spatially flat. Express your result as

$$H(t) = \sqrt{\Omega_\Lambda^0} H_0 \coth\left(\frac{3H_0\sqrt{\Omega_\Lambda^0}}{2} t\right), \tag{1.8}$$

where H_0 is the Hubble rate at present. Check that for $\Lambda = 0$ you obtain an expected result. Which one? Derive explicitly $a(t)$ from it.

c) Consider again the matter-dominated epoch. Verify that the cosmic time t is related with the cosmological redshift z as follows:

$$t(z) = \frac{2}{3\sqrt{\Omega_\Lambda^0} H_0} \sinh^{-1}\left(\sqrt{\frac{\Omega_\Lambda^0}{\Omega_m^0}}(1+z)^{-3/2}\right). \tag{1.9}$$

What is the limit $\Omega_\Lambda^0 \to 0$ of this expression? And what is the limit for $z \to \infty$? Explain your results.

d) Let us compute the age of the universe in the ΛCDM model for the present values of the cosmological parameters. If $\Omega_k^0 \neq 0$, the age cannot be given as a simple analytical formula. However, for $\Omega_k^0 = 0$ you can easily use the previous results to obtain an analytical expression for the age of the universe. Prove the following beautiful formula:

$$t_0 = \frac{2}{3\sqrt{\Omega_\Lambda^0} H_0} \sinh^{-1}\left(\sqrt{\frac{\Omega_\Lambda^0}{\Omega_m^0}}\right) = \frac{2}{3H_0} \frac{\tanh^{-1}\sqrt{\Omega_\Lambda^0}}{\sqrt{\Omega_\Lambda^0}}, \tag{1.10}$$

where the second equality is the most convenient one. Check it!

e) Verify in detail that for $\Lambda \to 0$ you recover the age of the Einstein-de Sitter universe given in Exercise **2** above, and also the first correction to this result for small values of Λ. Namely, show that

$$t_0 = \frac{2}{3} H_0^{-1}\left(1 + \frac{1}{3}\Omega_\Lambda^0 + \ldots\right). \tag{1.11}$$

Take the ratio of this result with the exact result, and check if it departs significantly from one for the current value of Ω_Λ^0.

f) In contrast to the exact formula (1.10) (which is quite nice but also quite opaque) the approximate formula (1.11) is numerically crude, but is qualitatively very useful since it transparently shows that for a non-vanishing and positive Λ the age of the universe is larger than without it. However, can you feel the intuitive meaning of this result beyond the mathematical result? Explain physically why an universe with $\Lambda > 0$ is necessarily older than another one with $\Lambda = 0$ (with the same matter content). Justify your answer, clearly showing that you fully understand the physical reason.

g) Evaluate Eq. (1.10) with the PLANCK results for the cosmological parameters and check that your result coincides very approximately with the age of the universe computed by direct numerical integration in Exercise **2.c)**. Compare also with the result quoted by the published paper on PLANCK observations.

Solution:

1. a) The first part can be derived using the Hubble rate definition, or on the other hand, using the Friedmann equations. For the time being we will only proceed using the definition $H \equiv \dot{a}/a$ (the second way involves steps which are quite similar as those we will do in next subsection). If we take a time derivative,

$$\dot{H} = \frac{\ddot{a}}{a} - \frac{\dot{a}}{a^2}\dot{a} \implies H^2 + \dot{H} = \frac{\ddot{a}}{a} \tag{1.12}$$

To provide an explicit expression for $\dot{H}$ in terms of the pressure and density, we may use the Friedmann equations. Solving for $\dot{H}$ and considering (1.1) and (1.2),

$$\begin{aligned}
\dot{H} = \frac{\ddot{a}}{a} - H^2 &= -\frac{4\pi G_N}{3}\left(\rho_m + 3p_m - 2\rho_\Lambda\right) - \frac{8\pi G_N}{3}\left(\rho_m + \rho_\Lambda\right) + \frac{k}{a^2} \\
&= -\frac{4\pi G_N}{3}\left(\rho_m + 3p_m - 2\rho_\Lambda + 2\rho_m + 2\rho_\Lambda\right) + \frac{k}{a^2} \\
&= -\frac{4\pi G_N}{3}\left(3\rho_m + 3p_m\right) + \frac{k}{a^2} \\
&= -4\pi G_N\left(\rho_m + p_m\right) + \frac{k}{a^2}.
\end{aligned} \tag{1.13}$$

We can also arrive to this expression taking a time derivative in (1.1), and using the energy conservation equation, $\dot{\rho}_m = -3H(\rho_m + p_m)$ (since now Λ is a constant, $\dot{\rho}_\Lambda = 0$; G_N is also assumed to be a constant),

$$2H\dot{H} = \frac{8\pi G_N}{3}\dot{\rho}_m + 2\frac{k}{a^3}\dot{a} \implies \dot{H} = -4\pi G_N(\rho_m + p_m) + \frac{k}{a^2}. \tag{1.14}$$

[1]In any case, with the expansion the term proportional to the spatial curvature k does not play any role. Assuming an equation of state $p_m = w_m\rho_m$ ($w_m > 0$), then $\dot{H} \sim -4\pi G_N(1 + w_m)\rho_m$. Since ρ_m decreases with expansion, this means that $\dot{H}$ becomes less negative with time, i.e. $\dot{H}$ increases with expansion.

b) Let us try to find a first integral equation of the system of equations formed by (1.1) and (1.2). For this purpose, we will use the relation that we have found previously, i.e. $\ddot{a}/a = H^2 + \dot{H}$. Now we may use (1.1) and (1.2), whereas $\dot{H}$ can be found taking a time derivative of (1.1) (it is done in the r.h.s. of (1.14), but we must be aware that in that case we set $\Lambda =$ constant, and now we are in the case $\Lambda = \Lambda(t)$ and then $\dot{\rho}_\Lambda \neq 0$). Joining all expressions, it follows that:

$$-\frac{4\pi G_N}{3}\left(\rho_m + 3p_m - 2\rho_\Lambda\right) = \frac{8\pi G_N}{3}\left(\rho_m + \rho_\Lambda\right) - \frac{k}{a^2} + \frac{4\pi G_N}{3H}\left(\dot{\rho}_m + \dot{\rho}_\Lambda\right) + \frac{k}{a^2}. \tag{1.15}$$

Solving for $\dot{\rho}_\Lambda + \dot{\rho}_m$, we arrive to the desired equation:

$$\dot{\rho}_\Lambda + \dot{\rho}_m + 3H\left(\rho_m + p_m\right) = 0. \tag{1.16}$$

In fact, any of these three equations (1.1), (1.2) and (1.3), are not independents. Let us show that any one of them can be derived from the other two. For example, we just have proved that (1.1) + (1.2) $\implies$ (1.3).

[1]The second way we have mentioned to prove (1.12) consists in to use this last expression for $\dot{H}$ and add H^2 given by the first Friedmann equation. We end up with $-(4\pi G_N/3)(\rho_m + 3p_m - 2\rho_\Lambda)$, which is effectively $\ddot{a}/a$.

Otherwise, if we start with (1.1) and (1.3): we take a time derivative to (1.1) and then we use (1.3), we obtain the well-known equation:

$$\dot{H} = -4\pi G_N \left(\rho_m + p_m\right) + \frac{k}{a^2}. \tag{1.17}$$

If we use this result (which, remember, comes from the combination of (1.1) and (1.3)) plus the equation (1.1), in the identity $\ddot{a}/a = H^2 + \dot{H}$, we end up with (1.2):

$$\begin{aligned} \frac{\ddot{a}}{a} &= \frac{8\pi G_N}{3}\left(\rho_m + \rho_\Lambda\right) - \frac{k}{a^2} - 4\pi G_N \left(\rho_m + p_m\right) + \frac{k}{a^2} \\ &= \frac{4\pi G_N}{3}\left(2\rho_m + 2\rho_\Lambda - 3\rho_m - 3p_m\right) \\ &= -\frac{4\pi G_N}{3}\left(\rho_m + 3p_m - 2\rho_\Lambda\right). \end{aligned} \tag{1.18}$$

And finally, let us proceed starting with (1.2) and (1.3). Because in equation (1.1) does not appear the pressure, we must exclude it. For instance, from (1.3) we have:

$$\dot{\rho}_\Lambda + \dot{\rho}_m + 3H(\rho_m + p_m) = 0 \implies 3p_m = -\frac{1}{H}\left(\dot{\rho}_\Lambda + \dot{\rho}_m\right) - 3\rho_m. \tag{1.19}$$

Substituting into equation (1.2) yields

$$\begin{aligned} \frac{\ddot{a}}{a} &= -\frac{4\pi G_N}{3}\left(\rho_m - \frac{1}{H}\left(\dot{\rho}_\Lambda + \dot{\rho}_m\right) - 3\rho_m - 2\rho_\Lambda\right) \\ &= \frac{4\pi G_N}{3}\left(\frac{a}{\dot{a}}\left(\dot{\rho}_m + \dot{\rho}_\Lambda\right) + 2\left(\rho_m + \rho_\Lambda\right)\right). \end{aligned} \tag{1.20}$$

If we multiply both sides of the equation by $a\dot{a}$, we get:

$$\dot{a}\ddot{a} = \frac{4\pi G_N}{3}\left(a^2\left(\dot{\rho}_m + \dot{\rho}_\Lambda\right) + 2a\dot{a}\left(\rho_m + \rho_\Lambda\right)\right). \tag{1.21}$$

It turns out that both side of the equation are total derivatives, this can be rewritten as:

$$\frac{1}{2}\frac{\mathrm{d}\dot{a}^2}{\mathrm{d}t} = \frac{4\pi G_N}{3}\frac{\mathrm{d}}{\mathrm{d}t}\left(a^2\left(\rho_m + \rho_\Lambda\right)\right). \tag{1.22}$$

Setting the integration constant equal to $-k$, leads to

$$\dot{a}^2 = \frac{8\pi G_N}{3}a^2\left(\rho_m + \rho_\Lambda\right) - k \implies H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3}\left(\rho_m + \rho_\Lambda\right) - \frac{k}{a^2}. \tag{1.23}$$

For strictly constant Λ (assuming $G_N =$ constant), we have that $\dot{\rho}_\Lambda = \dot{\Lambda}/(8\pi G_N) = 0 \Longrightarrow \rho_\Lambda =$ constant. Moreover, for non-relativistic matter with equation of state w_m (constant), $\rho_m + p_m = (1 + w_m)\rho_m$. Then the energy conservation equation becomes: $\dot{\rho}_m = -3H(1 + w_m)\rho_m$. Now we may use: $\dot{\rho}_m = \dot{a}\rho_m' = aH\rho_m'$ (the prime here denotes a derivative respect the scale factor: $()' \equiv \mathrm{d}/\mathrm{d}a$), obtaining:

$$aH\rho_m' = -3H(1 + w_m)\rho_m \implies \int \frac{\mathrm{d}\rho_m}{\rho_m} = -3(1 + w_m)\int \frac{\mathrm{d}a}{a}. \tag{1.24}$$

Performing the integrals and solving for ρ_m, we obtain:

$$\rho_m(a(t)) = \rho_m^0 \left(\frac{a_0}{a(t)}\right)^{3(1+w_m)}. \tag{1.25}$$

To get an expression explicitly function of t, we should find the scale factor $a(t)$ and substitute in (1.25). This could be find solving the first Friedmann equation (1.1):

$$\frac{\dot{a}}{a} = \sqrt{\frac{8\pi G_N}{3}\left(\rho_m(a) + \rho_\Lambda\right) - \frac{k}{a^2}} \tag{1.26}$$

Notice that for $k = 0$ (flat universe), we can directly find an expression for ρ_m. In this particular case, the Hubble rate is related with ρ_m via the first Friedmann equation,

$$H = \sqrt{\frac{8\pi G_N}{3}\left(\rho_m + \rho_\Lambda\right)}. \tag{1.27}$$

Plugging this expression into the energy conservation equation and integrating, yields:

$$\int \frac{\mathrm{d}\rho_m}{\rho_m\sqrt{\frac{8\pi G_N}{3}\left(\rho_m + \rho_\Lambda\right)}} = -3(1 + w_m)\int \mathrm{d}t. \tag{1.28}$$

We can solve this integral with `Wolframalpha`, for instance. The result is:

$$-\frac{2}{\sqrt{\frac{8\pi G_N}{3}\rho_\Lambda}}\tanh^{-1}\left(\sqrt{1 + \frac{\rho_m}{\rho_\Lambda}}\right) = -3(1 + w_m)t - \tilde{C}, \tag{1.29}$$

where $\tilde{C}$ is some constant of integration. Solving for ρ_m we finally arrive to:

$$\rho_m(t) = \rho_\Lambda\left[\tanh^2\left(\sqrt{6\pi G_N\rho_\Lambda}(1 + w_m)t + C\right) - 1\right]; \tag{1.30}$$

here $\tilde{C}$ has been redefined.

c) The Einstein equations with a cosmological constant Λ, are:

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}. \tag{1.31}$$

Here G_N is the gravitational constant and $G_{\mu\nu} \equiv R_{\mu\nu} - (1/2)g_{\mu\nu}R$ is the Einstein tensor, which satisfies $\nabla^\mu G_{\mu\nu} = 0$ due to the Bianchi identities. Notice that (1.31) can be rewritten in the following way:

$$G_{\mu\nu} = 8\pi G(T_{\mu\nu} + \rho_\Lambda g_{\mu\nu}) \equiv 8\pi G_N\tilde{T}_{\mu\nu}, \quad \text{with} \quad \rho_\Lambda \equiv \frac{\Lambda}{8\pi G_N}. \tag{1.32}$$

Of course, the Bianchi identities are still satisfied, which means that in the presence of a variable ρ_Λ what is conserved now is $\tilde{T}_{\mu\nu} = T_{\mu\nu} + \rho_\Lambda g_{\mu\nu}$ instead $T_{\mu\nu}$.

The Cosmological Principle states that the distribution of matter in the universe is homogeneous and isotropic when viewed on a large enough scale. However, this only means that $\rho_\Lambda \neq \rho_\Lambda(\mathbf{x})$ but $\rho_\Lambda = \rho_\Lambda(t)$ is allowed. So if we assume that ρ_Λ is a spatially homogeneous function of the cosmic time, it is still compatible with the Cosmological Principle.

Note that this possibility has a price: in order to still fulfil the Bianchi identities we would need either a time dependent gravitational constant, $G = G(t)$, or to admit the possibility that matter exchanges energy with vacuum (hence that matter is not self-conserved; see next subsection), or a combination of the two possibilities [2].

d) The FLRW metric for a spatially flat universe is:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j. \tag{1.33}$$

The stress-energy tensor of a perfect fluid with 4-velocity u^μ reads

$$T^\mu_\nu = (\rho_m + p_m)u^\mu u_\nu - p_m\delta^\mu_\nu, \tag{1.34}$$

where, for a comoving observer, $u^\mu = \delta^\mu_0$ and $u_\mu = \delta^0_\mu$ (since the metric is just the FLRW, which is comoving to the (cosmic) medium, the spatial components of the four-velocity vanish, i.e. $u^i = u_i = 0$). Adding the extra piece, we deal with:

$$\tilde{T}^\mu_\nu = (\rho_m + p_m)u^\mu u_\nu - p_m\delta^\mu_\nu + \rho_\Lambda\delta^\mu_\nu, \tag{1.35}$$

which must satisfy the local covariant conservation law: $\nabla_\mu\tilde{T}^\mu_\nu = 0$. It is easier if we contract this equation with u^ν. Then:

$$\begin{aligned}
0 &= u^\nu\nabla_\mu\tilde{T}^\mu_\nu = u^\nu\nabla_\mu\Big((\rho_m + p_m)u^\mu u_\nu - p_m\delta^\mu_\nu + \rho_\Lambda\delta^\mu_\nu\Big) \\
&= u^\nu\nabla_\mu(\rho_m + p_m)u^\mu u_\nu + u^\nu(\rho_m + p_m)\Big((\nabla_\mu u^\mu)u_\nu + u^\mu\nabla_\mu u_\nu\Big) \\
&\quad - u^\nu\nabla_\mu p_m\delta^\mu_\nu + u^\nu\nabla_\mu\rho_\Lambda\delta^\mu_\nu \\
&= u^\mu\nabla_\mu\rho_m + u^\mu\nabla_\mu p_m + (\rho_m + p_m)\nabla_\mu u^\mu - u^\nu\nabla_\nu p_m + u^\nu\nabla_\nu\rho_\Lambda \\
&= u^\mu\nabla_\mu\rho_m + (\rho_m + p_m)\nabla_\mu u^\mu + u^\nu\nabla_\nu\rho_\Lambda.
\end{aligned} \tag{1.36}$$

In this last development we have used that u^μ is normalized to 1 ($u^\mu u_\mu = 1$, it is a 4-velocity) and also that its integral curves are geodesics, i.e. $u^\mu\nabla_\mu u_\nu = 0$. Since $u^\mu = \delta^\mu_0$, we get:

$$\dot{\rho}_\Lambda + \dot{\rho}_m + (\rho_m + p_m)\nabla_\mu u^\mu = 0. \tag{1.37}$$

Now we just have to compute the four-divergence $\nabla_\mu u^\mu$. Using $\Gamma^\mu_{\mu\sigma} = \partial_\sigma \ln\sqrt{-g}$, it is easy to see that

$$\nabla_\mu u^\mu = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}u^\mu\right). \tag{1.38}$$

For our particular spacetime background, $g = -a^6 \implies \sqrt{-g} = a^3$, and $u^\mu = \delta^\mu_0$. Therefore,

$$\nabla_\mu u^\mu = \frac{1}{a^3}\partial_\mu\left(a^3\delta^\mu_0\right) = \frac{1}{a^3}\frac{\mathrm{d}a^3}{\mathrm{d}t} = \frac{1}{a^3}3a^2\dot{a} = 3H. \tag{1.39}$$

Coming back to (1.37) with this last result, the desired equation follows:

$$\dot{\rho}_\Lambda + \dot{\rho}_m + 3H(\rho_m + p_m) = 0. \tag{1.40}$$

e) From equation (1.2), we have that the deceleration parameter of the universe in the presence of matter and vacuum energy is given by:

$$\begin{aligned}
q &= -\frac{\ddot{a}a}{a^2} = \frac{4\pi G}{3}\left(\rho_m + 3p_m - 2\rho_\Lambda\right)\frac{a^2}{\dot{a}^2} = \frac{4\pi G}{3H^2}\left(\rho_m + 3p_m - 2\rho_\Lambda\right) \\
&= \frac{1}{2}\frac{8\pi G}{3H^2}\left(\rho_m + 3p_m - 2\rho_\Lambda\right) = \frac{1}{2\rho_c}\left(\rho_m + 3p_m - 2\rho_\Lambda\right),
\end{aligned} \tag{1.41}$$

where the definition of the critical density $\rho_c \equiv 3H^2/(8\pi G)$ has been used. It should be pointed out, that our starting point already takes into account that the equation of state for the vacuum component is already $w_\Lambda = -1$, i.e. $p_\Lambda = -\rho_\Lambda$. Thus we can write (1.41) as:

$$q = \frac{1}{2\rho_c}\left(\rho_m + 3p_m + \rho_\Lambda + 3p_\Lambda\right). \tag{1.42}$$

In the presence of several fluids with *constant* equation of state parameters ω_n ($p_n = w_n \rho_n$; for radiation, matter, vacuum energy etc.), this last expression generalizes to:

$$\begin{aligned} q &= \frac{1}{2\rho_c}\sum_n (\rho_n + 3p_n) = \frac{1}{2\rho_c}\sum_n (\rho_n + 3w_n\rho_n) \\ &= \frac{1}{2}\sum_n (1+3w_n)\frac{\rho_n}{\rho_c} = \sum_n (1+3w_n)\frac{\Omega_n(z)}{2}, \end{aligned} \tag{1.43}$$

where we have defined the various cosmological parameters $\Omega_n(z) = \rho_n(z)/\rho_c(z)$, satisfying the sum rule $\sum_n \Omega_n(z) = 1$ at any value of the redshift z.

If we evaluate this formula for $z = 0$ in the ΛCDM model, we must sum over $w_m = 0$ and $w_\Lambda = -1$, so that the current value of the acceleration parameter reads:

$$q_0 = \sum_{\omega_n = 0,-1} (1+3w_n)\frac{\Omega_n^0}{2} = \frac{\Omega_m^0}{2} - \Omega_\Lambda^0. \tag{1.44}$$

In particular, for a spatially flat universe $\Omega_k^0 = 0$, and then the cosmic sum rule implies: $\Omega_m^0 + \Omega_\Lambda^0 = 1 \Longrightarrow \Omega_\Lambda^0 = 1 - \Omega_m^0$. Substituting into (1.44) yields:

$$q_0\Big|_{\text{flat}} = \frac{\Omega_m^0}{2} - \Omega_\Lambda^0 = \frac{\Omega_m^0}{2} - (1-\Omega_m^0) = \frac{3\Omega_m^0 - 2}{2}. \tag{1.45}$$

According to the present data [1], $\Omega_m^0 = 0.309 \pm 0.006$, which gives the following value of q_0: $q_0 = -0.537 \pm 0.009 < 0$. This parameter is a dimensionless measure of the cosmic acceleration of the expansion of the space in a FLRW universe. The minus sign and name 'deceleration parameter' are historical; at the time of definition $\ddot{a}$ was thought to be negative (decelerating) giving rise to a positive q. However, present data actually suggest that the expansion of the universe is accelerating ($\ddot{a} > 0$).

2. a) From the definition of the Hubble rate, we have:

$$H(a) = \frac{\dot{a}}{a} = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} \quad \Longrightarrow \quad t - t_1 = \int_{t_1}^{t} \mathrm{d}t = \int_{a_1}^{a(t)} \frac{\mathrm{d}a}{aH(a)}. \tag{1.46}$$

We have the relation between the redshift and the scale factor: $1 + z = a_0/a$, where a_0 is the actual value of the scale factor that we take to be one. Differentiating:

$$\frac{\mathrm{d}a}{a} = \frac{1}{a}\left(-a^2 \mathrm{d}z\right) = -a\mathrm{d}z = -\frac{\mathrm{d}z}{1+z}. \tag{1.47}$$

Then the cosmic time in terms of the cosmological redshift, $t = t(z)$, will be given by (notice that the minus sign will flip the order of integration):

$$t - t_1 = \int_z^{z_1} \frac{\mathrm{d}z}{(1+z)H(z)}. \tag{1.48}$$

The expression of $H(z)$ comes from the Friedmann equation. Let us rewrite it in some other convenient way. If consider the definition of some cosmological parameters, as the curvature density $\rho_k \equiv -3k/(8\pi G_N a^2)$, or the critical density $\rho_c^0 \equiv 3H_0^2/(8\pi G_N)$, we can write (1.1) as follows:

$$\begin{aligned} H^2 &= \frac{8\pi G_N}{3}\left(\rho_m + \rho_\Lambda\right) - \frac{k}{a^2} = \frac{8\pi G_N}{3}\left(\rho_m + \rho_\Lambda - \frac{3k}{8\pi G_N a^2}\right) \\ &= \frac{8\pi G_N}{3}\left(\rho_m + \rho_\Lambda + \rho_k\right) = H_0^2 \frac{8\pi G_N}{3H_0^2}\left(\rho_m + \rho_\Lambda + \rho_k\right) \\ &= \frac{H_0^2}{\rho_c^0}\left(\rho_m + \rho_\Lambda + \rho_k\right). \end{aligned} \tag{1.49}$$

Notice that in our development radiation has not been considered. Now we plug the relations $\rho_m(a) = \rho_m^0 a^{-3}$ (this was found in class) and $\rho_k(a) = \rho_k^0 a^{-2}$ (and this comes from its own definition), and define the cosmological parameters Ω_i:

$$\begin{aligned} H^2 &= \frac{H_0^2}{\rho_c^0}\left(\rho_m^0 a^{-3} + \rho_\Lambda + \rho_k^0 a^{-2}\right) = H_0^2\left(\Omega_m^0 a^{-3} + \Omega_\Lambda + \Omega_k^0 a^{-2}\right) \\ &= H_0^2\left(\Omega_m^0 (1+z)^3 + \Omega_\Lambda + \Omega_k^0 (1+z)^2\right). \end{aligned} \tag{1.50}$$

It should be pointed out that in general $\Omega_\Lambda \equiv \rho_\Lambda/\rho_c^0 \neq \Omega_\Lambda^0$; note that $\Omega_\Lambda = \Omega_\Lambda^0$ only holds if $\Lambda =$ constant (as it is assumed in the ΛCDM model). Finally, (1.48) becomes

$$t - t_1 = H_0^{-1}\int_z^{z_1} \frac{\mathrm{d}z}{(1+z)\left[\Omega_m^0(1+z)^3 + \Omega_\Lambda + \Omega_k^0(1+z)^2\right]^{1/2}}. \tag{1.51}$$

b) In some intermediate step in (1.50), we have:

$$H^2 = H_0^2\left(\Omega_m^0 a^{-3} + \Omega_r^0 a^{-4} + \Omega_\Lambda + \Omega_k^0 a^{-2}\right). \tag{1.52}$$

where the radiation contribution has been taken into account. For the Einstein-de Sitter's Universe we have $\Omega_\Lambda = \Omega_k = 0$, and $\Omega_m = 1$, $\Omega_r = 0$ during the MDE or $\Omega_m = 0$, $\Omega_r = 1$ during the RDE. Expression (1.52) allows us to compute the $a = a(t)$ in each epoch. It follows that:

$$\begin{aligned} &\text{MDE:} \quad \frac{\dot a}{a} = H_0\sqrt{\Omega_m^0}a^{-3/2} \implies a(t) = \left(\frac{3}{2}H_0\Omega_m^{0\;1/2}\right)^{2/3} t^{2/3}, \\ &\text{RDE:} \quad \frac{\dot a}{a} = H_0\sqrt{\Omega_r^0}a^{-2} \implies a(t) = \left(2H_0\Omega_r^{0\,1/2}\right)^{1/2} t^{1/2}. \end{aligned} \tag{1.53}$$

Now let us compute the age of some universes. To do so, we consider the expression (1.51), and although in general this integral cannot be solved by quadrature, there are some interesting cases that can be worked out easily. If we take $t_1 \simeq 0$ (the beginning of the universe) and $t = t_0$ (the actual value of the cosmic time, i.e. the age of the universe), then we integrate from $z = 0$ to $z_1 = \infty$. At the beginning $a_1 = 0$ which means $z_1 = \infty$, whereas at present $a = a_0 = 1 \implies z = 0$. Then it follows that

$$t_0 = H_0^{-1}\int_0^{\infty} \frac{\mathrm{d}z}{(1+z)\left[\Omega_m^0(1+z)^3 + \Omega_\Lambda + \Omega_k^0(1+z)^2\right]^{1/2}}. \tag{1.54}$$

i) Einstein-dS's Universe. In this case $\Omega_m = 1$ and $\Omega_\Lambda = 0$. Due to the cosmic sum rule, we have $\Omega_k = 0$ (this universe only has matter). Then, from (1.54)

$$t_0 = H_0^{-1}\int_0^{\infty}\frac{\mathrm{d}z}{(1+z)^{5/2}} = \frac{2}{3}H_0^{-1}. \tag{1.55}$$

ii) Milne's Universe. In this case $\Omega_m \simeq 0$ and $\Omega_\Lambda \simeq 0$. Due to the cosmic sum rule, we have $\Omega_k \simeq 1$, i.e. pure curvature. In this case,

$$t_0 = H_0^{-1} \int_0^\infty \frac{\mathrm{d}z}{(1+z)^2} = H_0^{-1}. \tag{1.56}$$

iii) Inflationary Universe. In this case $\Omega_m = 0$ and $\Omega_\Lambda = 1 \Longleftrightarrow \Omega_k = 0$. Therefore,

$$t_0 = H_0^{-1} \int_0^\infty \frac{\mathrm{d}z}{1+z} = H_0^{-1} \times \infty. \tag{1.57}$$

Physically, this means that a universe that has always been dominated by dark energy (or the cosmological constant), has not had any beginning. He has always been inflating, i.e. if $\Omega_\Lambda = 1$, we have an eternal inflationary universe. Solving the Friedmann equation in this particular case, we obtain: $a(t) \sim \exp(H_0 t)$. Thus he also never stops to inflate.

c) Let us compute numerically the realistic age of our universe assuming the standard ΛCDM model of cosmology. To this end, we will integrate numerically the expression (1.54), taking into account the latest data released by the PLANCK satellite (February 2015) [1] on the cosmological parameters and assuming zero spatial curvature, i.e. $H_0 = 67.7 \pm 0.5$ km $\cdot\, s^{-1}$/Mpc, $\Omega_m^0 = 0.309 \pm 0.006$, $\Omega_\Lambda^0 = 0.691 \pm 0.006$ and $\Omega_k^0 \simeq 0$. Using `Mathematica` we obtain:

$$t_0\Big|_{\Lambda\text{CDM}} = H_0^{-1} \int_0^\infty \frac{\mathrm{d}z}{(1+z)\left[\Omega_m^0(1+z)^3 + \Omega_\Lambda^0\right]^{1/2}} \simeq 13.820(5) \text{ Gyr}. \tag{1.58}$$

which is of the order of the numerical age of the Einstein-de Sitter's Universe, i.e. $t_0\big|_{\text{E-dS}} = 9.636(3)$ Gyr , but the result with nonvanishing Λ is larger and hence can be compatible with the age of the older globular clusters (which is not the case with the EdS result!).

Notice that the expression that we have used in our computations, radiation has not been considered. However, it does not matter because the radiation-dominated epoch lasted very little compared with the others, say the matter-dominated epoch or dark energy-dominated epoch. The contribution of this epoch to the total age is negligible.

Now we can use expression (1.10) (see problem **3.d)** for its derivation), to cross-check the numerical computation. This reads,

$$t_0 = \frac{2}{3H_0} \frac{\tanh^{-1}\sqrt{\Omega_\Lambda^0}}{\sqrt{\Omega_\Lambda^0}}. \tag{1.59}$$

Plugging the required experimental values mentioned above, we obtain: $t_0 = 13.820(5)$ Gyr, in agree with the previous numerical results.

d) If we repeat the numerical computation done in the subsection above, without neglecting the spatial curvature term Ω_k (according to PLANCK data, we have taken $\Omega_k \simeq 0.0008$), we obtain $t_{0k} = 13.814(7)$ Gyr ($\lesssim t_0$), which differs less than a 0.05% from the previous value where we neglected curvature.

3. Let us compute analytically the age of the universe of the ΛCDM model. We will take again vanishing spatial curvature, i.e. $k \simeq 0$. The main reason because we neglect the spatial curvature term, is due to inflation. The huge and faster expansion of the micro-causal connected universe that took place during inflation, led to a spatially flat universe. Inflation was the process that eliminated any possible residual curvature, and although it is actually an extension of the current ΛCDM model, we believe in. Anyway, from the actual experimental data [1] we know that $\Omega_k^0 < 10^{-3}$, such that this approximation is completely justified (from theoretical as well as experimental point of view).

a) The equation (1.1) for a spatially flat universe reads:

$$H^2 = \frac{8\pi G_N}{3}\left(\rho_m + \rho_\Lambda\right) \implies \frac{3}{2}H^2 = 4\pi G_N\left(\rho_m + \rho_\Lambda\right). \tag{1.60}$$

Previously we have found an expression for $\dot{H}$ (see (1.13)). For $k = 0$ and in the matter-dominated epoch (i.e. we neglect the pressure $p_m \simeq 0$), this becomes: $\dot{H} \simeq -4\pi G_N \rho_m$. Therefore,

$$\dot{H} + \frac{3}{2}H^2 = -4\pi G_N \rho_m + 4\pi G_N\left(\rho_m + \rho_\Lambda\right) = 4\pi G_N \frac{\Lambda}{8\pi G_N} = \frac{\Lambda}{2}. \tag{1.61}$$

Solving this differential equation we can find the Hubble rate $H = H(t)$ in the MDE.

b) Let us solve the equation, but first let us make appear the desired parameters. It follows that:

$$\Omega_\Lambda^0 \equiv \frac{\rho_\Lambda}{\rho_c^0} = \frac{\Lambda}{8\pi G_N}\frac{8\pi G_N}{3H_0^2} = \frac{\Lambda}{3H_0^2}, \tag{1.62}$$

where H_0 is the Hubble rate at present. Then the equation under consideration can be rewritten in the following way:

$$\begin{aligned} \dot{H} &= \frac{\Lambda}{2} - \frac{3}{2}H^2 = \frac{\Lambda}{2}\left(1 - \frac{3H_0^2}{\Lambda}\frac{H^2}{H_0^2}\right) = \frac{\Lambda}{2}\left(1 - \frac{H^2}{\Omega_\Lambda^0 H_0^2}\right) \\ &= \frac{3H_0^2\Omega_\Lambda^0}{2}\left(1 - \frac{H^2}{\Omega_\Lambda^0 H_0^2}\right). \end{aligned} \tag{1.63}$$

Integrating and then solving for $H(t)$, we arrive to desired expression:

$$H(t) = \sqrt{\Omega_\Lambda^0} H_0 \coth\left(\frac{3H_0\sqrt{\Omega_\Lambda^0}}{2}t\right). \tag{1.64}$$

Comment: it seems that the integral should give a $\tanh^{-1}(\dots)$, however it is only true if its argument is less than one in absolute value. For all real $|x| > 1$, $\mathrm{d}(\coth^{-1} x)/\mathrm{d}x = (1-x^2)^{-1}$, which is our case since $0 < \Omega_\Lambda^0 < 1$.

For $\Lambda \to 0$, i.e. $\Omega_\Lambda^0 \to 0$, we may use the Taylor expansion $\coth x = x^{-1} + x/3 + \dots$,

$$H(t) = \sqrt{\Omega_\Lambda^0} H_0 \left(\frac{2}{3H_0\sqrt{\Omega_\Lambda^0}t} + \frac{H_0\sqrt{\Omega_\Lambda^0}}{2}t + \dots\right)\Bigg|_{\Omega_\Lambda^0=0} = \frac{2}{3t}. \tag{1.65}$$

If we evaluate this expression at present time, we find that $t_0 = 2/(3H_0)$, as expected. A matter-dominated universe with $\Lambda = 0$ corresponds to the Einstein-dS's universe, whose age is given by (see (1.55)): $t_0 = 2/(3H_0)$, in agree with what we have found taking the limit $\Lambda \to 0$. Now we may recover the scale factor $a(t)$ (1.53):

$$\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2}{3t} \quad \Longrightarrow \quad a(t) \propto t^{2/3}. \tag{1.66}$$

c) To find the relation between the cosmic time t and the cosmological redshift z, we use the expression (1.51) found in the previous problem:

$$t - t_1 = H_0^{-1} \int_z^{z_1} \frac{\mathrm{d}z}{(1+z)\left[\Omega_m^0(1+z)^3 + \Omega_\Lambda^0 + \Omega_k^0(1+z)^2\right]^{1/2}}. \tag{1.67}$$

If consider the matter-dominated epoch, i.e. neglecting radiation and curvature, and we integrate from the beginning ($t_1 = 0 \Longleftrightarrow z_1 = \infty$) to some redshift z, we have:

$$\begin{aligned} t(z) &= H_0^{-1} \int_{1+z}^{\infty} \frac{\mathrm{d}z'}{(1+z')\left[\Omega_m^0(1+z')^3 + \Omega_\Lambda^0\right]^{1/2}} \\ &\equiv H_0^{-1} \int_x^{\infty} \frac{\mathrm{d}x'}{x'\left[\Omega_m^0 x'^3 + \Omega_\Lambda^0\right]^{1/2}} = \frac{1}{\sqrt{\Omega_\Lambda^0} H_0} \int_x^{\infty} \frac{\mathrm{d}x'}{x'\left[1 + (\Omega_m^0/\Omega_\Lambda^0)x'^3\right]^{1/2}}, \end{aligned} \tag{1.68}$$

where we have defined $x' \equiv 1 + z'$. Using `Wolframalpha` to solve the integral, we arrive to:

$$t(z) = \frac{2}{3\sqrt{\Omega_\Lambda^0} H_0} \sinh^{-1}\left(\sqrt{\frac{\Omega_\Lambda^0}{\Omega_m^0}}(1+z)^{-3/2}\right). \tag{1.69}$$

If we take the limit $\Omega_\Lambda^0 \to 0$, we can Taylor expand: $\sinh^{-1} x = x - x^3/6 + \ldots$,

$$\begin{aligned} t(z) &= \frac{2}{3\sqrt{\Omega_\Lambda^0} H_0} \left(\sqrt{\frac{\Omega_\Lambda^0}{\Omega_m^0}}(1+z)^{-3/2} - \frac{1}{6}\left(\frac{\Omega_\Lambda^0}{\Omega_m^0}\right)^{3/2}(1+z)^{-9/2} + \ldots\right)\Bigg|_{\Omega_\Lambda^0=0} \\ &= \frac{2}{3\sqrt{\Omega_m^0} H_0}(1+z)^{-3/2} \quad \Longrightarrow \quad (1+z)^{-1} = \left(\frac{3}{2} H_0 \Omega_m^{0\ 1/2}\right)^{2/3} t^{2/3}, \end{aligned} \tag{1.70}$$

as we expect. That is the expression for the scale factor $a(t)$ that we found in problem **2** subsection **b)**. In that case we took the normalization so that $a_0 = 1$, in such a way that $(1+z)^{-1} = a(t)$. We get again the expression (1.53) for the MDE,

$$a(t) = \left(\frac{3}{2} H_0 \Omega_m^{0\ 1/2}\right)^{2/3} t^{2/3}. \tag{1.71}$$

Otherwise, if we take the limit $z \to \infty$ (remember that bigger values of z means that we are looking to the past, therefore $z \to \infty$ corresponds to the beginning of the universe; we should obtain $t \to 0$, where $a = 0$),

$$t(z \to \infty) = \frac{2}{3\sqrt{\Omega_\Lambda^0} H_0} \sinh^{-1}(0) = 0, \tag{1.72}$$

since $\sinh^{-1}(0) = 0$. This can be a kind of check.

d) Let us compute the age of the universe in the ΛCDM model for the present values of the cosmological parameters, assuming $\Omega_k^0 = 0$ (otherwise we can not find an analytical expression; after all, we must use the previous results which carry the assumption $\Omega_k^0 = 0$). Evaluating (1.69) in $z = 0$ (that is the actual redshift with the used normalization, i.e. $a_0 = 1$), we end up with:

$$t_0 = \frac{2}{3\sqrt{\Omega_\Lambda^0} H_0} \sinh^{-1}\left(\sqrt{\frac{\Omega_\Lambda^0}{\Omega_m^0}}\right). \tag{1.73}$$

Using the cosmic sum rule, $\Omega_m^0 + \Omega_\Lambda^0 = 1 \Longrightarrow \Omega_m^0 = 1 - \Omega_\Lambda^0$, and using the relation $\sinh^{-1} x = \ln(x + \sqrt{1+x^2})$, it follows that (for simplicity and for the moment we exclude the proportionality factor):

$$\begin{aligned}
t_0 \propto \sinh^{-1}\left(\sqrt{\frac{\Omega_\Lambda^0}{1-\Omega_\Lambda^0}}\right) &= \ln\left(\sqrt{\frac{\Omega_\Lambda^0}{1-\Omega_\Lambda^0}} + \sqrt{1 + \frac{\Omega_\Lambda^0}{1-\Omega_\Lambda^0}}\right) \\
&= \ln\left(\sqrt{\frac{\Omega_\Lambda^0}{1-\Omega_\Lambda^0}} + \frac{1}{\sqrt{1-\Omega_\Lambda^0}}\right) = \ln\left(1 + \sqrt{\Omega_\Lambda^0}\right) - \ln\sqrt{1-\Omega_\Lambda^0} \\
&= \ln\left(1 + \sqrt{\Omega_\Lambda^0}\right) - \frac{1}{2}\ln\left(1 - \Omega_\Lambda^0\right) \\
&= \ln\left(1 + \sqrt{\Omega_\Lambda^0}\right) - \frac{1}{2}\ln\left[\left(1 - \sqrt{\Omega_\Lambda^0}\right)\left(1 + \sqrt{\Omega_\Lambda^0}\right)\right] \\
&= \ln\left(1 + \sqrt{\Omega_\Lambda^0}\right) - \frac{1}{2}\ln\left(1 - \sqrt{\Omega_\Lambda^0}\right) - \frac{1}{2}\ln\left(1 + \sqrt{\Omega_\Lambda^0}\right) \\
&= \frac{1}{2}\ln\left(\frac{1+\sqrt{\Omega_\Lambda^0}}{1-\sqrt{\Omega_\Lambda^0}}\right) = \tanh^{-1}\sqrt{\Omega_\Lambda^0},
\end{aligned} \tag{1.74}$$

since $\tanh^{-1} x = (1/2)\ln\left(\frac{1+x}{1-x}\right)$. Restoring the proportional constant,

$$t_0 = \frac{2}{3H_0}\frac{\tanh^{-1}\sqrt{\Omega_\Lambda^0}}{\sqrt{\Omega_\Lambda^0}}. \tag{1.75}$$

e) For $\Lambda \to 0$, i.e. $\Omega_\Lambda^0 \to 0$, we may use the Taylor expansion $\tanh^{-1} x = x + x^3/3 + \dots$, in (1.74)

$$\begin{aligned}
t_0 &= \frac{2}{3}H_0^{-1}\frac{1}{\sqrt{\Omega_\Lambda^0}}\left(\sqrt{\Omega_\Lambda^0} + \frac{1}{3}(\Omega_\Lambda^0)^{3/2} + \dots\right) \\
&= \frac{2}{3}H_0^{-1}\left(1 + \frac{1}{3}\Omega_\Lambda^0 + \dots\right).
\end{aligned} \tag{1.76}$$

If we strictly take $\Omega_\Lambda^0 = 0$, we recover the age of the Einstein-de Sitter universe: $t_0 = 2/(3H_0)$, as we could expect. We had proven it in the previous subsection **b)** of this problem **3** (see (1.65)).

Taking the ratio of this result with the exact result we check that for the current value of Ω_Λ^0, it differs from one but not significantly:

$$\frac{t_0\big|_{\text{approx}}}{t_0\big|_{\text{exact}}} = 0.8578. \tag{1.77}$$

f) Expression (1.76) states that the age of the universe with a non-vanishing and positive Λ is larger than those without it. Intuitively, that is because the cosmic expansion of an universe with a positive non-vanishing cosmological constant is accelerating. This means that the cosmic expansion was slower in the past, in such a way that it took the universe longer to expand to its present size. Although both have the same matter content, that universe with $\Lambda \neq 0$ take longer to reach its present rate of expansion.

g) The first part has already been done in problem **2**, and in fact, we have seen that the age of the universe computed by direct numerical integration coincides exactly with that we obtain using the analytical formula (1.75). See exercise **2.c)**. We have obtained: $t_0 = 13.820(5)$ Gyr, which is quite admissible if we compare this value with that deduced from PLANCK observations [1]: $t_0|_{\text{PLANCK}} = 13.799 \pm 0.021$ Gyr.

2 More on FLRW cosmologies

■ Dark Energy and scalar fields

6. Let us consider scalar fields in cosmology. We have derived on the blackboard the explicit expressions for the energy density and pressure, and we found

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \qquad p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{2.1}$$

a) These equations follow easily in the Minkowskian spacetime and assuming that ϕ does not depend on the space coordinates (why?). Show that these equations also hold for the FLRW metric, which means you have to make explicit use of the homogeneity and isotropy of spacetime.

b) Use the above equations to determine the equation of motion for ϕ in the FLRW metric.

c) Recompute the density and pressure in the case when ϕ is *not* homogeneous but is still isotropic.

d) Show that in the last case the equation of state $\omega_\phi = p_\phi/\rho_\phi$ reads

$$\omega_\phi = -1 + \frac{\dot{\phi}^2/V(\phi) + \frac{1}{3}(\nabla\phi)^2/V(\phi)}{1 + \frac{1}{2}\dot{\phi}^2/V(\phi) + \frac{1}{2}(\nabla\phi)^2/V(\phi)}. \tag{2.2}$$

Explain what is the meaning of $(\nabla\phi)^2$ in this context, and check that the above result boils down to the standard one (which we discussed on the blackboard) for homogeneous scalar fields.

e) Explain how space inhomogeneities could make that scalar field ϕ to appear as being phantom-like, without really being a phantom field. What is the quantitative condition on the time and space variations of the field so as to fulfil this phantom-like behavior? Compare it with the standard quintessence case. Could a time-independent scalar field look quintessence or phantom-like?

6. The expression for the energy-momentum tensor of the scalar field ϕ that we found is:

$$T^\phi_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi - V(\phi)\right). \tag{2.3}$$

If $\nabla_\mu\phi$ is timelike, we can define a 4-velocity $u_\mu = \nabla_\mu\phi/|\nabla\phi|$, where the factor $|\nabla\phi| \equiv (\nabla_\sigma\phi\nabla^\sigma\phi)^{1/2}$ ensures the canonical normalization, i.e. $u_\mu u^\mu = 1$. Then $T^\phi_{\mu\nu}$ can be written as that of a perfect fluid with 4-velocity u^μ:

$$\begin{aligned} T^\phi_{\mu\nu} &= \nabla_\sigma\phi\nabla^\sigma\phi u_\mu u_\nu - g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}u_\alpha u_\beta\nabla_\sigma\phi\nabla^\sigma\phi - V(\phi)\right) \\ &= \nabla_\sigma\phi\nabla^\sigma\phi u_\mu u_\nu - g_{\mu\nu}\left(\frac{1}{2}\nabla_\sigma\phi\nabla^\sigma\phi - V(\phi)\right) \\ &\equiv (\rho_\phi + p_\phi)u_\mu u_\nu - p_\phi g_{\mu\nu}. \end{aligned} \tag{2.4}$$

We can identify $\rho_\phi + p_\phi = \nabla_\sigma\phi\nabla^\sigma\phi$, and $p_\phi = (1/2)\nabla_\sigma\phi\nabla^\sigma\phi - V(\phi)$. Then $\rho_\phi = (1/2)\nabla_\sigma\phi\nabla^\sigma\phi + V(\phi)$. Otherwise, we could have obtained these expressions looking at the 00- and ii-components, since $\rho_\phi = T^\phi_{00}$ and $p_\phi = T^\phi_{ii}$. However, this way is perhaps unnatural since we are already assuming that $T^\phi_{\mu\nu}$ takes that form of a perfect fluid.

a) ϕ does not depend on the space coordinates due to the cosmological principle. If the universe feels the presence of a scalar field ϕ giving rise to a cosmological constant effect, the scalar field must be invariant under translations and rotations in space.

The expressions for the density and pressure for the FLRW metric,

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left(\frac{1}{1-kr^2}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right), \tag{2.5}$$

are (since the metric is diagonal, we have $g^{tt} = (g_{tt})^{-1} = 1$):

$$\begin{aligned} \rho_\phi &= \frac{1}{2}\nabla_\sigma\phi\nabla^\sigma\phi + V(\phi) = \frac{1}{2}g^{tt}\dot\phi^2 + V(\phi) = \frac{1}{2}\dot\phi^2 + V(\phi), \\ p_\phi &= \frac{1}{2}\nabla_\sigma\phi\nabla^\sigma\phi - V(\phi) = \frac{1}{2}g^{tt}\dot\phi^2 - V(\phi) = \frac{1}{2}\dot\phi^2 - V(\phi), \end{aligned} \tag{2.6}$$

which coincide with the expressions for ρ_ϕ and p_ϕ in Minkowskian spacetime. Notice that it happens because we have assumed a homogeneous field —as required by the Cosmological Principle— and after all, the mentioned principle also invites us to look for homogenous and isotropic spacetimes, which is the case of the FLRW metric. This fact that ensures that ρ_ϕ and p_ϕ take the same form in both backgrounds.

b) Notice that we have written $T^\phi_{\mu\nu}$ in a perfect fluid form. Then we have the energy conservation equation: $\nabla^\mu T^\phi_{\mu 0} = 0 \Longrightarrow \dot\rho_\phi + 3H(\rho_\phi + p_\phi) = 0$, which has been derived for the case we had $\Lambda = \Lambda(t)$, some problem back (see problem **1** subsection **d)**). Inserting the corresponding expressions, we have:

$$\dot\rho_\phi = \frac{1}{2}2\dot\phi\ddot\phi + V'(\phi)\dot\phi = \left(\ddot\phi + V'(\phi)\right)\dot\phi, \qquad \rho_\phi + p_\phi = \dot\phi^2 \tag{2.7}$$

and the energy conservation equation becomes

$$\ddot\phi + 3H\dot\phi + V'(\phi) = 0 \tag{2.8}$$

i.e. the equation of motion for ϕ in the FLRW metric.

c) For a *non*-homogeneous field but still isotropic, we have that $\partial_x\phi = \partial_y\phi = \partial_z\phi$ (all directions must be equivalent from the point of view of the scalar field). It can also be implemented by considering $\phi = \phi(t, r)$ (the field is invariant under $SO(3)$).

Notice that in this case, gradients induce a non-vanishing energy flux, T_i^0, or momentum density, T_0^i, in such a way that $T_{\mu\nu}^{\phi}$ can not be written in a perfect fluid form. Therefore, we must proceed more generally using the general expression for $T_{\mu\nu}^{\phi}$ (see (2.3)), taking $\rho_\phi = T_{00}^{\phi}$, $p_\phi = T_{ii}^{\phi}$ ($\equiv (1/3)\sum_i T_{ii}^{\phi}$ due to isotropy). If follows,

$$\begin{aligned} \rho_\phi &= \dot{\phi}^2 - \left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - V(\phi)\right) = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + V(\phi), \\ p_\phi &= (\partial_i\phi)^2 + \left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - V(\phi)\right) = \frac{1}{3}(\vec{\nabla}\phi)^2 + \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - V(\phi) \\ &= \frac{1}{2}\dot{\phi}^2 - \frac{1}{6}(\vec{\nabla}\phi)^2 - V(\phi). \end{aligned} \tag{2.9}$$

We have defined: $(\vec{\nabla}\phi)^2 \equiv (\partial_x\phi)^2 + (\partial_y\phi)^2 + (\partial_z\phi)^2$. Of course, all development has been done in Minkowski space, for simplicity (just as before).

d) In this last case, the equation of state $\omega_\phi = p_\phi/\rho_\phi$ reads

$$\begin{aligned} \omega_\phi &= \frac{\frac{1}{2}\dot{\phi}^2 - \frac{1}{6}(\vec{\nabla}\phi)^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + V(\phi)} = -1 + 1 + \frac{\frac{1}{2}\dot{\phi}^2 - \frac{1}{6}(\vec{\nabla}\phi)^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + V(\phi)} \\ &= -1 + \frac{\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + V(\phi) + \frac{1}{2}\dot{\phi}^2 - \frac{1}{6}(\vec{\nabla}\phi)^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + V(\phi)} \\ &= -1 + \frac{\dot{\phi}^2 + \frac{1}{3}(\vec{\nabla}\phi)^2}{V(\phi) + \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2} = -1 + \frac{\dot{\phi}^2/V(\phi) + \frac{1}{3}(\vec{\nabla}\phi)^2/V(\phi)}{1 + \frac{1}{2}\dot{\phi}^2/V(\phi) + \frac{1}{2}(\vec{\nabla}\phi)^2/V(\phi)}. \end{aligned} \tag{2.10}$$

Perhaps the best way to understand the meaning of $(\vec{\nabla}\phi)^2$ in this context, is to look at the energy density, which we have found to be $\rho_\phi = (\dot{\phi}^2 + (\vec{\nabla}\phi)^2)/2 + V(\phi)$. Because now homogeneity is lost, we see that $(\vec{\nabla}\phi)^2$ is the cost of energy due to move the field in space, as well as the kinetic energy is somehow the energy associated to move the field in time. Note that for homogeneous scalar fields ($\vec{\nabla}\phi = 0$), we recover the standard expression mentioned in class:

$$\omega_\phi \to -1 + \frac{\dot{\phi}^2/V(\phi)}{1 + \frac{1}{2}\dot{\phi}^2/V(\phi)} \simeq -1 + \frac{\dot{\phi}^2}{V(\phi)} \equiv -1 + \epsilon. \tag{2.11}$$

e) Let us find a condition on space inhomogeneities giving rise to a phantom-like equation of state (i.e. $w_\phi \lesssim -1$), but still with a scalar field ϕ having a positive kinetic term. If we enforce w_ϕ to be smaller than minus one, we can find a condition involving $\dot{\phi}$ and $\vec{\nabla}\phi$. From (2.10), it follows:

$$-1 + \frac{\dot{\phi}^2/V(\phi) + \frac{1}{3}(\vec{\nabla}\phi)^2/V(\phi)}{1 + \frac{1}{2}\dot{\phi}^2/V(\phi) + \frac{1}{2}(\vec{\nabla}\phi)^2/V(\phi)} \lesssim -1 \quad \Longleftrightarrow \quad \dot{\phi}^2 \lesssim -\frac{1}{3}(\vec{\nabla}\phi)^2. \tag{2.12}$$

Notice however, that we must require to have $\dot{\phi}^2 > 0$, which is only satisfied if the gradient is purely imaginary: $\vec{\nabla}\phi = i\mathrm{Im}(\vec{\nabla}\phi)$. Then the condition is:

$$\dot{\phi}^2 \lesssim \frac{1}{3}\left[\mathrm{Im}(\vec{\nabla}\phi)\right]^2. \tag{2.13}$$

If we perform a Taylor expansion, w_ϕ becomes:

$$w_\phi \simeq -1 + \frac{1}{V(\phi)}\left(\dot{\phi}^2 + \frac{1}{3}(\vec{\nabla}\phi)^2\right) \equiv -1 + \varepsilon, \tag{2.14}$$

with $\varepsilon < 0$ if (2.13) is satisfied. For the quintessence case we obtained $w_\phi \simeq -1+\delta$, with $\delta > 0$ in any case ($\delta = \dot{\phi}^2/V(\phi)$ for a homogeneous field). If we have a time-independent scalar field, $w_\phi \simeq -1 + (\vec{\nabla}\phi)^2/(3V(\phi))$, which behaves as quintessence if the gradient is real, or as a phantom field if the gradient is purely imaginary.

7. **a)** If we replace the standard kinetic term in the Lagrangian density by $\frac{1}{2}\xi g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$, being ξ some coefficient, we now deal with:

$$\mathcal{L} \to \mathcal{L}_\xi = \frac{1}{2}\xi g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi). \tag{2.15}$$

The expression for w_ϕ that we can derive using this Lagrangian density is completely equivalent to take (2.10) and make the replacements: $\dot{\phi} \to \sqrt{\xi}\dot{\phi}$ and $\vec{\nabla}\phi \to \sqrt{\xi}\vec{\nabla}\phi$. We end up with:

$$\begin{aligned} w_\phi^\xi &= -1 + \frac{\xi\dot{\phi}^2/V(\phi) + \frac{1}{3}\xi(\vec{\nabla}\phi)^2/V(\phi)}{1+\frac{1}{2}\xi\dot{\phi}^2/V(\phi)+\frac{1}{2}\xi(\vec{\nabla}\phi)^2/V(\phi)} \\ &\simeq -1 + \frac{\xi}{V(\phi)}\left(\dot{\phi}^2 + \frac{1}{3}(\vec{\nabla}\phi)^2\right). \end{aligned} \tag{2.16}$$

For instance, for a homogeneous field $w_\phi^\xi = -1 + \xi\dot{\phi}^2/V(\phi)$. Thus if $\xi < 0$, we have $w_\phi^\xi \lesssim -1$ (and the scalar field becomes a phantom field). We do not like this condition because it implies we have a negative kinetic term, giving rise to deep problems in QFT, e.g. unitarity could not be preserved or particle states do not have the standard energy-momentum relation ($E^2 \neq m^2 + \mathbf{p}^2$). Also, with a negative kinetic term we can have an arbitrary large negative energy. Nothing of this kind is observed, which casts serious doubt on possible existence of phantom fields [1].

b) The equation of state for the dark energy (DE) derived from PLANCK observations, is [2]: $w = -1.019^{+0.075}_{-0.080}$, which corresponds to a phantom-like behavior ($w \lesssim -1$). Due to the serious problems that have been mentioned just before, maybe the most realistic possibility to consider is to mimic the phantom behavior with a *non*-phantom scalar field. In Ref.[1] some ideas are given to mimic a phantom field through a dynamical cosmological term. The student can also try to do Exercise 5 in the list, of which we do not provide an explicit solution here since the solutions can be found in the arXiv. preprints cited there, see e.g. Ref.[3] below and even more specifically Ref.[4]. In that exercise one finds specific examples of dynamical vacuum models that can mimic a phantom field. So true phantom fields are actually not needed!

References: [1] P.A.R. Ade et al., PLANCK Collab. *Cosmological parameters* [arXiv:1502.01589].

[2] J. Solà, *Cosmological constant and vacuum energy: old and new ideas*, J. Phys. Conf. Ser. (2013) **453** 012015 [e-Print: arXiv:1306.1527].

[3] A. Gómez-Valent, J. Solà, S. Basilakos, *Dynamical vacuum energy in the expanding Universe confronted with observations: a dedicated study*, JCAP **01** (2015) 004 [e-Print: arXiv:1409.7048]

[4] S. Basilakos, J. Solà, *Effective equation of state for running vacuum: "mirage" quintessence and phantom dark energy* Mon.Not.Roy.Astron.Soc. 437 (2014) 3331 [e-Print: arXiv:1307.4748]