BSM Exercices

1. Evaluate the one loop contribution to m_h^2, the squared Higgs mass, induced by the top quark in the SM.

   Notation:

   $$\mathcal{L}_{\text{Yukawa}} = -\frac{y_t}{\sqrt{2}} H \bar{t}_L t_R + \text{ h.c.}$$

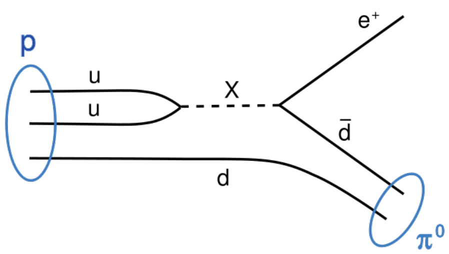
   with $H = v + h$.

2. Show that *both* the quadratically and logarithmically divergent corrections obtained in 1) cancel if we assume that there are N new scalar particles described by :

   $$\mathcal{L}_{\text{scalar}} = -\frac{\lambda}{2} h^2 (|\Phi_L|^2 + |\Phi_R|^2) - h(\mu_L |\Phi_L|^2 + \mu_R |\Phi_R|^2) - m_L |\Phi_L|^2 - m_R |\Phi_R|^2 \, .$$

   and we properly choose N and the Lagrangian parameters.

3. Baryon number is violated in the minimal $SU(5)$ GUT through interactions mediated by new gauge bosons X, Y such as the one shown in the figure:

   Assuming gauge couplings close to the electroweak ones (*e.g.*, $\sim$ e), estimate τ_p, the proton life-time for $M_X \sim 10^{15}$ GeV. Compare with the present experimental value [1] for the $e^+\pi^0$ mode:

   $$\tau_{p \to e^+ \pi^0} > 8.2 \; 10^{33} \text{ years } \; (90\% \, \text{C.L.})$$

   Note: 1 GeV^{-1} $\sim 6.6 \; 10^{-25} s$.

4. Consider an extension of the Standard Model where a scalar neutral singlet Φ has been added. Having the same quantum numbers, after EWSB this extra field will in general mix with the SM Higgs field.

    - Is it possible to avoid that?
    - What are the consequences of this mixing for Higgs physics?
    - If the new scalar particle is not very heavy, could it be produced at the LHC? If so, how could it be detected?

References

[1] **Super-Kamiokande** Collaboration, H. Nishino et al., *Search for Proton Decay via* $p \to e^+\pi^0$ *and* $p \to \mu^+ pi^0$ *in a Large Water Cherenkov Detector*, *Phys. Rev. Lett.* **102** (2009) 141801, [arXiv:0903.0676].

1

INTRODUCTION TO SUPERSYMMETRY

1.1 The unreasonable effectiveness of the Standard Model

The standard model (SM) of particle physics is well-known to be unreasonably effective, since it is in accord with all the experimental data. However, the consistency of the model relies on the Higgs field having a vacuum expectation value (VEV) of 246 GeV even though this is highly unstable under quantum loop corrections. This instability can be seen by computing the loop corrections to the Higgs mass term. The fact that these corrections diverge quadratically with the high-energy cutoff is the signal that this instability is a severe problem. Much of the recent interest in supersymmetry (SUSY) has been driven by the possibility that SUSY can cure this instability.

The largest contribution to the Higgs mass correction in the SM of particle physics comes from the top quark loop. The top quark acquires a mass from the VEV, $\langle H^0 \rangle$, of the, real, neutral component of the Higgs field (denoted by H^0). Given the coupling of the Higgs to the top quark:

$$\mathcal{L}_{\text{Yukawa}} = -\frac{y_t}{\sqrt{2}} H^0 \overline{t_L} t_R + h.c. \tag{1.1}$$

(where t_L and t_R are the left-handed and right-handed components of the top quark, y_t is the top Yukawa coupling, and $h.c.$ denotes the hermitian conjugate) and expanding H^0 around its VEV

$$H^0 = \langle H^0 \rangle + h^0 = v + h^0 \tag{1.2}$$

(here h^0 represents the quantum fluctuations around the VEV) we have that the top mass is

$$m_t = \frac{y_t \, v}{\sqrt{2}} \,. \tag{1.3}$$

Given the coupling in eqn (1.1), we can easily evaluate the Feynman diagram in Fig. 1.1.

The contribution to the Higgs mass squared corresponding to Fig. 1.1 is

$$-i\delta m_h^2|_{\text{top}} = (-1)N_c \int \frac{d^4k}{(2\pi)^4} \text{Tr}\left[\frac{-iy_t}{\sqrt{2}} \frac{i}{\not{k} - m_t} \left(\frac{-iy_t^*}{\sqrt{2}}\right) \frac{i}{\not{k} - m_t}\right]$$

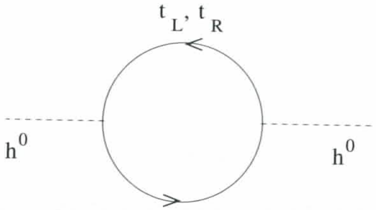


FIG. 1.1. The top loop contribution to the Higgs mass term.

$$= -2N_c|y_t|^2 \int \frac{d^4k}{(2\pi)^4} \frac{k^2+m_t^2}{(k^2-m_t^2)^2} \,. \tag{1.4}$$

After a Wick rotation ($k_0 \to ik_4$, $k^2 \to -k_E^2$) we can perform the angular integration and impose a hard momentum cutoff ($k_E^2 < \Lambda^2$), which yields:

$$-i\delta m_h^2|_{\rm top} = \frac{iN_c|y_t|^2}{8\pi^2} \int_0^{\Lambda^2} dk_E^2 \, \frac{k_E^2(k_E^2-m_t^2)}{(k_E^2+m_t^2)^2} \,. \tag{1.5}$$

Changing variables to $x = k_E^2 + m_t^2$ results in

$$\begin{aligned}\delta m_h^2|_{\rm top} &= -\frac{N_c|y_t|^2}{8\pi^2} \int_{m_t^2}^{\Lambda^2} dx \left(1 - \frac{3m_t^2}{x} + \frac{2m_t^4}{x^2}\right) \\ &= -\frac{N_c|y_t|^2}{8\pi^2}\left[\Lambda^2 - 3\,m_t^2 \ln\left(\frac{\Lambda^2+m_t^2}{m_t^2}\right) + \ldots\right],\end{aligned} \tag{1.6}$$

where ... indicates finite terms in the limit $\Lambda \to \infty$. So we find that there are quadratically and logarithmically divergent corrections which (in the absence of a severe fine-tuning) push the natural value of the Higgs mass term (and hence the Higgs VEV) up toward the cutoff. Another way of saying this is that the SM can only be an effective field theory with a cutoff near 1 TeV, and some new physics must come into play near the TeV scale which can stabilize the Higgs VEV. SUSY is the leading candidate for such new physics.

There is a simple way to stabilize the Higgs VEV by canceling the divergent corrections to the Higgs mass term.[1] Suppose there are N new scalar particles ϕ_L and ϕ_R that are lighter than a TeV with the following interactions:

$$\begin{aligned}\mathcal{L}_{\rm scalar} = &-\frac{\lambda}{2}(h^0)^2(|\phi_L|^2+|\phi_R|^2) - h^0(\mu_L|\phi_L|^2+\mu_R|\phi_R|^2) \\ &-m_L^2|\phi_L|^2 - m_R^2|\phi_R|^2 \,.\end{aligned} \tag{1.7}$$

The interactions in eqn (1.7) produce two new corrections to the Higgs mass term, which are shown in Figs 1.2 and 1.3.

[1]This approach was discussed, for example, in ref. [1].

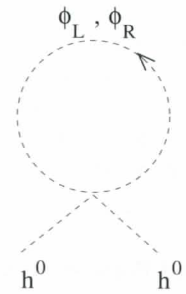


FIG. 1.2. Scalar boson contribution to the Higgs mass term via the quartic coupling.

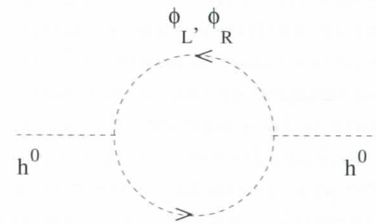


FIG. 1.3. Scalar boson contribution to the Higgs mass term via the trilinear coupling.

The contribution to the Higgs mass squared corresponding to Fig. 1.2 is

$$-i\delta m_h^2|_2 = -i\lambda N \int \frac{d^4k}{(2\pi)^4}\left[\frac{i}{k^2-m_L^2} + \frac{i}{k^2-m_R^2}\right] \tag{1.8}$$

by a similar series of manipulations as above we find

$$\delta m_h^2|_2 = \frac{\lambda N}{16\pi^2}\left[2\Lambda^2 - m_L^2 \ln\left(\frac{\Lambda^2+m_L^2}{m_L^2}\right) - m_R^2 \ln\left(\frac{\Lambda^2+m_R^2}{m_R^2}\right) + \ldots\right]. \tag{1.9}$$

The contribution to the Higgs mass squared corresponding to Fig. 1.3 is

$$-i\delta m_h^2|_3 = N \int \frac{d^4k}{(2\pi)^4}\left[\left(-i\mu_L \frac{i}{k^2-m_L^2}\right)^2 + \left(-i\mu_R \frac{i}{k^2-m_R^2}\right)^2\right], \tag{1.10}$$

which yields

$$\delta m_h^2|_3 = -\frac{N}{16\pi^2}\left[\mu_L^2 \ln\left(\frac{\Lambda^2+m_L^2}{m_L^2}\right) + \mu_R^2 \ln\left(\frac{\Lambda^2+m_R^2}{m_R^2}\right) + \ldots\right]. \tag{1.11}$$

Notice that if $N = N_c$ and $\lambda = |y_t|^2$ the quadratic divergences in eqns (1.6) and (1.9) are canceled. If we also have $m_t = m_L = m_R$ and $\mu_L^2 = \mu_R^2 =$

$2\lambda m_t^2$ the logarithmic divergences in eqns (1.6), (1.9), and (1.11) are canceled as well. SUSY is a symmetry between fermions and bosons that will guarantee just these conditions.[2] The cancellation of the logarithmic divergence is more than is needed to resolve the hierarchy problem; it is the consequence of powerful non-renormalization theorems that we will encounter in Chapter 8.

1.2 SUSY algebra

Interest in symmetries that extend Poincaré symmetry dates back to the 1960s when the suggestion [2, 3] of an approximate $SU(6)$ symmetry[3] of the hadron spectrum motivated Coleman and Mandula [4] to prove a "no-go" theorem. Their theorem stated that the only symmetry of the scattering matrix (S-matrix) that included Poincaré symmetry (with certain assumptions) was the product of Poincaré symmetry and an internal symmetry group. The proof shows that additional symmetry generators that transform as Lorentz tensors would over-constrain the S-matrix. For example, in two body scattering, Poincaré symmetry restricts the S-matrix element to be a function of only one variable: the scattering angle. The existence of an additional tensor symmetry generator would mean that the scattering could only occur at particular scattering angles, which means that the S-matrix would not be analytic (violating one of the prime assumptions). The extension of the Poincaré algebra to a "graded-Lie algebra" (i.e. algebras with anticommutators and spinor generators) by Golfand and Likhtman [5] allowed for the nontrivial possibility of a symmetry between bosons and fermions[4]: SUSY! Haag et. al. [7] extended the Coleman-Mandula theorem to allow for graded-Lie algebras and showed that SUSY is the only possible extension of the Poincaré algebra, and found the most general form of the SUSY algebra.

The algebra of the SUSY generators can be used directly to prove some interesting results.[5] In addition to the usual Poincaré generators (translations, boosts, and rotations) the generators of SUSY include complex, anticommuting spinors[6] Q and their conjugates $Q^\dagger$:

$$\{Q_\alpha, Q_\beta\} = \{Q^\dagger_{\dot\alpha}, Q^\dagger_{\dot\beta}\} = 0 \,. \tag{1.12}$$

The nontrivial extension of Poincaré symmetry arises because the anticommutator of Q and $Q^\dagger$ gives a translation generator (the momentum operator P_μ):

$$\{Q_\alpha, Q^\dagger_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu , \tag{1.13}$$

where

(2.56)

[2]As we will see in more detail in Section 2.6.

[3]$SU(6)$ arises by considering three flavors of quarks with two spins (up and down) as one fundamental multiplet.

[4]A very detailed history of SUSY is given in ref. [6].

[5]For excellent reviews, see refs [11, 9].

[6]It is often useful to keep track of the spinor indices, $\alpha = 1, 2$, of Q and $Q^\dagger$ separately by putting a dot ($\dot{}$) on the indices of all conjugates, writing instead $Q^\dagger_{\dot\alpha}$.

$$\sigma^\mu_{\alpha\dot\alpha} = (1, \sigma^i) \,, \quad \overline{\sigma}^{\mu\dot\alpha\alpha} = (1, -\sigma^i) \,. \tag{1.14}$$

Here the σ^i are the usual Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{1.15}$$

The SUSY generators commute with translations:

$$[P_\mu, Q_\alpha] = [P_\mu, Q^\dagger_{\dot\alpha}] = 0 \,. \tag{1.16}$$

The SUSY algebra is invariant under a multiplication of Q_α by a phase, so in general there is one linear combination of $U(1)$ charges, called the R-charge, that does not commute with Q and $Q^\dagger$:

$$[Q_\alpha, R] = Q_\alpha \,, \; [Q^\dagger_{\dot\alpha}, R] = -Q^\dagger_{\dot\alpha} \,. \tag{1.17}$$

The corresponding R-symmetry group is called $U(1)_R$.

Note that from eqn (1.13) it follows that the energy (Hamiltonian operator) is given by the sum of squares of SUSY generators

$$H = P^0 = \frac{1}{4}(Q_1 Q_1^\dagger + Q_1^\dagger Q_1 + Q_2 Q_2^\dagger + Q_2^\dagger Q_2) \,. \tag{1.18}$$

Single particle states fall into irreducible representations of the SUSY algebra called supermultiplets. Since Q is a spinor, when it acts on a bosonic state it produces a fermionic state, that is supermultiplets contain both bosons and fermions.

A boson and a fermion in the same supermultiplet are called superpartners. Since $P^\mu P_\mu$ commutes with Q and $Q^\dagger$ all particles in a supermultiplet have the same mass. Since gauge generators also commute with Q and $Q^\dagger$, all particles in a supermultiplet have the same gauge charge.

If we define the operator $\mathbf{F}$ which counts the fermion number of a state then

$$(-1)^{\mathbf{F}} \, |\text{boson}\rangle = +1 \, |\text{boson}\rangle \,, \tag{1.19}$$

$$(-1)^{\mathbf{F}} \, |\text{fermion}\rangle = -1 \, |\text{fermion}\rangle \,, \tag{1.20}$$

which implies

$$\{(-1)^{\mathbf{F}}, Q_\alpha\} = 0 \,. \tag{1.21}$$

Now consider the subspace of states $|i\rangle$ in a supermultiplet with a momentum p_μ. Completeness requires

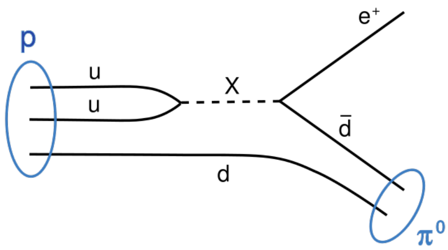
$$\sum_i |i\rangle\langle i| = 1 \,. \tag{1.22}$$

If we calculate the trace of energy operator weighted by +1 for bosons and -1 for fermions we find (using eqn (1.18)):

$$\sum_i \langle i|(-1)^{\mathbf{F}} P^0 |i\rangle = \frac{1}{4}\left(\sum_i \langle i|(-1)^{\mathbf{F}} Q Q^\dagger |i\rangle + \sum_i \langle i|(-1)^{\mathbf{F}} Q^\dagger Q |i\rangle\right)$$

BSM Exercices: Solutions

3 Baryon number is violated in the minimal $SU(5)$ GUT through interactions mediated by new gauge bosons X, Y such as the one shown in the figure:



Assuming gauge couplings close to the electroweak ones ($e.g.$, $\sim$ e), estimate τ_p, the proton life-time for $M_X \sim 10^{15}$ GeV. Compare with the present experimental value for the $e^+\pi^0$ mode:

$$\tau_{p\to e^+\pi^0} > 8.2\ 10^{33}\ \text{ years } \ (90\%\,\text{C.L.})$$

Note: 1 $\text{GeV}^{-1} \sim 6.6\ 10^{-25} s$.

The width is proportional to the square of the amplitude. Neglecting phase space effects (ie, $m_p >> m_\pi$)

$$\Gamma \sim m_p^{\#} \left(\frac{g^2}{M_X^2}\right)^2 \tag{0.1}$$

and by dimensional analysis $m_p^{\#} = m_p^5$. Then using $g^2 \sim \alpha\, 4\pi$ and taking into account that $\tau \sim \Gamma^{-1}$, we get

$$\tau \sim m_p^{-5} \left(\frac{M_X^2}{4\pi\alpha}\right)^2 \tag{0.2}$$

We can now substitute $m_p \sim 1$ GeV, $M_X \sim 10^{15}$ GeV

$$\tau \sim \frac{1}{(4\pi\alpha)^2}\, 10^{60}\,\text{GeV}^{-1} \tag{0.3}$$

Taking $\alpha \sim \frac{1}{137}$ and using 1 $\text{GeV}^{-1} \sim 6.6\ 10^{-25} s$,

$$\tau \sim \left(\frac{137}{4\pi}\right)^2 6.6\,10^{35}\, s \sim \left(\frac{137}{4\pi}\right)^2 6.6\,10^{35}\, \frac{1}{365\times 24\times 3600}\,\text{years} \tag{0.4}$$

and we get

$$\tau_{p\to e^+\pi^0}\Big|_{th} \sim 2.5\,10^{30}\,\text{years} \tag{0.5}$$

shorter than the lower experimental limit and then excluded.

1 BSM Exercices

1. Consider an arbitrary set of scalar multiplets Φ_i of isospin J_i having a neutral component that takes a vev, v_i. Show that the ρ parameter is given by

$$\rho_{\text{tree}} = \frac{M_W^2}{M_Z^2 \cos^2\theta_W} = \frac{\sum [J_i(J_i+1) - Y_i^2] v_i^2}{2\sum Y_i^2 v_i^2}$$

Gauge boson mass terms are originated from

$$\sum (D_\mu \Phi_i)^\dagger (D^\mu \Phi_i) \tag{1.1}$$

The covariant derivative is given by:

$$\begin{aligned} D_\mu &= \partial_\mu - igJ_1W_\mu^1 - igJ_2W_\mu^2 - igJ_3W_\mu^3 - ig'YB_\mu && (1.2)\\ &= \partial_\mu - ig\frac{1}{\sqrt{2}}(J_+W^- + J_-W^+) - igJ_3W^3 - ig'YB_\mu && (1.3)\end{aligned}$$

where $J_\pm = J_1 \pm iJ_2$ and $W^\pm = \dfrac{1}{\sqrt{2}}(W^1 \pm iW^2)$. M_W^2 is given by the W^+W^- coefficient[1], ie

$$\begin{aligned} M_W^2 &= g^2 \sum \langle JM|\, \frac{1}{2}(J_+J_- + J_-J_+)\, |JM\rangle_i \; v_i^2 && (1.4)\\ &= g^2 \sum \langle JM|\, (J_1^2 + J_2^2)\, |JM\rangle_i \; v_i^2 && (1.5)\\ &= g^2 \sum \langle JM|\, (J^2 - J_3^2)\, |JM\rangle_i \; v_i^2 && (1.6)\\ &= g^2 \sum [J_i(J_i+1) - Y_i^2]\; v_i^2 && (1.7)\\ & && (1.8)\end{aligned}$$

where $J^2 = J_1^2 + J_2^2 + J_3^2$ and we have used

$$\begin{aligned} J^2\, |JM\rangle &= J(J+1)\, |JM\rangle && (1.9)\\ J_3\, |JM\rangle &= M\, |JM\rangle && (1.10)\end{aligned}$$

and

$$Q|JM\rangle_i = (J_3 + Y)|JM\rangle_i = 0 \tag{1.11}$$

for the neutral scalar components, the ones that adquire a vev.

[1]Remember that $W^+W^- = \frac{1}{2}(W^1W^1 + W^2W^2)$

The neutral bosons can be evaluated similarly. The mass matrix in the (W^3_μ, B_μ) basis is given by:

$$\mathcal{M}_0^2 = 2\sum_i \begin{pmatrix} g^2 & g'g \\ g'g & g'^2 \end{pmatrix} Y_i^2 v_i^2 \tag{1.12}$$

where we have used Eq.(1.11). $\mathcal{M}_0^2$ is diagonalized by a rotation of angle θ_W

$$\mathcal{O}_W = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \tag{1.13}$$

i.e,

$$\mathcal{O}_W\, \mathcal{M}_0^2\, \mathcal{O}_W^T = \begin{pmatrix} M_Z^2 & 0 \\ 0 & 0 \end{pmatrix} \tag{1.14}$$

Then,

$$\mathcal{M}_0^2 = \mathcal{O}_W^T \begin{pmatrix} M_Z^2 & 0 \\ 0 & 0 \end{pmatrix} \mathcal{O}_W \tag{1.15}$$

In particular,

$$\mathcal{M}_0^2 \Big|_{11} = \cos^2\theta_W M_Z^2 \tag{1.16}$$

Using Eq. (1.12), we complete the proof.

Getting $\rho = 1$

For a unique multiplet, $\rho = 1$ implies that $J(J+1) = 3Y^2$

- For semi-integer isospin, $d = 2J+1$ is even and $2Y = \sqrt{\frac{(d+1)(d-1)}{3}}$ has to be odd. Smallest multiplet: (d=2, Y = 1/2), Higgs doublet
- For integer isospin, $d = 2J+1$ is odd and $2Y = \sqrt{\frac{(d+1)(d-1)}{3}}$ has to be even. Smallest multiplet: (d=7, Y = 2), Higgs septuplet.