Problems for SM/Higgs (I)

1

Draw all possible Feynman diagrams (at the lowest level in perturbation theory) for the processes $e^+e^- \to \mu^+\mu^-,\ \nu_e\bar{\nu}_e,\ \gamma\gamma,\ ZZ,\ W^+W^-$. Likewise, draw all possible Feynman diagrams (at lowest order and at the partonic level) for the processes $pp \to d\bar{d},\ t\bar{t},\ \gamma\gamma,\ ZZ,\ W^+W^-,\ W^+Z$.

2

Using

$$\mathscr{L}_{\rm kin}^{\rm Higgs} = (D_\mu H)^\dagger (D^\mu H) \tag{1}$$

where

$$D_\mu H = \left(\partial_\mu - \frac{1}{2} ig\ \vec{\sigma}\cdot\vec{W}_\mu - iq_Y^{(H)} g'\ B_\mu\right) H \tag{2}$$

and $q_Y^{(H)} = -1/2$, check that the mass eigenstates of the gauge bosons are

$$\begin{aligned} W_\mu^\pm &= \frac{W_\mu^1 \mp iW_\mu^2}{\sqrt{2}} \\ Z_\mu &= \cos\theta_W W_\mu^3 - \sin\theta_W B_\mu \\ A_\mu &= \sin\theta_W W_\mu^3 + \cos\theta_W B_\mu \end{aligned} \tag{3}$$

with

$$\tan\theta_W = \frac{g'}{g}, \qquad e = g\sin\theta_W \tag{4}$$

Check also that the mass eigenvalues are

$$M_W^2 = \frac{1}{4} g^2 v^2, \qquad M_Z^2 = \frac{1}{4}(g^2 + g'^2)v^2, \qquad M_A = 0 \tag{5}$$

where v is the Higgs VEV.

3

Consider the lifetimes of the μ and τ leptons, the different channel in which they can decay and the different branching ratios. More precisely, discuss

1. Why does the μ have just one decay mode ($\mu \to e\bar{\nu}_e\nu_\mu$) while the τ has several ones.
2. Using $m_\mu \simeq 106$ MeV, $m_\tau \simeq 1777$ MeV and $T_\mu \simeq 2.2\times 10^{-6}$ sec, $T_\tau \simeq 2.9\times 10^{-13}$ sec, estimate the fraction of τ decays into $e\bar{\nu}_e\nu_\tau$ and compare with the observed BR($\tau \to e\bar{\nu}_e\nu_\tau$)$\simeq$ (17.85 ± 0.005) %. What do you expect for BR($\tau \to \mu\bar{\nu}_\mu\nu_\tau$)? And for BR($\tau \to$ hadrons)?

4

1. Show that the SM does not violate CP unless the number of generations is ≥ 3. In order to do that, count the number of real parameters (angles) and phases of the V_{CKM} if there were just two generations of quarks.

2. Do the same exercise for the lepton sector, assuming that neutrinos have Majorana masses: count the number of the number of angles and phases of the U_{MNS} matrix (the equivalent of V_{CKM} in the lepton sector) for two and three generations.

   Could it be possible to have CP violation in the lepton sector with just two generations?

   Note: the leptonic Lagrangian with Majorana masses for the neutrinos has the form

$$-\mathscr{L}_{\text{lep}} = L^T Y_e \bar{H} e^c + \frac{1}{2} \nu_L^T \mathcal{M} \nu_L + \text{ h.c.} \tag{6}$$

   Here Y_e can be assumed to be diagonal (by appropriate rotation of L_L and e^c). The MNS matrix is the unitary matrix that diagonalizes $\mathcal{M}$, i.e. $\mathcal{M} = U^* D_\nu U^\dagger$.

5

Determine what types of heavy particle could generate by tree-level interchange the d=5 operator

$$\frac{\kappa_{ij}}{\Lambda}(L_{L_i} H)(L_{L_j} H), \tag{7}$$

which leads to Majorana masses for left-handed neutrinos. Find all different possibilities, giving for each the quantum numbers of the heavy particle being exchanged.

Solutions to Problems for SM/Higgs (I)

1

I think this is straightforward

2

Using $\langle H\rangle = (v/\sqrt{2}, 0)$, the relevant piece of $\mathscr{L}_{\rm kin}^{\rm Higgs}$ is

$$\frac{v^2}{2}\ \left\{\left(\frac{1}{2}ig\ \vec{\sigma}\cdot\vec{W}_\mu - \frac{1}{2}ig'\ B_\mu\right)\left(-\frac{1}{2}ig\ \vec{\sigma}\cdot\vec{W}^\mu + \frac{1}{2}ig'\ B\mu\right)\right\}_{11} \tag{8}$$

Taking into account

$$\sigma^a\sigma^b = i\sum_c \epsilon_{abc}\sigma_c + \delta_{ab}\ 1 \tag{9}$$

we see that

$$\begin{aligned} \frac{v^2}{2}\{\quad\}_{11} &= \frac{v^2}{8}\left\{g'^2B_\mu^2 + g^2\sum_{a=1}^{3}(W_\mu^a)^2 - 2gg'W_3^\mu B_\mu\right\} \\ &= \frac{v^2}{8}(W_\mu^1, W_\mu^2, W_\mu^3, B_\mu)\begin{pmatrix} g^2 & 0 & 0 & 0 \\ 0 & g^2 & 0 & 0 \\ 0 & 0 & g^2 & -gg' \\ 0 & 0 & -gg' & g'^2 \end{pmatrix}\begin{pmatrix} W_\mu^1 \\ W_\mu^2 \\ W_\mu^3 \\ B_\mu \end{pmatrix} \end{aligned} \tag{10}$$

Diagonalizing these matrix one finds the correct mass eigenvalues and eigenstates. Since W_μ^1, W_μ^2 are degenerate, the $W_\mu^\pm$ states are as good as W_μ^1, W_μ^2:

$$(W_\mu^1)^2 + (W_\mu^2)^2 = 2W_\mu^+W_\mu^- \tag{11}$$

However the $W_\mu^\pm$ states have physical meaning since they have definite electric charges. You can check this by considering the infinitesimal transformation

$$W_\mu^a \to W^a + f^{abc}W_\mu^b\alpha^c + \frac{1}{g}\partial_\mu\alpha^a \tag{12}$$

with $\alpha^1 = \alpha^2 = 0, \alpha^3 = \text{const}$.

3

1. The first point is trivial.

2. The amplitude for the muon decay $\mu \to e\bar{\nu}_e\nu_\mu$ is proportional to the Fermi constant, $A \sim G_F$, so that the decay probability goes like $|A|^2 \sim G_F^2 \sim [\text{mass}]^4$. Neglecting the electron mass,

$m_e \ll m_\mu$, the only relevant mass scale is m_μ and, therefore, the muon width $\Gamma_\mu = 1/T_\mu = c\ G_F^2 m_\mu^5$, where c is some numerical coefficient that takes care of the kinematics.

The partial width of the τ into the $e\bar{\nu}_e\nu_\tau$ channel is also controlled by G_F and, within the same approximation of neglecting $m_e \ll m_\tau$, will go as $\Delta\Gamma_\tau = c\ G_F^2 m_\tau^5$ with the same constant c. Knowing the total widths $1/T_\mu$, $1/T_\tau = \Gamma_\tau$, we get

$$\mathrm{BR}(\tau \to e\bar{\nu}_e\nu_\mu) = \frac{\Delta\Gamma_\tau}{\Gamma_\tau} = \frac{\Gamma_\mu}{\Gamma_\tau}\frac{\Delta\Gamma_\tau}{\Gamma_\mu} = \frac{T_\tau}{T_\mu}\left(\frac{m_\tau}{m_\mu}\right)^5 = 0.18 \tag{13}$$

which is a very good estimate of the measured BR.

The decay $\tau \to \mu\bar{\nu}_\mu\nu_\tau$ will proceed along a similar G_F coupling and the corresponding BR will be of the same order, except a little bit lower due to the phase space factors (since $m_\mu > m_e$). This can be checked in the PDG review of particle properties.

Concerning BR($\tau \to$ hadrons), one can follow similar steps. For kinematic reasons, there is just one hadronic channel (at the quark level) $\tau \to d\bar{u}\nu_\tau$. The amplitude of this decay is roughly similar to that of $\tau \to e\bar{\nu}_e\nu_\tau$ and $\tau \to \mu\bar{\nu}_\mu\nu_\tau$. However the quarks have three colors, so there are three different hadronic final states. Hence the BR for $\tau-$hadronic decays should be roughly three times BR($\tau \to d\bar{u}\nu_\tau$) and BR($\tau \to d\bar{u}\nu_\tau$).

Consequently, the total BR for hadronic $\tau-$decays should be about 60%, which is not far from the experimental value.

4

(a) Starting in a basis where the kinetic (and thus gauge) interactions are flavor-diagona, the quark part of the SM Lagrangian reads

$$\begin{aligned}\mathscr{L} &\supset \sum_i \bar{Q}_i \not{D} Q_i + \sum_i \bar{d}_i \not{D} d_i + \sum_i \bar{u}_i \not{D} u_i \\ &- \left(Q_i^T (Y_u)_{ij} H d_j^c + Q_i^T (Y_u)_{ij} \bar{H} u_j^c + h.c.\right)\end{aligned} \tag{14}$$

The Y_d, Y_u matrices (in generation space) can be diagonalized using two unitary matrices:

$$Y_d = U_{d_L}^T D_{Y_d} U_{d_R}, \qquad Y_u = U_{u_L}^T D_{Y_u} U_{u_R}. \tag{15}$$

Replacing these expressions in $\mathscr{L}$ and redefining $U_{d_L} Q \to Q$, $U_{d_R} d^c \to d^c$, $U_{u_R} u^c \to d^c$, the Lagrangian reads

$$\mathscr{L} \to -\left(Q^T D_{Y_d} H d^c + Q^T U_{d_L}^* U_{u_L}^T D_{Y_u} \bar{H} u^c + \text{h.c.}\right) + \mathscr{L}_{\text{kin/gauge}} + \cdots \tag{16}$$

(Note that $\mathscr{L}_{\text{kin/gauge}}$ remains invariant.). The matrix

$$U \equiv \left(U_{d_L}^* U_{u_L}^T\right)^T = U_{u_L} U_{d_L}^\dagger \tag{17}$$

is essentially the CKM matrix, but it can be simplified using field redefinitions.
In principle U is a complex 3×3 complex matrix (thus having 18 real parameters), but the unitarity condition $U^\dagger U = 1$ eliminates nine of them, so U has 9 (real) parameters. As any other unitary matrix, it can be parametrized in the following way:

$$U = \Phi_1 U' \Phi_2 \tag{18}$$

where $\Phi_{1,2}$ are diagonal (3×3) matrices whose entries are phases:

$$\Phi_1 = \text{diag}(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}), \qquad \Phi_2 = \text{diag}(e^{i\beta_1}, e^{i\beta_2}, e^{i\beta_3}), \tag{19}$$

and U' contains the remaining parameters. One of the phases in $\Phi_{1,2}$, say α_3 is irrelevant, since multiplying Φ_1 by $e^{-i\alpha_3} \times 1$ and Φ_2 by $e^{i\alpha_3} \times 1$ does not affect U', but cancels the third entry of Φ_1 . So we can take

$$\Phi_1 = \text{diag}(e^{i\alpha_1}, e^{i\alpha_2}, 1), \qquad \Phi_2 = \text{diag}(e^{i\beta_1}, e^{i\beta_2}, e^{i\beta_3}), \tag{20}$$

In summary we have taken out 5 out of the 9 parameters in U. Thus U' still has 4 parameters, and they must include one phase since if U were real, it would be an orthogonal matrix. But orthogonal 3×3 matrices have just three parameters (e.g. the three Euler angles).
Now, looking at eq.(16), the phases in Φ_1 (Φ_2) can be re-absorbed in a redefinition of u^c (Q). Then the latter phase re-appears in the first term of eq.(16), but it can be re-absorbed in a redifinition of d^c since the Yukawa matrix of the $d-$quarks has diagonal form. But the phase in U' *cannot* be re-absorbed. This U' with 4 parameters is the CKM matrix.

Consequently the mass matrix for the $u-$quarks has a phase which cannot be re-absorbed, and thus the theory is not CP-conserving.

Following the same steps for the 2-generation case we arrive at a U' matrix which has just one parameter. SInce an orthogonal 2×2 matrix has just one parameter (a rotation angle), one concludes that, for two generations, CP cannot be violated in the quark sector of the SM.

(b) For the leptonic case we have

$$\mathscr{L}_{\text{lep}} = -\left(L^T Y_e \bar{H} e^c + \frac{1}{2}\nu_L^T \mathcal{M} \nu_L + \text{h.c.}\right) + \mathscr{L}_{\text{kin/gauge}} + \cdots \tag{21}$$

Y_e, $\mathcal{M}$ are matrices in generation-space (we drop indices). recall that Y_e can be assumed to be diagonal (by appropriate rotation of L_L and e^c). The MNS matrix is essentially the unitary matrix that diagonalizes $\mathcal{M}$, i.e.

$$\mathcal{M} = U^* D_\nu U^\dagger. \tag{22}$$

U can be factorized in a way similar to eq.(18), i.e.

$$U = \Phi_1 V \Phi_2 \tag{23}$$

where we choose

$$\Phi_1 = \text{diag}(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}), \qquad \Phi_2 = \text{diag}(e^{i\beta_1}, e^{i\beta_2}, 1), \tag{24}$$

So the neutrino mass term of the Lagrangian reads

$$\nu_L^T\ \Phi_1^* V^* \Phi_2^* D_\nu \Phi_2^* V^\dagger \Phi_1^*\ \nu_L + \text{h.c.} \tag{25}$$

Now we can redefine $\Phi_1^* \nu_L \to \nu_L$ (actually we must redefine the whole L doublets, i.e. $\Phi_1^* L_L \to L_L$). Then a diagonal phase appears in the first term of (21), which can be re-absorbed in a redefinition of e^c since Y_e is diagonal.
At the end of the day, the part of the U matrix, eq.(23), that cannot be re-absorbed is:

$$V_{MNS} = V \Phi_2 \tag{26}$$

where V has a structure similar to U_{CKM}. So in this case there are two extra phases that can trigger CP violation.

Following the same steps for the case of two generations one arrives at a V_{MNS} with the same structure as eq.(26). So there is a phase that cannot be absorbed and, consequently, CP can vilated in the neutrino sector, even with just two generations. (I think this is correct!).

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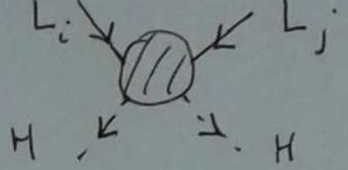
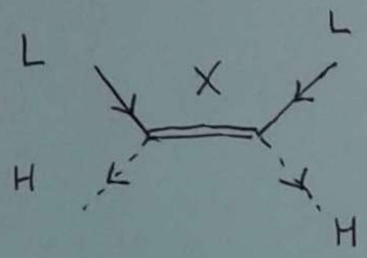
Sol. Problem 5

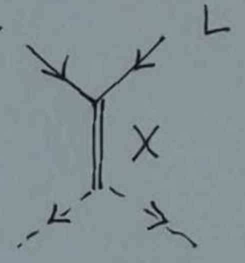
In order to generate an effective operator

$\frac{\kappa}{\Lambda} \, L\bar{H} L\bar{H}$



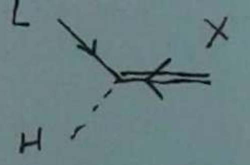
from a tree-level diagram there are two possibilities





Let us identify what could X be.

In the first case I need a vertex



$L\bar{H}X$ + h.c.

I can re-name $X \to X^c$ and vice-versa, so that the vertex is

$L\bar{H}X^c$ + h.c.

The vertex $L\bar{H}X^c$ must be Lorentz & $SU(2)\times U(1)_Y$ invariant

$\Rightarrow$ X is a right-handed spinor

$$Y_X = 0$$

X must be a singlet or a triplet under $SU(2)$ since

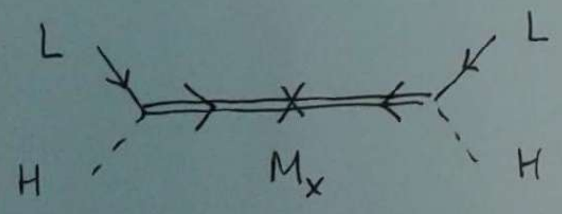
$$\underset{L}{2} \times \underset{\bar{H}}{2} = 3, 0$$

I.e. $X \equiv N_R$ or T_R

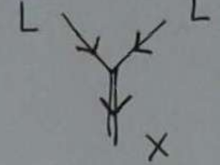
1_0 $\quad$ 3_0

SU(2) singlet $\quad$ Hypercharge $\quad$ SU(2) triplet

Finally notice that in order to connect the two vertices of the diagram the X-propagator needs a Majorana-mass insertion : $\frac{1}{2}M_X XX$ + h.c. (Otherwise the fermionic arrows do not match) :



Let us now consider the second case. First, I need a vertex



$L\,L\,\bar{X}$

Lorentz & $SU(2) \times U(1)_Y$ invariance requires

$X \equiv$ scalar (it cannot be a vector since I cannot form a vector with two left-handed spinors. To construct a vector I need a right-handed one: $\psi_L \gamma^\mu \xi_R$)

$$Y_X = -1$$

X must be a singlet or a triplet under $SU(2)$:

$$\underset{L_i \cdot L_j}{2 \times 2} = 3,\, 0$$

In addition I need a vertex

$\bar{H}\,\bar{H}\,X$

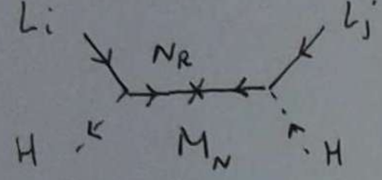
Note that $\bar{H}\bar{H}X$ is indeed Lorentz & $U(1)_Y$ invariant. What about $SU(2)$? If X is a singlet, then I have to form a singlet with $\bar{H} \cdot \bar{H}$. Since $\bar{H}$ is a $SU(2)$ doublet, the invariant combination is

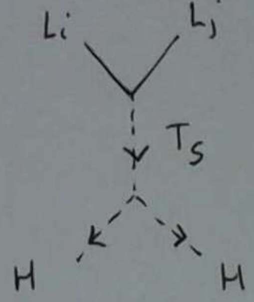
$$\varepsilon_{ij}\, \bar{H}_i\, \bar{H}_j = 0$$

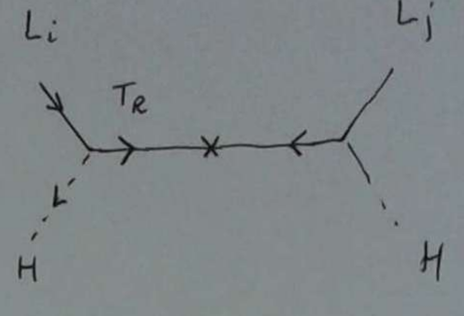
$\Rightarrow$ X = triplet $\equiv$ T_S

3_{-1}

In summary:







type-I seesaw

type-II seesaw

type-III seesaw