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QCD

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Quantum Chromodynamics (QCD)



The theory of quarks, gluons and their interactions

It's central to all modern colliders
(and QCD is what we are made of)

Outline

1. QCD Lagrangean, and IR divergences in e^+e^- .
2. pQCD at hadron colliders
3. New methods in pQCD: helicity formalism and generalized unitarity
4. The collinear limit of QCD
5. Parton distribution functions
6. Jets

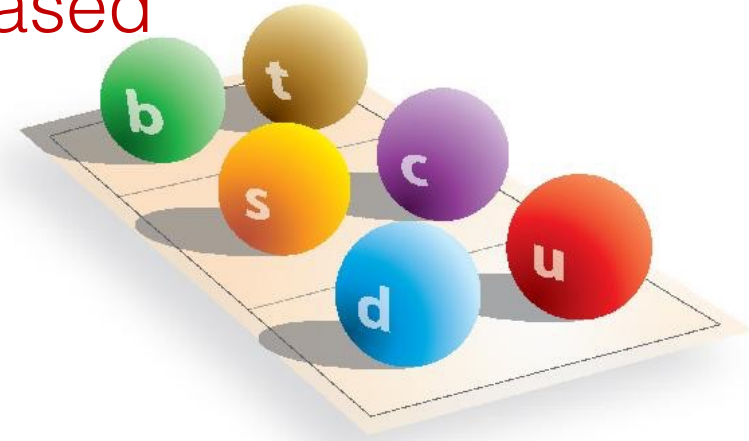
Particle physics
gives me a
hadron

QCD and e^+e^- colliders

The ingredients of QCD

QCD is a gauge invariant QFT, based on a local SU(3) symmetry group

- ▶ Quarks (and anti-quarks): six flavours
 - they come in 3 colours
- ▶ Gluons: massless gauge bosons
 - a bit like photons in QED
 - but there are 8 of them, and they are colour charged
- ▶ And the coupling $\alpha_s(\mu)$
 - that's not so small and runs fast
 - at the LHC, in the range 0.08 @ 5 TeV to O(1) at 0.5 GeV



Quark Lagrangean + colour

The quark part of the Lagrangean

$$\mathcal{L}_q = \bar{\psi}_i \left(\delta_{ij} (i \not{\partial} - m) + g_S T_{ij}^a A^a \right) \psi_j$$

- ▶ where quarks carry three colours $\psi_i = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$
- ▶ SU(3) local gauge symmetry: 8 ($= 3^2 - 1$) generators $T_{ij}^1 \dots T_{ij}^8$ corresponding to 8 gluons $A_\mu^1 \dots A_\mu^8$
- ▶ The fundamental representation: $\mathbf{T}^a = \frac{1}{2} \lambda^a$, Traceless and Hermitian

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda^8 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{-2}{\sqrt{3}} \end{pmatrix}$$

Gluon Lagrangean

The gluon part of the Lagrangean

$$\mathcal{L}_g = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a}$$

where the field tensor is

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + i g_S (-i f_{abc}) A_\mu^a A_\nu^c$$

$$[\mathbf{T}^a, \mathbf{T}^b] = i f_{abc} \mathbf{T}^c$$

f_{abc} are the structure constant of SU(3): antisymmetric in all indices.

Needed for gauge invariance of the Lagrangean

Gluon propagator: $\frac{1}{k^2 + i0} d^{\mu\nu}(k)$

Feynman gauge $d^{\mu\nu}(k) = -g^{\mu\nu}$ simpler but requires ghosts

Axial gauge $d^{\mu\nu}(k, n) = -g^{\mu\nu} + \frac{k^\mu n^\nu + n^\mu k^\nu}{n \cdot k}, \quad n^2 = 0$

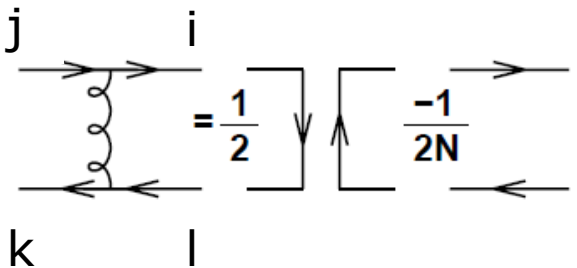
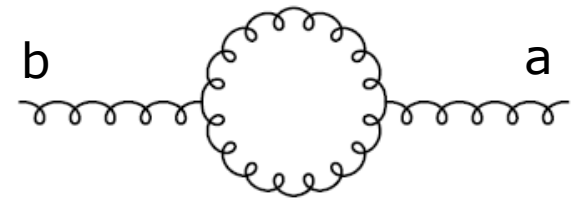
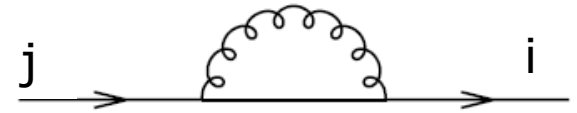
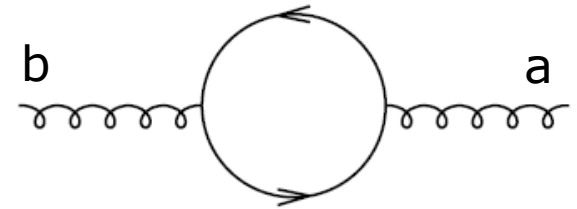
Colour algebra

$$\text{Tr}(\mathbf{T}^a \mathbf{T}^b) = T_R \delta^{ab}, \quad T_R = \frac{1}{2}$$

$$\sum_a T_{ik}^a T_{kj}^a = C_F \delta_{ij}, \quad C_F = \frac{N_C^2 - 1}{2N_C} = \frac{4}{3}$$

$$\sum_{c,d} f^{acd} f^{bcd} = C_A \delta^{ab}, \quad C_A = N_C = 3$$

$$T_{ij}^a T_{kl}^a = \frac{1}{2} \left(\delta_{jk} \delta_{il} - \frac{1}{N_C} \delta_{ij} \delta_{kl} \right), \quad \text{Fierz}$$

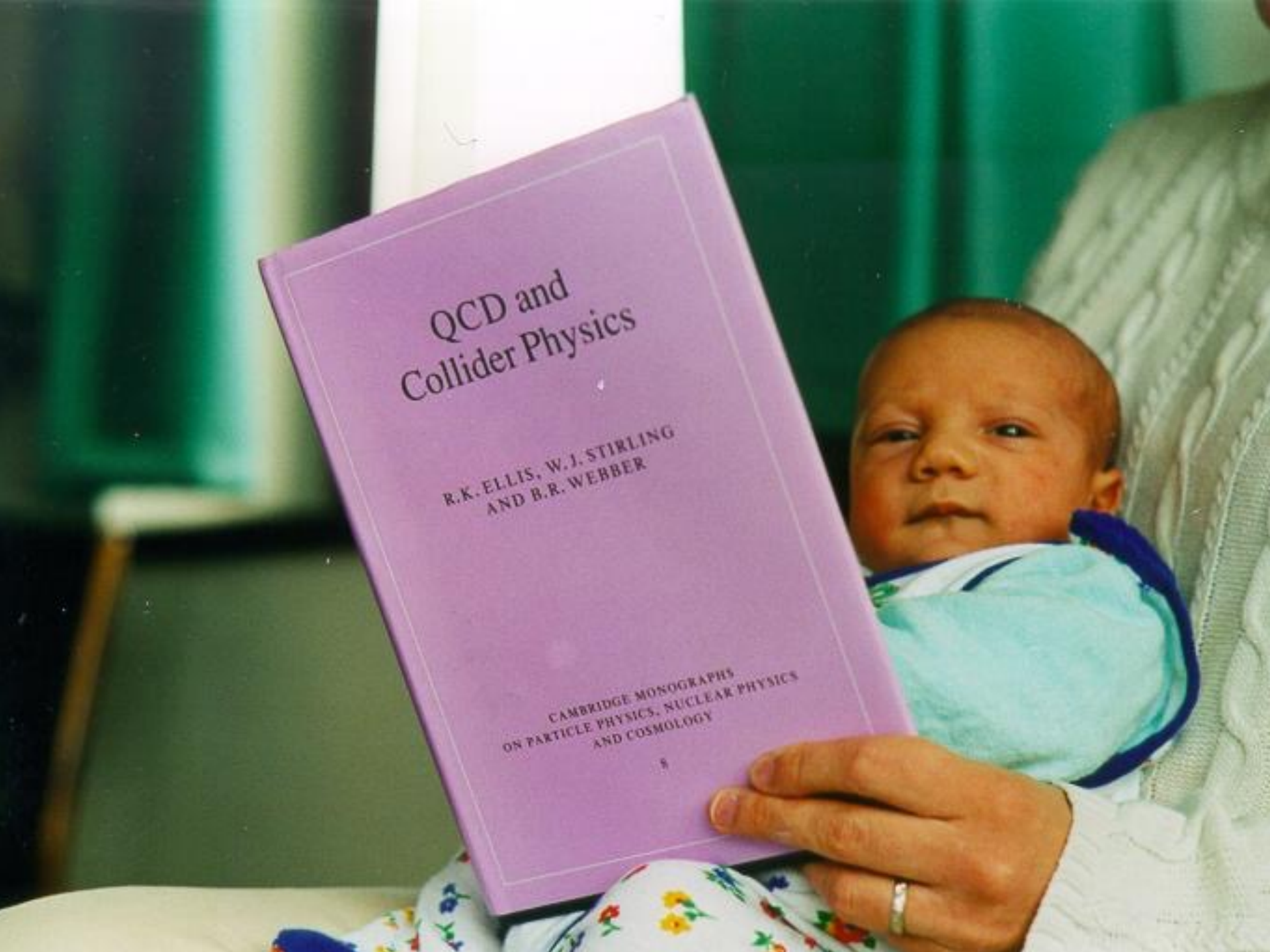


QCD and
Collider Physics

R.K. ELLIS, W.J. STIRLING
AND B.R. WEBBER

CAMBRIDGE MONOGRAPHS
ON PARTICLE PHYSICS, NUCLEAR PHYSICS
AND COSMOLOGY

5



Perturbation Theory

- Relies on the idea of order-by-order expansion in the small coupling $\alpha_S \ll 1$

$$\alpha_S + \alpha_S^2 + \alpha_S^3 + \dots$$

small

 smaller

 negligible?

incoming quark	$u(p)$			$\bar{u}(p)$	outgoing quark
incoming antiquark	$\bar{v}(p)$			$v(p)$	outgoing antiquark
incoming gluon	$\varepsilon_\mu^a(k)$			$\varepsilon_\mu^{a*}(k)$	outgoing gluon

	$i \frac{1}{\not{p} - m + i0} \delta_{ij}$	fermion propagator, momentum q in the direction of the fermion arrow
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	$i \frac{d_{\mu\nu}(k)}{k^2 + i0} \delta_{ab}$	gluon propagator, momentum k
--	--	--------------------------------

	$i \frac{1}{k^2 + i0} \delta_{ab}$	ghost propagator, momentum k
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	$i g_S T_{ij}^a \gamma_\mu$	fermionic vertex
--	-----------------------------	------------------

	$i g_S (-i f_{abc}) k^\mu$	ghost vertex
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	$i g_S (-i f_{abc}) [g_{\mu\nu}(k_1 - k_2)_\sigma + g_{\nu\sigma}(k_2 - k_3)_\mu + g_{\sigma\mu}(k_3 - k_1)_\nu]$	triple gluon vertex (outgoing momenta)
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	$-i g_S^2 [f_{abe} f_{cde} (g_{\mu\sigma} g_{\nu\rho} - g_{\mu\rho} g_{\nu\sigma}) + f_{ace} f_{bde} (g_{\mu\nu} g_{\sigma\rho} - g_{\mu\rho} g_{\nu\sigma}) + f_{ade} f_{cbe} (g_{\mu\sigma} g_{\nu\rho} - g_{\mu\nu} g_{\sigma\rho})]$	quartic gluon vertex
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How big is the coupling ?

All the SM couplings (including \overline{MS} mass/Yukawa) depend on the energy scale (obey Renormalization Group Equation RGE), and the QCD coupling **run fast**

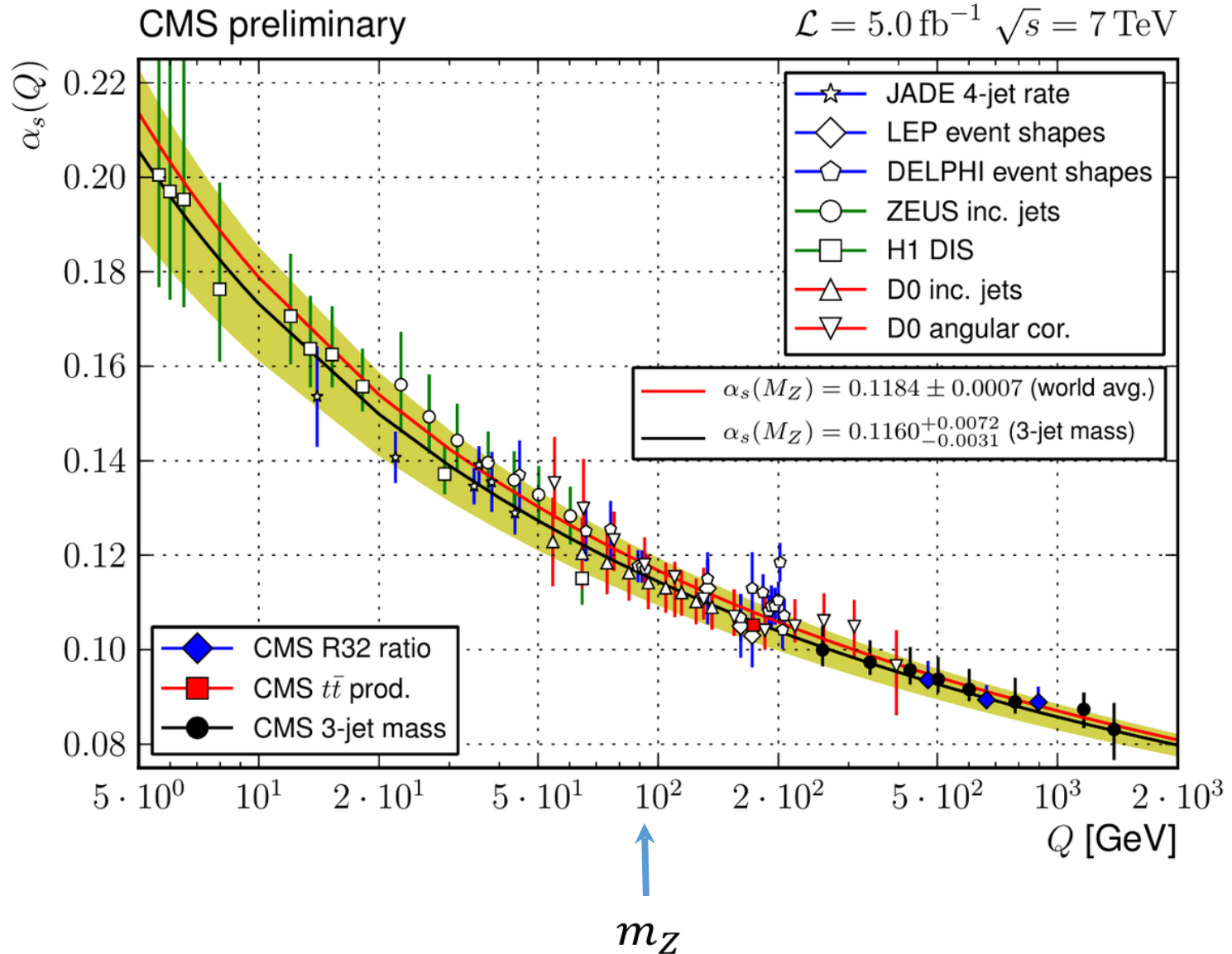
$$\frac{\partial a_S}{\partial \log \mu^2} = \beta(a_S) = -a_S^2 (b_0 + a_S b_1 + a_S^2 b_2 + \dots) , \quad a_S = \frac{\alpha_S}{\pi}$$

$$\frac{\partial \log m_q}{\partial \log \mu^2} = \gamma_m(a_S) = -a_S (g_0 + a_S g_1 + a_S^2 g_2 + \dots) ,$$

$$b_0 = \frac{1}{12} (11C_A - 2N_F) , \quad b_1 = \frac{1}{24} (17C_A^2 - (5C_A + 3C_F)N_F)$$

$$g_0 = 1 \quad g_1 = \frac{1}{16} \left(\frac{202}{3} - \frac{20}{9} N_F \right)$$

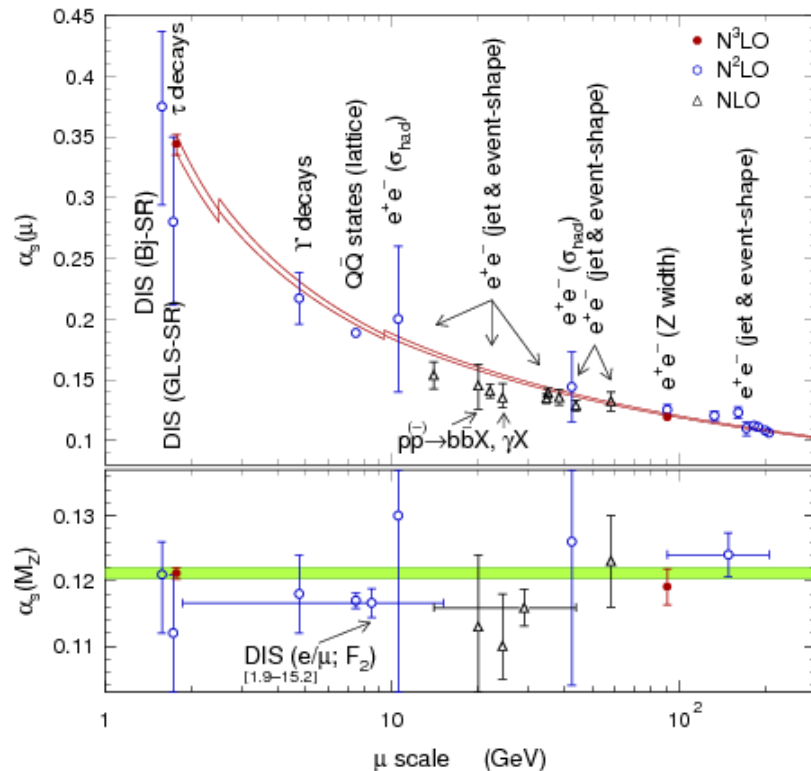
- Sign $\beta(\alpha_S) < 0$: **Asymptotic Freedom** due to gluon self-interactions
[Nobel Prize 2004, Gross, Politzer, Wilczek]
- At high scales: coupling becomes small, quarks and gluons are almost free, strong interactions are weak
- At low scales: coupling becomes large, quarks and gluons interact strongly, confined into hadrons, perturbation theory fails



Flavour thresholds

$$a_S^{(N_F)}(\mu_{\text{th}}) = a_S^{(N_F-1)}(\mu_{\text{th}}) \left[1 + \sum C_k(x) (a_S^{(N_F-1)}(\mu_{\text{th}}))^k \right]$$

$$m_q^{(N_F)}(\mu_{\text{th}}) = m_q^{(N_F-1)}(\mu_{\text{th}}) \left[1 + \sum H_k(x) (a_S^{(N_F-1)}(\mu_{\text{th}}))^k \right], \quad x = \log(\mu_{\text{th}}^2/m_q^2)$$



$$C_1 = \frac{x}{6}, \quad C_2 = -\frac{11}{72} + \frac{19}{24}x + \frac{x^2}{36}$$

$$H_1 = 0, \quad H_2 = -\frac{89}{432} + \frac{5}{36}x - \frac{x^2}{12}$$

- The $\beta(\alpha_S)$ and $\gamma_m(\alpha_S)$ functions depend on N_F
- Interpret it in the context of **Effective Theories** with different number of active flavours, and match the couplings at threshold
- Matching is independent of μ_{th} (up to higher orders)

- α_S might become discontinuous, is that a problem ?
- Similar discussion for PDFs



Exercises:

1. Integrate analytically the one-loop and two-loop RGE for the strong coupling, and one-loop for a quark mass
2. Calculate $\alpha_S(10 \text{ GeV})$ and $\alpha_S(1 \text{ TeV})$ from $\alpha_S(m_Z) = 0.1184 \pm 0.0007$
3. If $m_b(m_b) = 4.2 \pm 0.1 \text{ GeV}$, what is $m_b(m_Z)$
4. Hint

$$a_S(\mu) = \frac{a_S(\mu_0)}{1 + b_0 a_S(\mu_0) \log \frac{\mu^2}{\mu_0^2}} \quad \alpha_S(\mu) = \frac{\pi}{b_0 \log \frac{\mu^2}{\Lambda_{\text{QCD}}^2}}$$

Then calculate Λ_{QCD} , the “fundamental” scale of QCD, at which coupling blows up (NB: it is not unambiguously defined at higher order)

The infrared problem in gauge theories

- **Soft divergences (=IR)** because gluons are massless and can be emitted with zero energy (same phenomenon as in QED with soft photons)
- **Collinear divergences (=mass singularities):** when either gluons or massless quarks are produced with parallel momenta
 - Formally could keep $m_q \neq 0$ but perturbative results will depend on large $\log(m_q)$, and are not trustworthy

Ultraviolet divergences are removed by renormalization

Soft and collinear divergences should cancel → results dominated by large virtualities

Theorems about cancellation of divergences

- **BN (Block-Nordsieck):** QED (with finite fermion mass) IR divergences cancel is sum over soft (unobserved) photons in the final state
- **KLN (Kinoshita, Lee, Nauenberg):** IR and collinear divergences cancel if sum over degenerate final and initial states ($\gamma^* \rightarrow$ hadrons need only sum in final state)

Definition of infrared and collinear safety

For an observable's distribution to be calculable in [fixed order] perturbation theory, the observable should be **infrared safe**, i.e. insensitive to the emission of soft or collinear gluons. In particular if \vec{p}_i is any momentum occurring in its definition, it must be invariant under the branching

$$\vec{p}_i \rightarrow \vec{p}_j + \vec{p}_k$$

whenever \vec{p}_j and \vec{p}_k are parallel (collinear) or one of them is small (soft)

[Ellis, Stirling, Webber, QCD and Collider Physics]

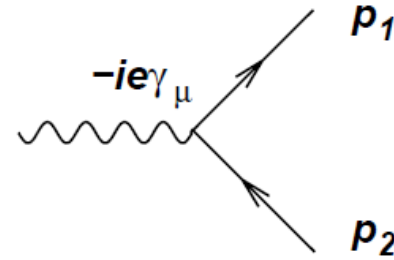
Examples

- Multiplicity of gluons **not IRC safe**, modified by soft/collinear splitting
- Energy of hardest particle **not IRC safe**, modified by collinear splitting
- Energy flow into a cone **is IRC safe**, soft emissions don't change energy flow and collinear emissions don't change its direction

e^+e^- : soft-collinear gluon amplitude

► At leading-order (LO):

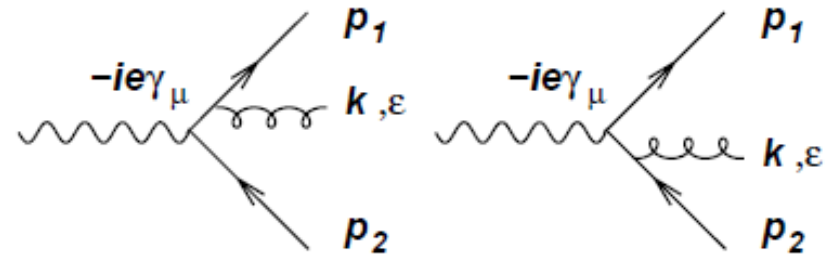
$$M_{q\bar{q}}^{(0)} = (-ie_q) \bar{u}(p_1) \gamma^\mu v(p_2)$$



► Then emit a gluon

$$M_{q\bar{q}g}^{(0)} = (-ie_q)(ig_s) \mathbf{T}^a \bar{u}(p_1) \left(\not{\epsilon}(k) \frac{i}{\not{p}_1 + \not{k}} \gamma^\mu - \gamma^\mu \frac{i}{\not{p}_2 + \not{k}} \not{\epsilon}(k) \right) v(p_2)$$

Using equation of motion $\not{p}_2 v(p_2) = 0$
 and $\not{p}_2 \not{\epsilon} = 2\epsilon \cdot p_2 - \not{\epsilon} \not{p}_2$
 in the soft ($\not{k} \rightarrow 0$) and
 collinear ($\not{k} v(p_2) \rightarrow 0$) limits



$$(\not{p}_2 + \not{k}) \not{\epsilon}(k) v(p_2) \simeq 2\epsilon \cdot p_2 v(p_2)$$

Then

$$M_{q\bar{q}g}^{(0)} \simeq (-ie_q)(ig_s) \mathbf{T}^a \bar{u}(p_1) \gamma^\mu v(p_2) \left(\frac{p_1 \cdot \epsilon}{p_1 \cdot k} - \frac{p_2 \cdot \epsilon}{p_2 \cdot k} \right)$$

e^+e^- : square amplitude

$$\begin{aligned} |M_{q\bar{q}g}^{(0)}|^2 &\simeq \sum_{a, pol} \left| i g_S \mathbf{T}^a M_{q\bar{q}}^{(0)} \left(\frac{p_1 \cdot \varepsilon}{p_1 \cdot k} - \frac{p_2 \cdot \varepsilon}{p_2 \cdot k} \right) \right|^2 \\ &= -|M_{q\bar{q}}^{(0)}|^2 g_S^2 C_F \left(\frac{p_1}{p_1 \cdot k} - \frac{p_2}{p_2 \cdot k} \right)^2 = |M_{q\bar{q}}^{(0)}|^2 g_S^2 C_F \frac{2p_1 \cdot p_2}{(p_1 \cdot k)(p_2 \cdot k)} \end{aligned}$$

Include phase space

$$d\Phi_{q\bar{q}g} |M_{q\bar{q}g}^{(0)}|^2 \simeq \left(d\Phi_{q\bar{q}} |M_{q\bar{q}}^{(0)}|^2 \right) \frac{d^3k}{2E(2\pi)^3} g_S^2 C_F \frac{2p_1 \cdot p_2}{(p_1 \cdot k)(p_2 \cdot k)}$$

Note **factorization** into **hard** and soft-collinear-gluon emission

e^+e^- : square amplitude

The squared matrix element in terms of energy and angle

$$\frac{2p_1 \cdot p_2}{(p_1 \cdot k)(p_2 \cdot k)} = \frac{4}{E^2(1 - \cos^2 \theta)}$$

- It diverges for $E \rightarrow 0$: **infrared (or soft) emission**
- It diverges for $\theta \rightarrow 0$ and $\theta \rightarrow \pi$: **collinear singularities**

Use **dimensional regularization** to integrate analytically over the soft and collinear region of the phase-space

$$\frac{d^3 k}{2E(2\pi)^3} \rightarrow \frac{d^{d-1} k}{2E(2\pi)^{d-1}} \quad d = 4 - 2\epsilon$$

Leads to poles in $1/\epsilon^2$, $1/\epsilon$, and a finite remainder

Isolating the poles in ϵ

- Slicing method: split phase-space in two regions

$$\begin{aligned}\int_0^1 \frac{f(x)}{x} &\rightarrow \int_0^1 x^{-1+\epsilon} f(x) \simeq f(0) \int_0^w x^{-1+\epsilon} + \int_w^1 \frac{f(x)}{x} \\ &= f(0) \left(\frac{1}{\epsilon} + \log w \right) + \int_w^1 \frac{f(x)}{x}\end{aligned}$$

- Subtraction method: add and subtract back an approximation having the same singular behaviour

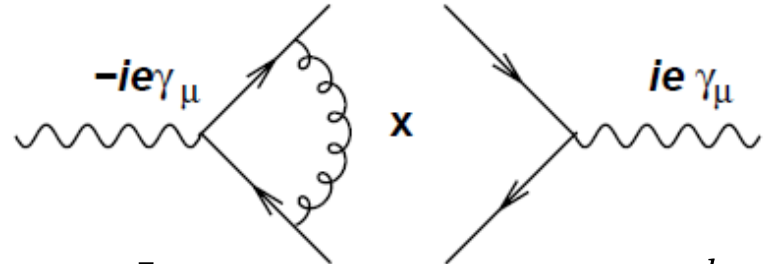
$$\int_0^1 x^{-1+\epsilon} f(x) = f(0) \int_0^1 x^{-1+\epsilon} + \int_0^1 \frac{f(x) - f(0)}{x}$$

e^+e^- : virtual amplitude

► The one-loop amplitude:

$$M_{q\bar{q}}^{(1)} = (-ie_q) g_S^2 C_F \bar{u}(p_1)$$

$$\times \left[\int_q \frac{\gamma^\nu (\not{q} - \not{p}_1) \gamma^\mu (\not{q} + \not{p}_2) \gamma_\nu}{[(q - p_1)^2 + i0][(q + p_2)^2 + i0](q^2 + i0)} \right] v(p_2) \quad \int_q = -i \int \frac{d^d q}{(2\pi)^d}$$



► Set the virtual gluon on-shell $\frac{1}{q^2 + i0} \rightarrow -2\pi i \theta(q_0) \delta(q^2) = -\tilde{\delta}(q)$

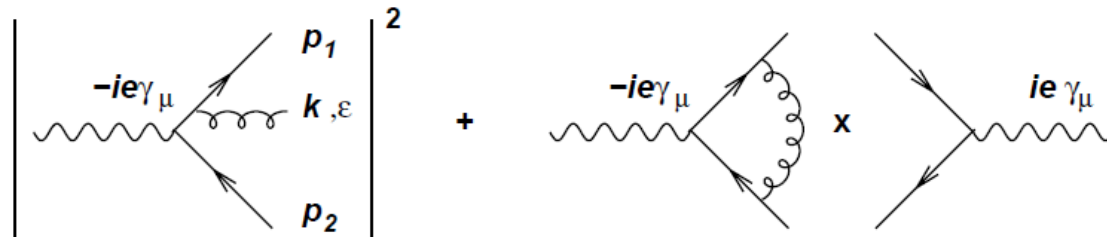
$$(\not{q} + \not{p}_2) \gamma^\nu v(p_2) = [2(q + p_2)^\nu - \gamma^\nu \not{q}] v(p_2)$$

$$M_{q\bar{q}}^{(1)} \simeq -g_S^2 C_F M_{q\bar{q}}^{(0)} \int_q \frac{p_1 \cdot p_2}{(q \cdot p_1)(q \cdot p_2)} \tilde{\delta}(q)$$

Total cross-section must be finite: if real part has poles in $1/\epsilon$, integration of the virtual part should exhibit the same poles of opposite sign (Unitarity, conservation of probability)

e^+e^- : total cross-section

The total cross-section is the sum of all real and virtual diagrams



- Corrections to σ_{tot} come from hard ($E \sim Q$) large-angle gluons, and large virtualities ($q \sim Q$): physics at short-distance
- Soft gluons are emitted on long timescale $\sim 1/(E \theta^2)$ relative to the collision scale ($1/Q$) and cannot influence the cross-section
- Transition to hadrons also occurs on long time scale ($1/\Lambda_{QCD}$) and then is factorized
- Correct renormalization scale for α_S is $\mu \sim Q$

Anatomy of a fixed order calculation

leading order (LO)

$$\sigma^{\text{LO}} = \int d\Phi_n(\{p_i\}) \times |M_n^{(0)}(\{p_i\})|^2 \times F_n(\{p_i\})$$

phase-space:
multidimensional
integral

tree-level Feynman graphs, can be
obtained by analytical/numerical
methods

selection cuts +
observable dependent
function

computable numerically by using e.g. MC methods
practical limitation: $m_{\text{max}} \sim 10$ at present

$$d\Phi_n(\{p_i\}) = \frac{1}{2s} \times \left(\prod \frac{1}{(2\pi)^{d-1}} \frac{d^{d-1}p_i}{2E_i} \right) \underbrace{\delta^{(d)}(p_1 + p_2 - \sum p_i)}_{\text{Momentum conservation}}$$

Initial state flux

Momentum conservation

Anatomy of a fixed order calculation

leading order (LO)

$$\sigma^{\text{LO}} = \int d\Phi_n(\{p_i\}) \times |M_n^{(0)}(\{p_i\})|^2 \times F_n(\{p_i\})$$

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next-to-leading order (NLO)

$$\sigma^{\text{NLO}} = \int_n d\sigma^{\text{V}} + \int_{n+1} d\sigma^{\text{R}}$$

virtual contribution

real radiation

new feature wrt LO:
combine n with $n+1$

A) real radiation

$$\int_{n+1} d\sigma^{\text{R}} = \int d\Phi_{n+1}(\{p_i\}) \times \underbrace{|M_{n+1}^{(0)}(\{p_i\})|^2 \times F_{n+1}(\{p_i\})}_{\text{split phase-space integrand in two parts}}$$

several well known/tested working methods (subtraction, dipole, slicing, mixed, ...)

split phase-space integrand in two parts

$(\dots)_{\text{div}} + (\dots)_{\text{fin}}$

- IR singular: analytically computable up to $O(\epsilon)$
- IR finite: computable numerically as LO
- cancels with virtual

IR finite: computable numerically as LO

B) virtual contribution

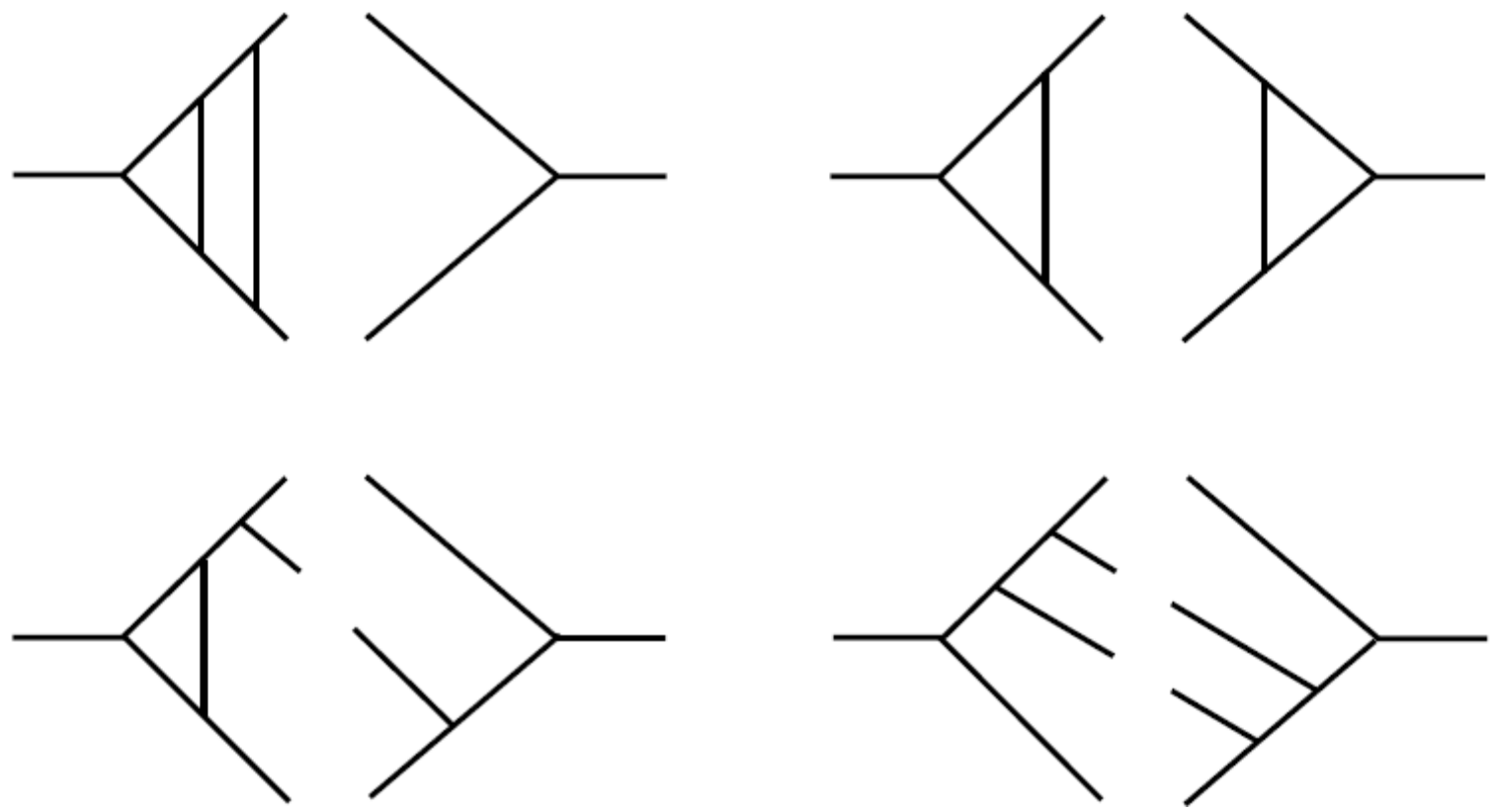
$$\int_n d\sigma^{\text{V}} = \int d\Phi_n(\{p_i\}) \times 2\text{Re}\langle M_n^{(0)}(\{p_i\}) | \underbrace{\int d^d q M_n^{(1)}(\{q, p_i\})}_{\text{loop integral}} \rangle \times F_n(\{p_i\})$$

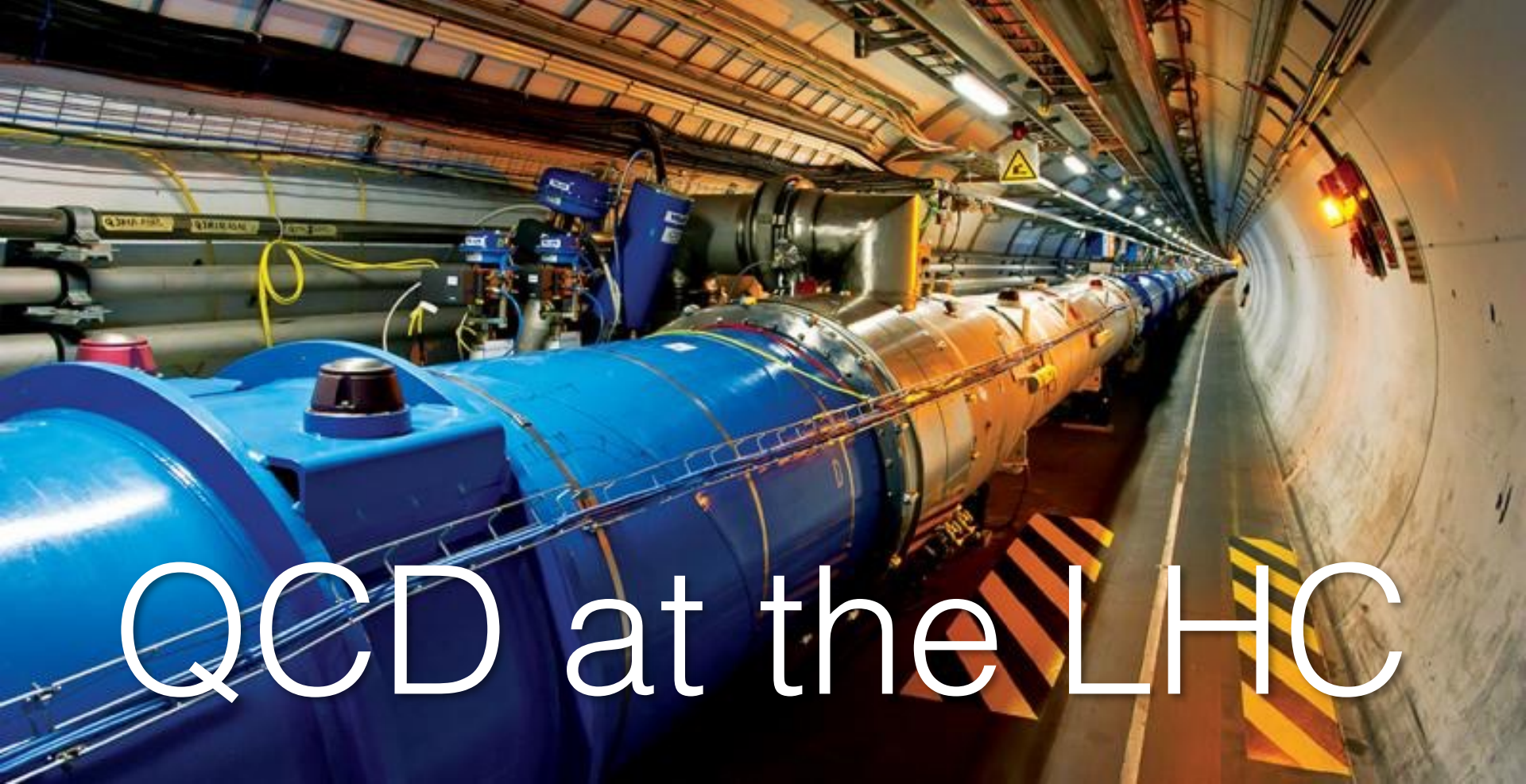
loop integral: in multiparton processes ($m \geq 5$) was regarded as main bottleneck

- hard to get in analytic form
- Feynman parametrization costs one extra Feynman parameter per extra parton
- numerical methods/reduction formalism (pentagons, hexagons \rightarrow boxes) have to eliminate/control numerical instabilities (many terms, Gram determinants)
- Many new developments in recent years

NNLO ingredients

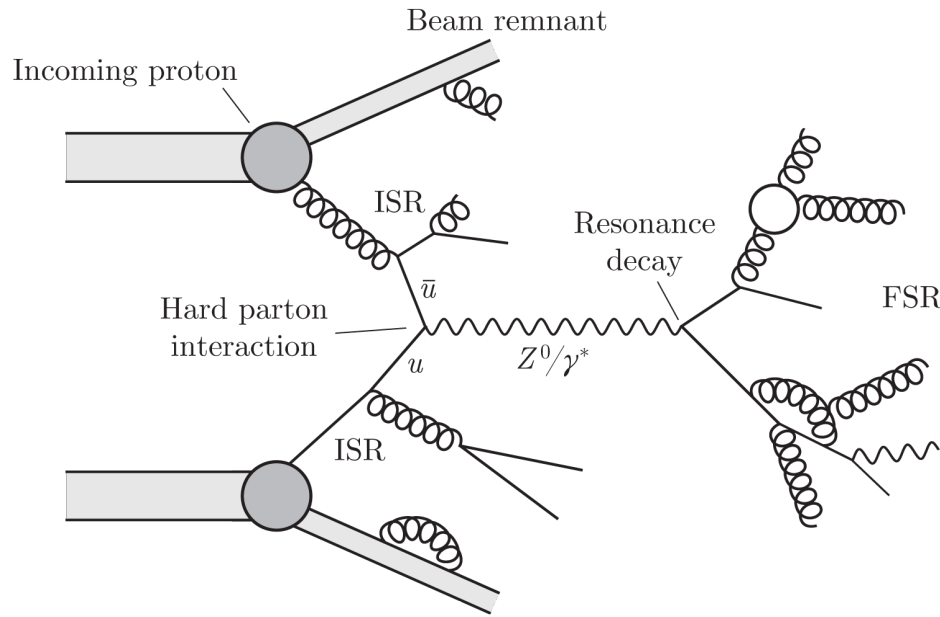
$$\sigma^{\text{NNLO}} = \int_m d\sigma^{\text{VV}} + \int_{m+1} d\sigma^{\text{VR}} + \int_{m+2} d\sigma^{\text{RR}}$$





QCD at the LHC

Factorization in hadronic collisions



- Factorize physics into **long distance** (hadronic $\sim M_{\text{had}}$), and **short distance** (partonic $Q \gg M_{\text{had}}$),
- factorization violation is power suppressed $\sim \mathcal{O}(M_{\text{had}}/Q)^q$

$$\sigma = \sum_{a,b} \int dx_1 dx_2 f_a(x_1, \mu_F) f_b(x_2, \mu_F) \hat{\sigma}_{ab}(x_1 p_A, x_2 p_B; \mu_F, \mu_R) + \mathcal{O}\left(\frac{1}{Q}\right)$$

Parton densities PDF (green text) points to $f_a(x_1, \mu_F)$ and $f_b(x_2, \mu_F)$.
 Hard scattering cross-section (brown text) points to $\hat{\sigma}_{ab}(x_1 p_A, x_2 p_B; \mu_F, \mu_R)$.
 Factorization and renormalization scales (green text) points to μ_F, μ_R .
 Partonic cms energy $\hat{s} = x_1 x_2 s$ (green text) points to $x_1 p_A, x_2 p_B$.
 Higher twist (brown text) points to $\mathcal{O}\left(\frac{1}{Q}\right)$.

Collinear factorization

theorem proven for

sufficiently inclusive

observables in the final state
of the scattering of colorless
hadrons

[Collins, Soper, Sterman]

- Often assumed that partonic scattering amplitudes factorize: **fixed order and resummations**
- **Monte Carlo** event generators are based on factorization
- In neither of these cases factorization is guaranteed.

pQCD for hard-scattering
processes based on
universality:

- the sole uncanceled IR divergences are due to partonic states whose momenta are collinear to the collider partons
- removed by redefinition of bare parton densities

The LHC is a hadronic machine working at higher energies than ever before

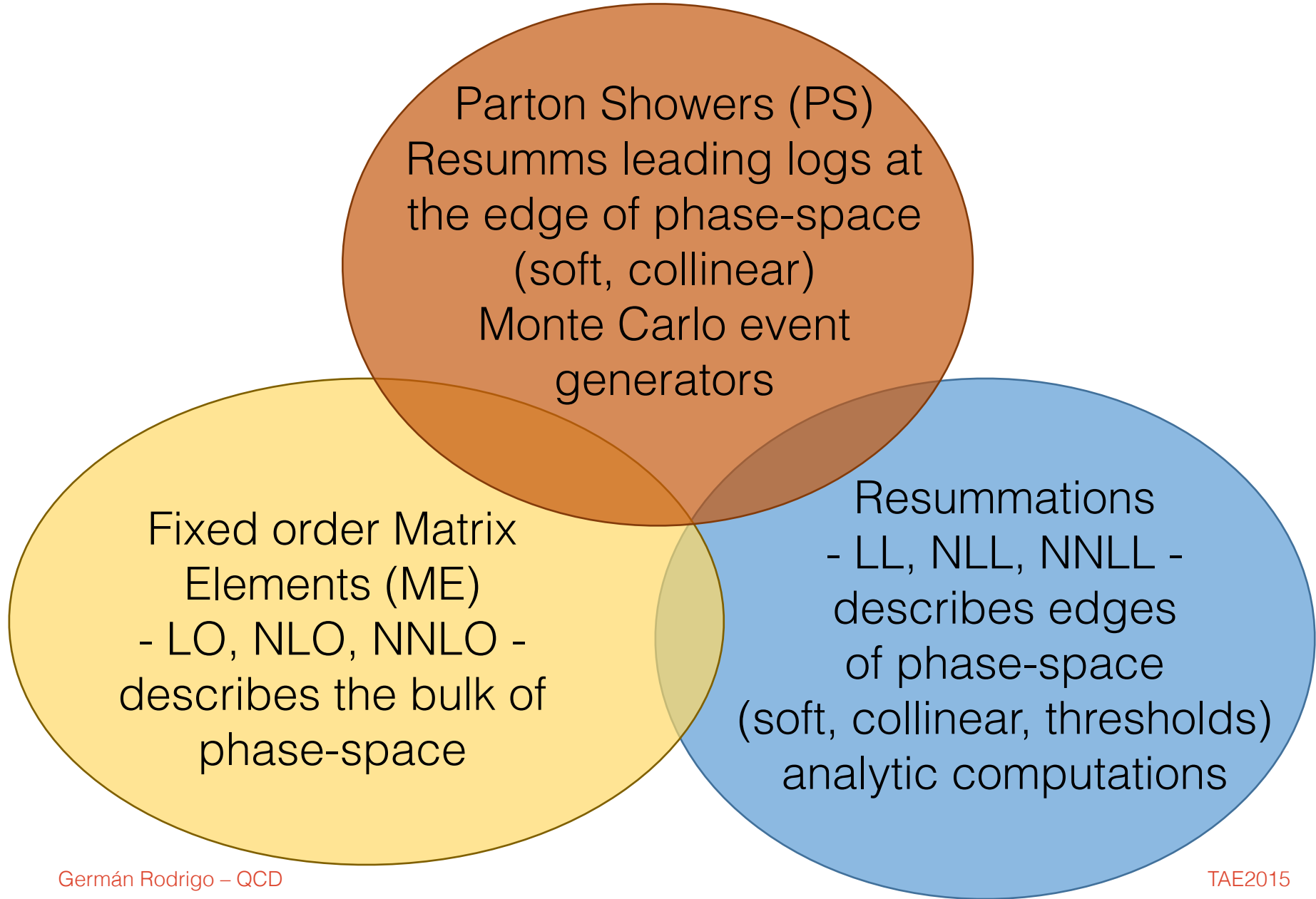
- larger phase-space for hard radiation
- higher multiplicities (external legs)
 - more powers of α_s
 - multi-particle final states are the signal for new physics
 - multi-scale processes: logs of the ratio of very different scales
- proton is not elementary:
 - need to know PDF accurately
 - new channels open at higher orders in pQCD



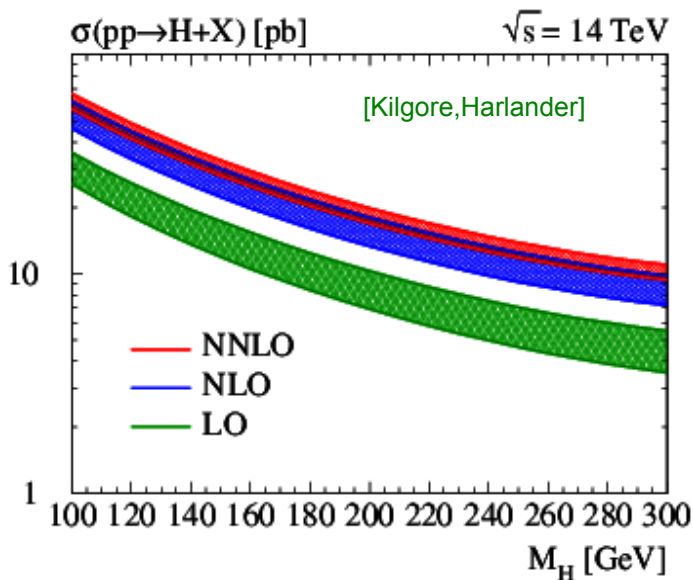
Huge radiative corrections

The absence so far of a clear signal BSM makes even more relevant the role of precision physics

The path to precision



Perturbative view: higher orders improve systematically the precision of the theoretical predictions (estimated by varying the renormalization/factorization scales) for background and signal



- LO: fails to describe normalization (up to a factor 2). Monte Carlo event generators (LO + parton showers) : improves the shape of distributions, but normalization still underestimated
- NLO: first reliable estimate of central value
- NNLO: first serious estimate of the theoretical error