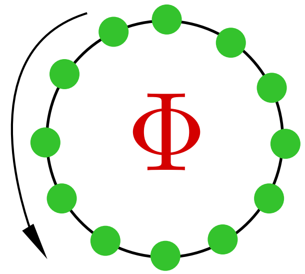
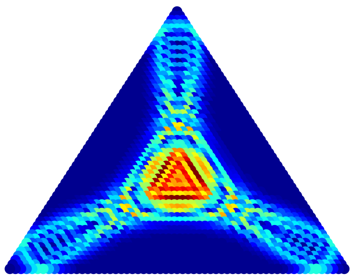
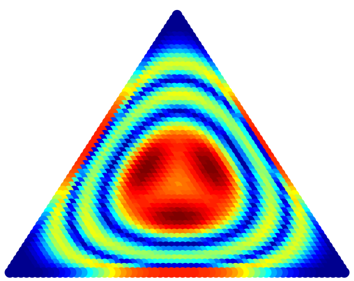
Superfluidity and Chaos in low dimensional circuits

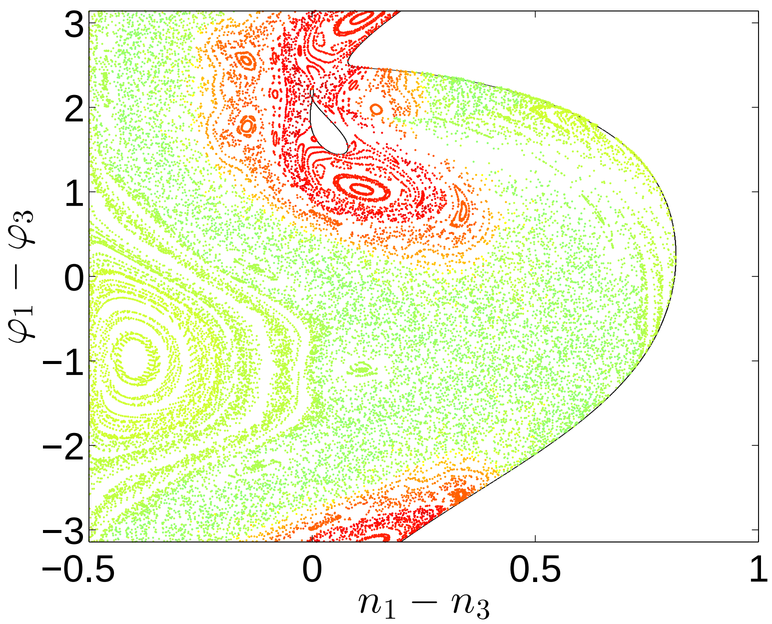
The qchaos group, Ben-Gurion University

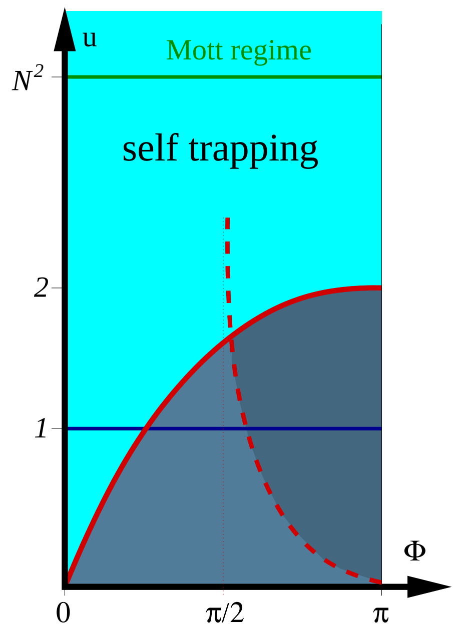
[**1**] Geva Arwas, Amichay Vardi, Doron Cohen [**PRA 2014**]
[**2**] Geva Arwas, Amichay Vardi, Doron Cohen [**arXiv 2014**]
[**3**] Additional collaborations (see next page)

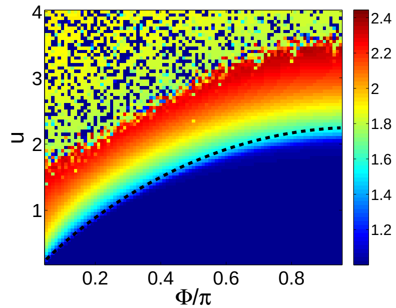


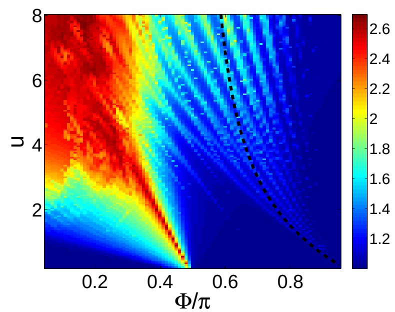












BHH - dimers and trimers

The Bose-Hubbard Hamiltonian (BHH):

$$\mathcal{H}_{\text{BHH}} = \frac{U}{2}\sum_{j=1}^{M} a_j^{\dagger}a_j^{\dagger}a_j a_j - \frac{K}{2}\sum_{j=1}^{M}\left(a_{j+1}^{\dagger}a_j + a_j^{\dagger}a_{j+1}\right) \qquad u \equiv \frac{NU}{K}$$

Dimer (M=2): minimal BHH; Bosonic Josephson junction; Pendulum physics [1,5].

Driven dimer: Landau-Zener dynamics [2], Kapitza effect [3], Zeno effect [4], Standard-map physics [5].

Trimer (M=3): minimal model for chaos; Coupled pendula physics.

Triangular trimer: minimal model with topology, Superfluidity [6], Stirring [7].

Coupled trimers: minimal model for mesoscopic thermalization [8,9].

[1] Chuchem, Smith-Mannschott, Hiller, Kottos, Vardi, Cohen (PRA 2010).

[2] Smith-Mannschott, Chuchem, Hiller, Kottos, Cohen (PRL 2009).

[3] Boukobza, Moore, Cohen, Vardi (PRL 2010).

[4] Khripkov, Vardi, Cohen (PRA 2012)

[5] Khripkov, Cohen, Vardi (JPA 2013, PRE 2013).

[6] Geva Arwas, Vardi, Cohen (PRA 2014).

[7] Hiller, Kottos, Cohen (EPL 2008, PRA 2008).

[8] Tikhonenkov, Vardi, Anglin, Cohen, PRL (2013).

[9] Christine Khripkov, Vardi, Cohen, NJP (2015).

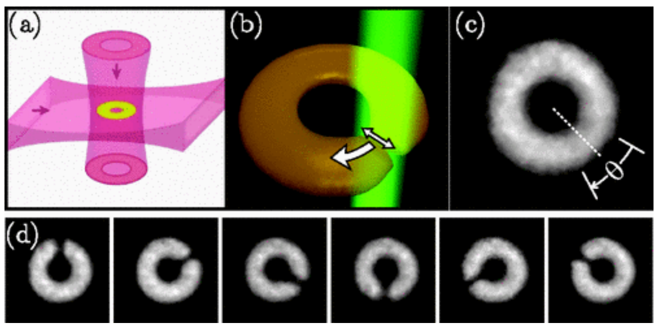
Scope

- The recent experimental realization of confining potentials with toroidal shapes [1] has opened a new arena of studying superfluidity in low dimensional circuits. In particular a discrete ring has been realized [2].

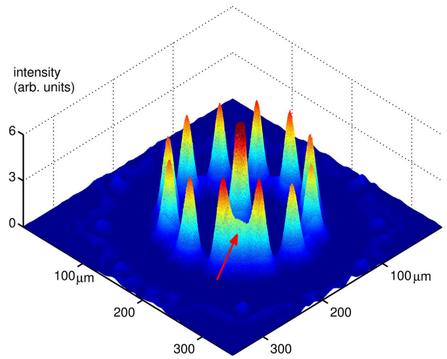
$$\mathcal{H}_{\mathrm{BHH}} = \frac{U}{2}\sum_{j=1}^{M} a_j^{\dagger} a_j^{\dagger} a_j a_j - \frac{K}{2}\sum_{j=1}^{M}\left(a_{j+1}^{\dagger} a_j + a_j^{\dagger} a_{j+1}\right)$$

- The hallmark of superfluidity is a metastable non-equilibrium steady-state current.
- The traditional paradigm is based on the Landau criterion and the BdG stability analysis [3-5].
- We challenge the traditional paradigm and highlight the role of chaos in the analysis.

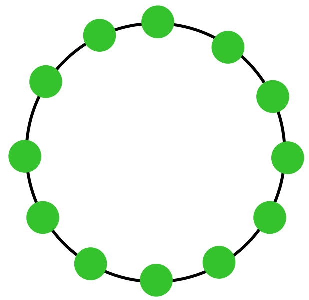
Torodial Ring [1]



Discrete ring [2]



Bode-Hubbard Model

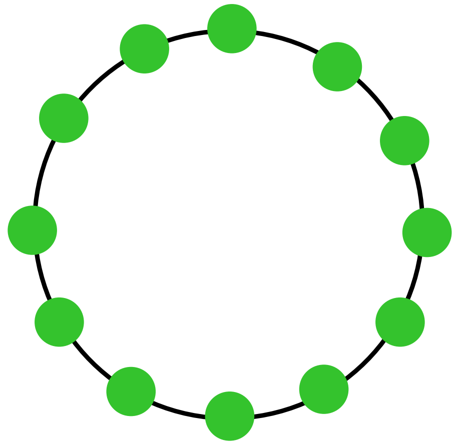


[1] K.C. Wright, R.B. Blakestad, C.J. Lobb, W.D. Phillips, G.K. Campbell, Phys. Rev. Lett. 110, 025302 (2013).
[2] L. Amico, D. Aghamalyan, F. Auksztol, H. Crepaz, R. Dumke, L.C. Kwek, Sci. Rep. 4, 4298 (2014).
[3] B. Wu and Q. Niu, New J. Phys. 5, 104 (2003).
[4] A. Smerzi, A. Trombettoni, P.G. Kevrekidis, A.R. Bishop, Phys. Rev. Lett. 89, 170402 (2002).
[5] F.S. Cataliotti, L. Fallani, F. Ferlaino, C. Fort, P. Maddaloni, M. Inguscio, New J. Phys. 5, 71 (2003).

The Model (non-rotating ring)

A Bose-Hubbard system with M sites and N bosons:

$$\mathcal{H} \;=\; \sum_{j=1}^{M}\left[\frac{U}{2}a_j^\dagger a_j^\dagger a_j a_j - \frac{K}{2}\left(a_{j+1}^\dagger a_j + a_j^\dagger a_{j+1}\right)\right]$$



In a semi-classical framework:

$$a_j \;=\; \sqrt{\boldsymbol{n}_j}\,\mathrm{e}^{i\boldsymbol{\varphi}_j} \quad , \quad [\boldsymbol{\varphi}_j, \boldsymbol{n}_i] \;=\; i\delta_{ij}$$

$$z \;=\; (\boldsymbol{\varphi}_1, \cdots, \boldsymbol{\varphi}_M,\; \boldsymbol{n}_1, \cdots, \boldsymbol{n}_M)$$

This is like M coupled oscillators with $\mathcal{H} = H(z)$

$$H(z) \;=\; \sum_{j=1}^{M}\left[\frac{U}{2}\boldsymbol{n}_j^2 - K\sqrt{\boldsymbol{n}_{j+1}\boldsymbol{n}_j}\cos\left(\boldsymbol{\varphi}_{j+1} - \boldsymbol{\varphi}_j\right)\right]$$

The dynamics is generated by the Hamilton equation:

$$\dot{z} \;=\; \mathbb{J}\boldsymbol{\partial}H \quad , \quad \mathbb{J} \;=\; \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}$$

(DNLS)

Classically there is a single dimensionless parameter:

$$u \;=\; \frac{NU}{K}$$

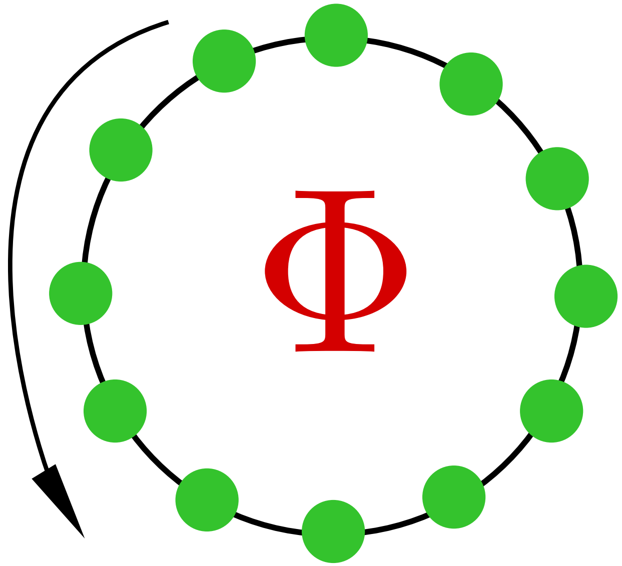
Rescaling coordinates:

$$\tilde{\boldsymbol{n}} \;=\; \boldsymbol{n}/N$$

$$[\boldsymbol{\varphi}_j, \tilde{\boldsymbol{n}}_i] \;=\; i\hbar\delta_{ij}$$

$$\hbar \;=\; \frac{1}{N}$$

The model (rotating ring)



In the rotating reference frame we have a Coriolis force,
which is like magntic field $\mathcal{B} = 2\mathsf{m}\Omega$. Hence is is like having flux

$$\Phi \;=\; 2\pi R^2 \mathsf{m}\,\Omega \;=\; \frac{M^2}{2\pi}\left(\frac{\mathsf{m}}{\mathsf{m}_{\text{eff}}}\right)\frac{\Omega}{K}$$

Note: there are optional experimental realizations.

$$\mathcal{H} \;=\; \sum_{j=1}^{M}\left[\frac{U}{2}a_j^{\dagger}a_j^{\dagger}a_j a_j - \frac{K}{2}\left(\mathrm{e}^{i(\Phi/M)}a_{j+1}^{\dagger}a_j + \mathrm{e}^{-i(\Phi/M)}a_j^{\dagger}a_{j+1}\right)\right]$$

Summary of model parameters:

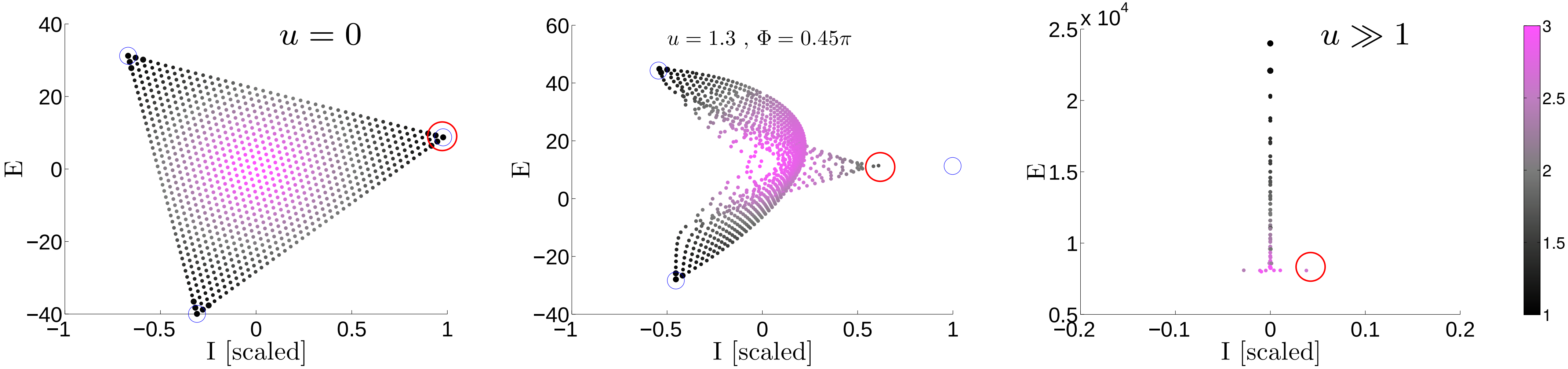
The "classical" dimensionless parameters of the DNLS are u and Φ.
The mumber of particles N is the "quantum" paramater.
The system has effectively $d = M-1$ degrees of freedom.

$M = 2$	Bosonic Josephson junction (Integrable)
$M = 3$	Minimal circuit (mixed chaotic phase-space)
$M \geq 4$	High dimensional chaos (Arnold diffusion)
$M \to \infty$	Continuous ring (Integrable)

The many-body spectrum

We characterize each eigenstate $|\alpha\rangle$ of the BHH by $(\mathcal{I}_\alpha, E_\alpha)$ and colorcode by $\mathcal{M}_\alpha$

The expected location of a vortex state, and the maximum current state, are encircled by ◯ and ◯



$$|m\rangle \;=\; \left(\tilde{a}_m^\dagger\right)^N |0\rangle \qquad m = 1...M$$

$$\mathcal{I}_m \;=\; N \times \left(\frac{K}{M}\right) \sin\left(\frac{1}{M}(2\pi m - \Phi)\right)$$

$$\mathcal{I}_\alpha \;\equiv\; -\left\langle \frac{\partial \mathcal{H}}{\partial \Phi} \right\rangle_\alpha$$

$$\rho_{ij} \;\equiv\; \frac{1}{N}\langle a_j^\dagger a_i \rangle_\alpha \;=\; \text{reduced probability matrix}$$

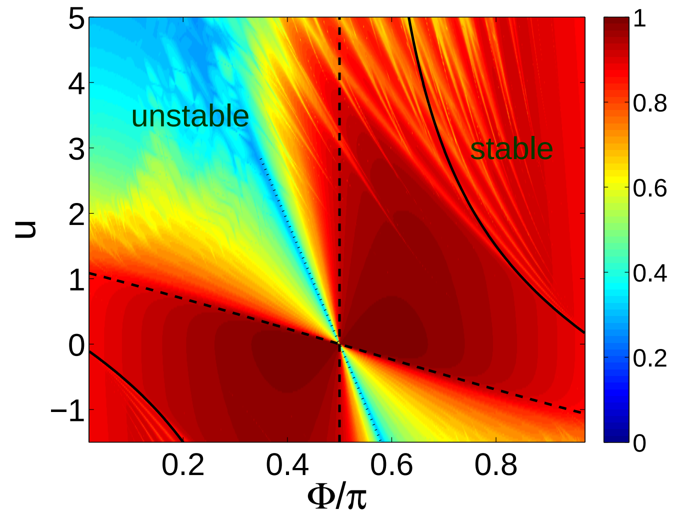
$$\mathcal{M}_\alpha \;\equiv\; [\text{trace}(\rho^2)]^{-1} \;\in\; [1, M]$$

$\mathcal{M}_\alpha = 1$ for coherent state (condensation).

$\mathcal{M}_\alpha \sim M$ for maximally fragmented or chaotic state.

Constructing the regime diagram:

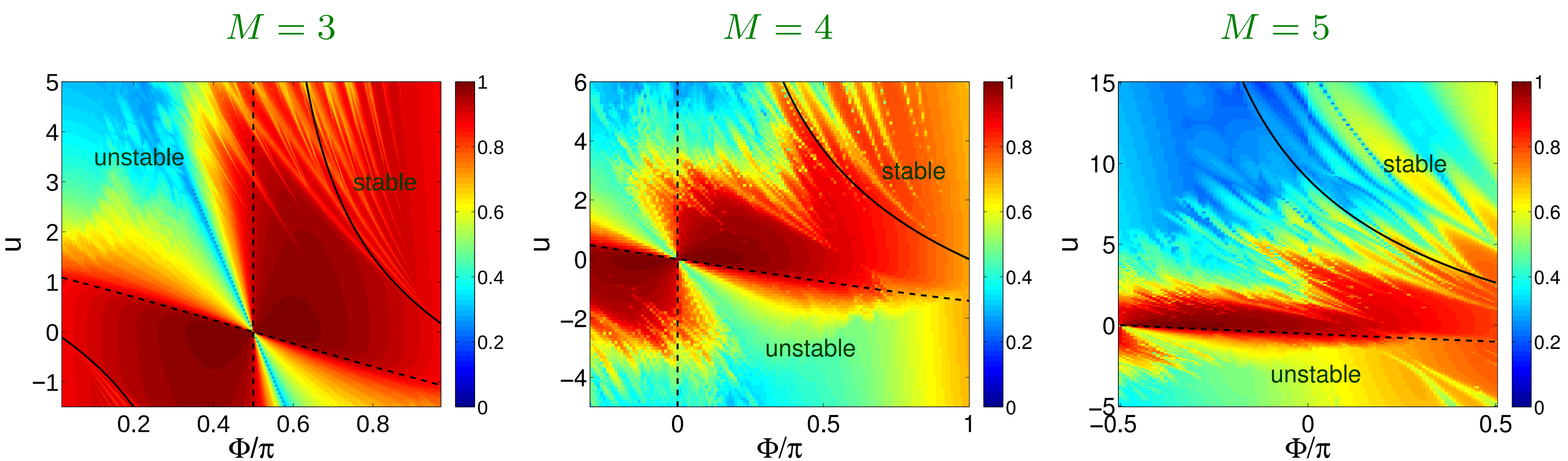
For every (Φ, u) value we plot $\max\{I_\alpha\}$



Regime diagram

The I of the maximum current state is imaged as a function of (Φ, u)

solid lines = spectral stability borders (Landau); dashed lines = dynamical stability borders (BdG)



The traditional paradigm associates vortex states with stationary fixed-points in phase space. Consequently the Landau criterion, and more generally the Bogoliubov de Gennes linear-stability-analysis, are conventionally used to determine the viability of superfluidity.

- We challenge the application of the traditional paradigm to low-dimensional circuits.
- We highlight the role of chaos in the “stability analysis”.
- We identify novel types of states that can support superfluidity.

Stability analysis of the excited vortex state

The dynamics is generated by the Hamilton equation: $\dot{z} = \mathbb{J}\boldsymbol{\partial}H(z)$ (DNLS)

Coherent states are supported by fixed-points of the classical Hamiltonian: $\boldsymbol{\partial}H(z) = 0$

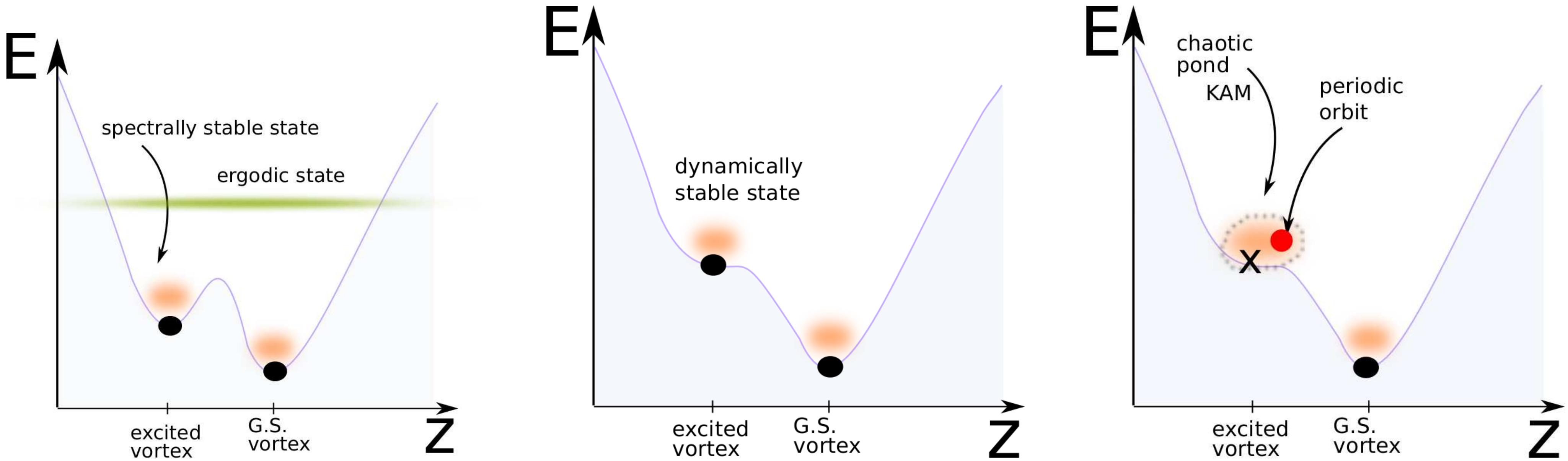
Technical note: The cyclic degree of freedom has to be separated (N is constant of motion).

Linear stability analysis (Bogoliubov de Gennes): $\dot{z} = \mathbb{J}\mathcal{A}z$ where $\mathcal{A}_{\nu,\mu} = \partial_\nu \partial_\mu H$

Spectral stability: Energy local extremal points (Landau criterion) – based on $\mathcal{A}$ diagonalization

Dynamical stability: Zero Lyapunov exponents (real BdG frequencies) – based on $\mathbb{J}\mathcal{A}$ diagonalization

Schematic illustration of the energy landscape $E = H(z)$



Stability of the "ground" vortex state

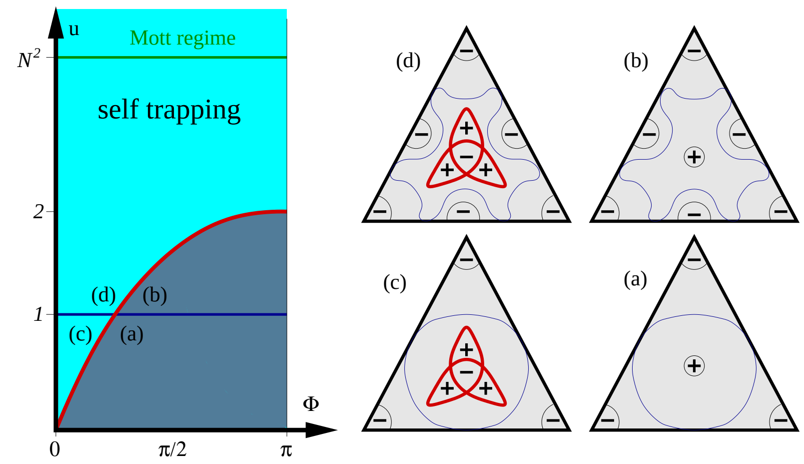
(digression)

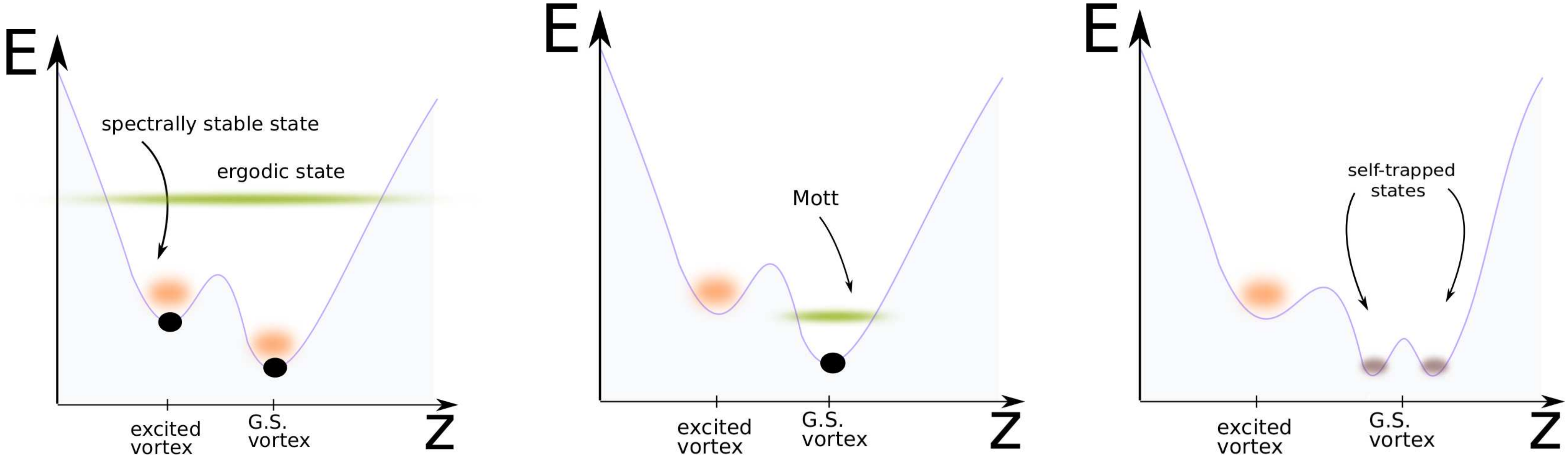
The ground-state vortex can destabilize as well:

Quantum transition: Mott transition for $u > N^2/M$

Classical transition: Self-trapping for $u >$ something

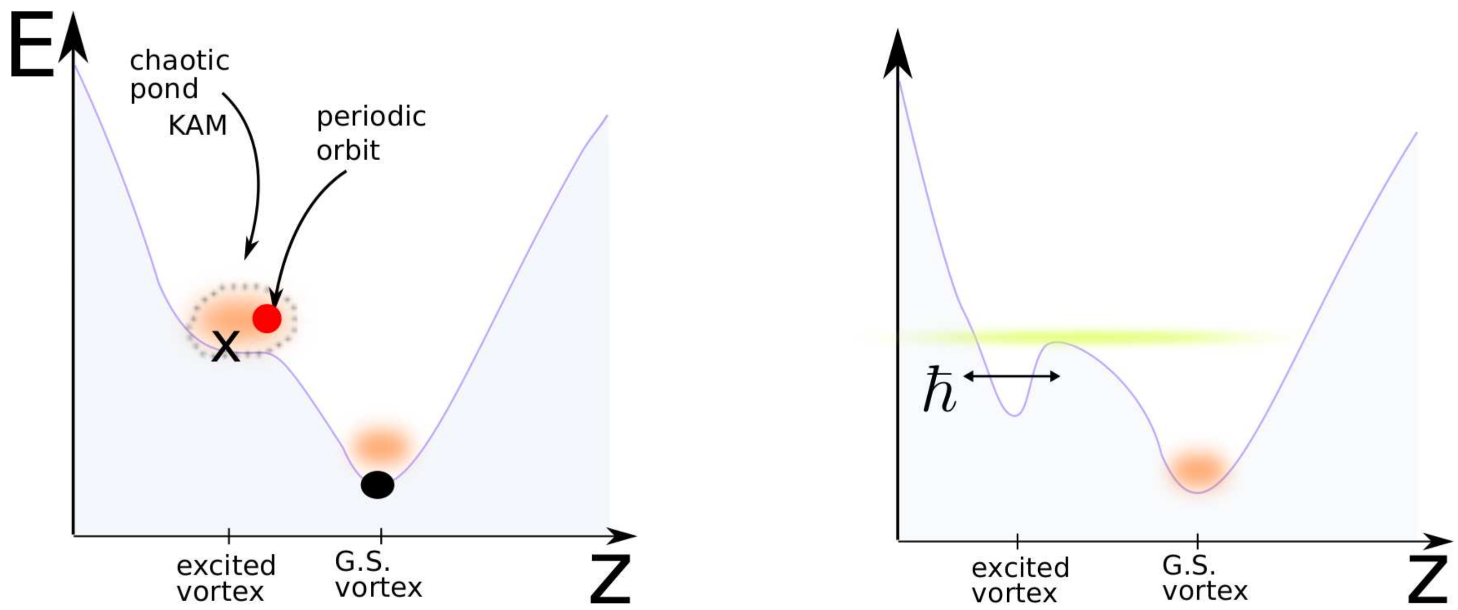
Note: upper state is like ground state for $U \mapsto -U$





Beyond the traditional view

- Dynamical instability of a vortex state does imply that superfluidity is diminished. Kolmogorov-Arnold-Moser (KAM) structures $\rightsquigarrow$ Chaotic and irregular vortex states.
- Dynamical stability of a vortex state does not imply in general strict stability. For $M > 3$ the KAM tori do not block transport (Arnold diffusion).
- One should take into account quantum fluctuations (uncertainty width of a coherent state). Stability is required within a Plank cell around the fixed-point. Regime-diagram is $\hbar$ dependent.



Regime Diagram for $M=3$

A stable vortex state carries current:

$$I_m \;=\; \frac{N}{M} K \,\sin\left(\frac{1}{M}(2\pi m - \Phi)\right)$$

Here: $M=3$; $m=1$; $I_m \sim \frac{N}{M}K$

Spectral stability (solid line):

$$u > \frac{3 - 12\sin^2\left(\frac{\Phi}{3} - \frac{\pi}{6}\right)}{4\sin\left(\frac{\Phi}{3} - \frac{\pi}{6}\right)}$$

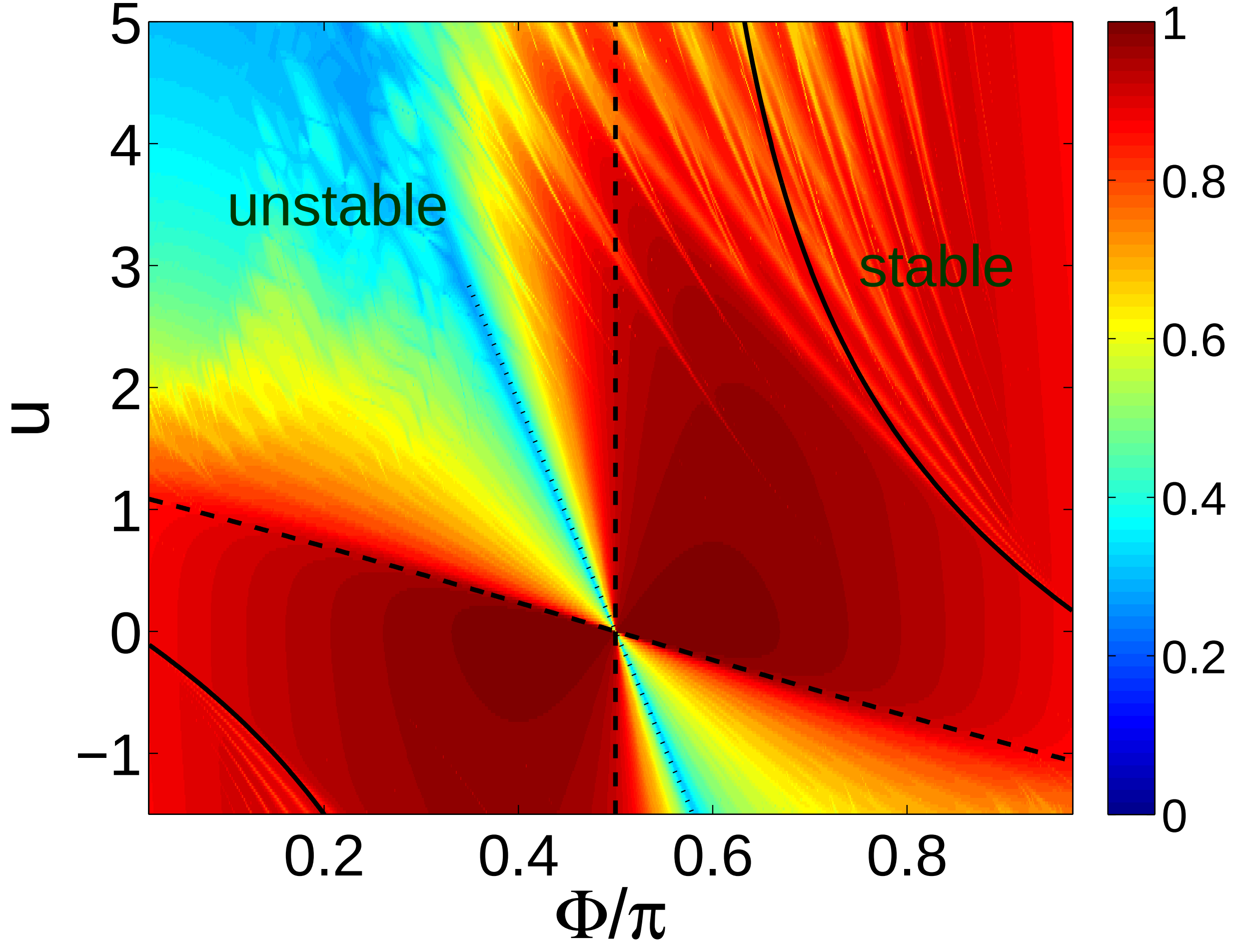
Dynamical instability (dashed line):

$$u > \frac{9}{4}\sin\left(\frac{\pi}{6} - \frac{\Phi}{3}\right) \quad \& \quad \Phi < \frac{\pi}{2}$$

Swap transition (dotted line):

$$u \;=\; 18\sin\left(\frac{\pi}{6} - \frac{\Phi}{3}\right)$$

I of maximal current state:



Spectral vs dynamical stability

Poincare section $n_2 = n_3$ at the vortex energy.

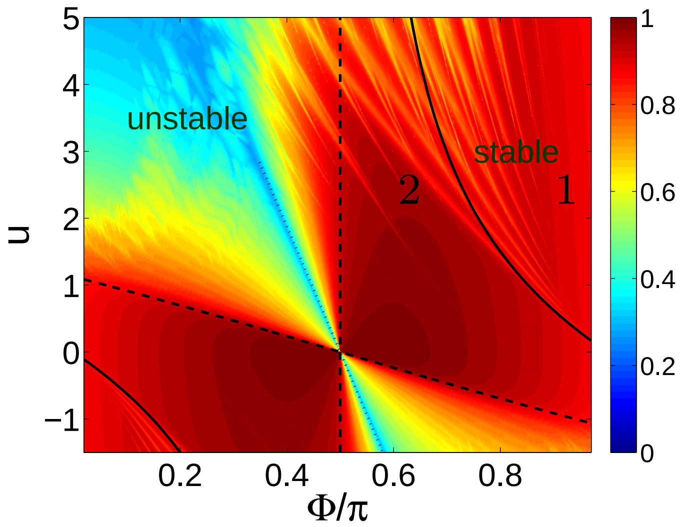
(1) Spectral stability; **(2)** Dynamical stability.

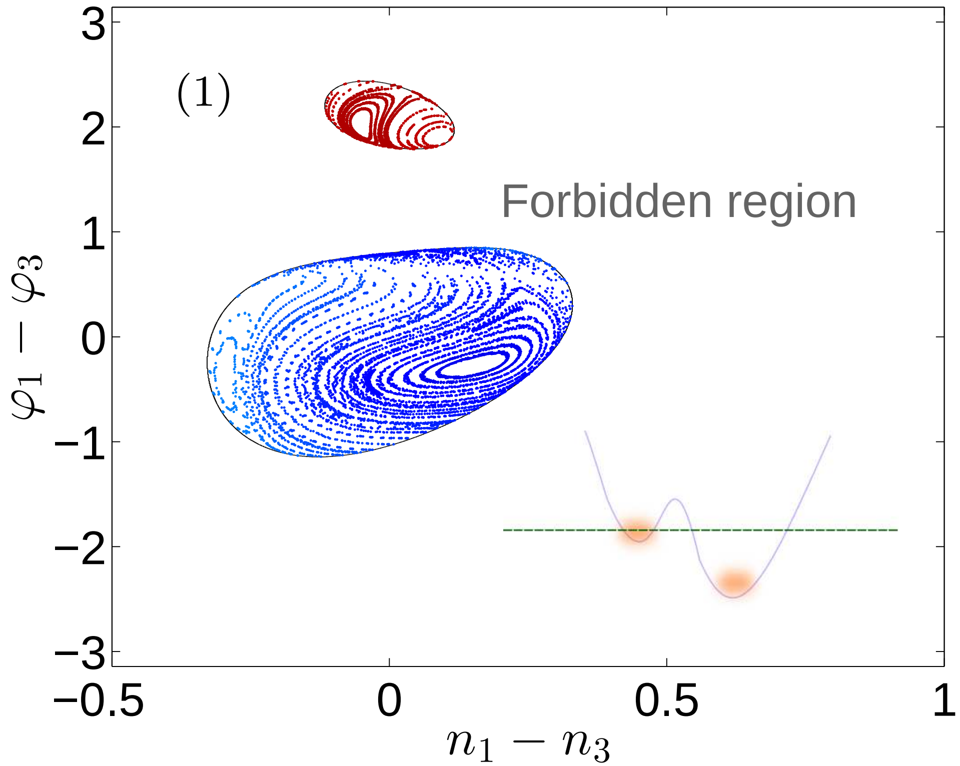
red trajectories = large positive current

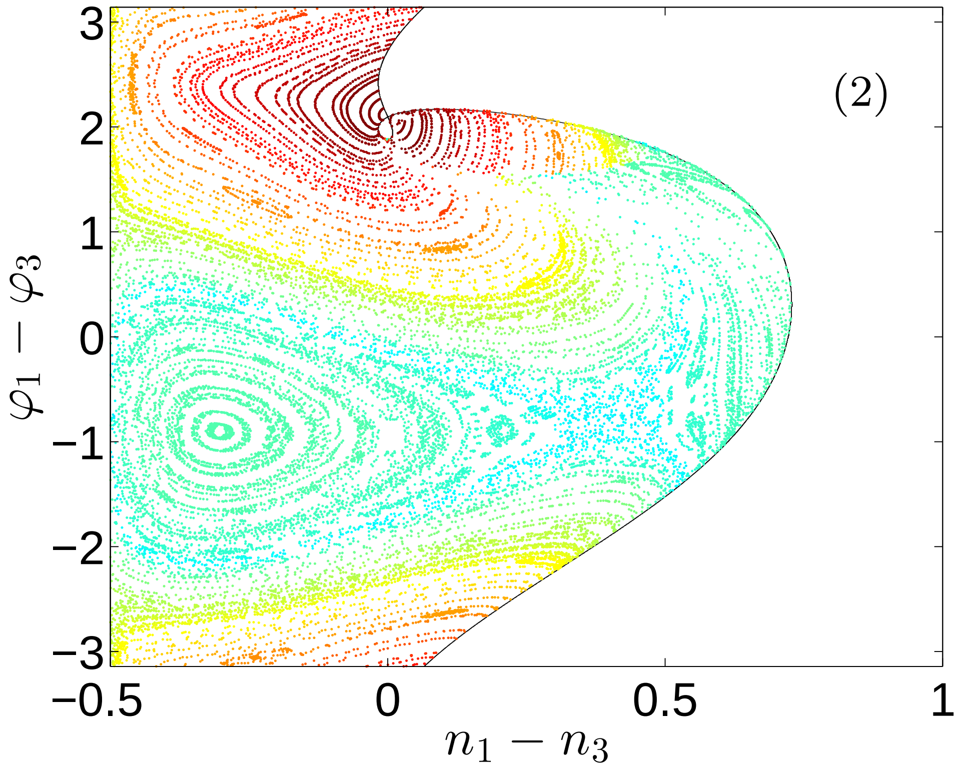
blue trajectories = large negative current

The Vortex fixed-points are located along the symmetry axis:

$$n_1 = n_2 = \cdots = N/M, \qquad \varphi_i - \varphi_{i-1} = \left(\frac{2\pi}{M}\right) m$$

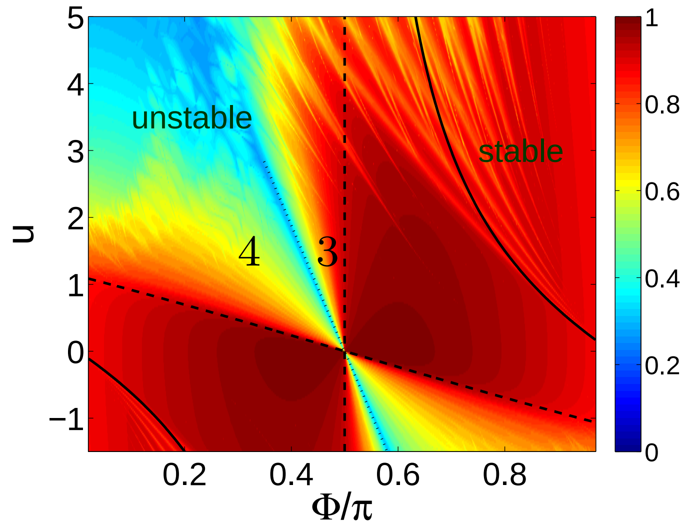






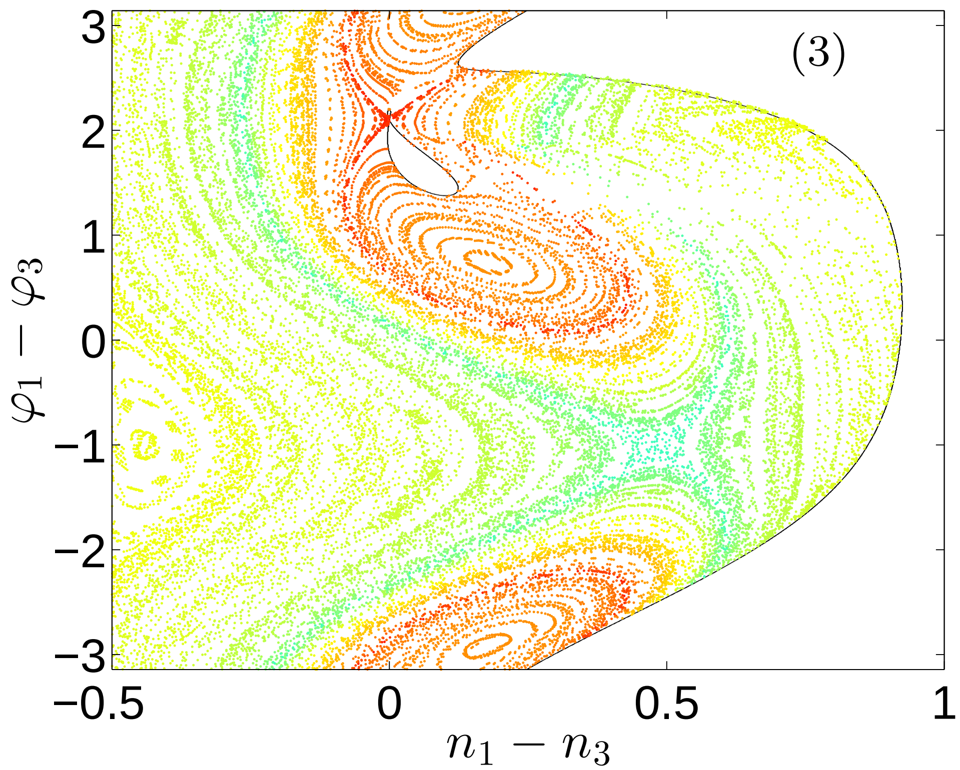
Swap transition

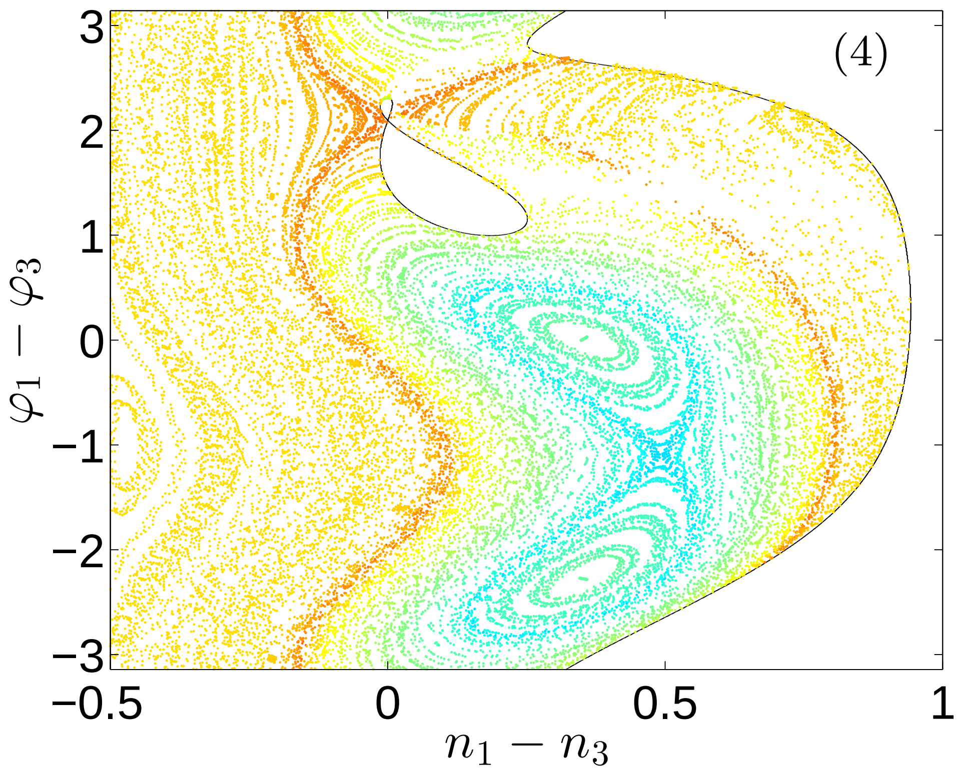
In **(3)** and **(4)** dynamical stability is lost $\rightsquigarrow$ chaotic motion.
But the chaotic trajectory is confined within a chaotic pond;
uni-directional chaotic motion; superfluidity persists!
At the separatrix swap-transition superfluidity diminishes.



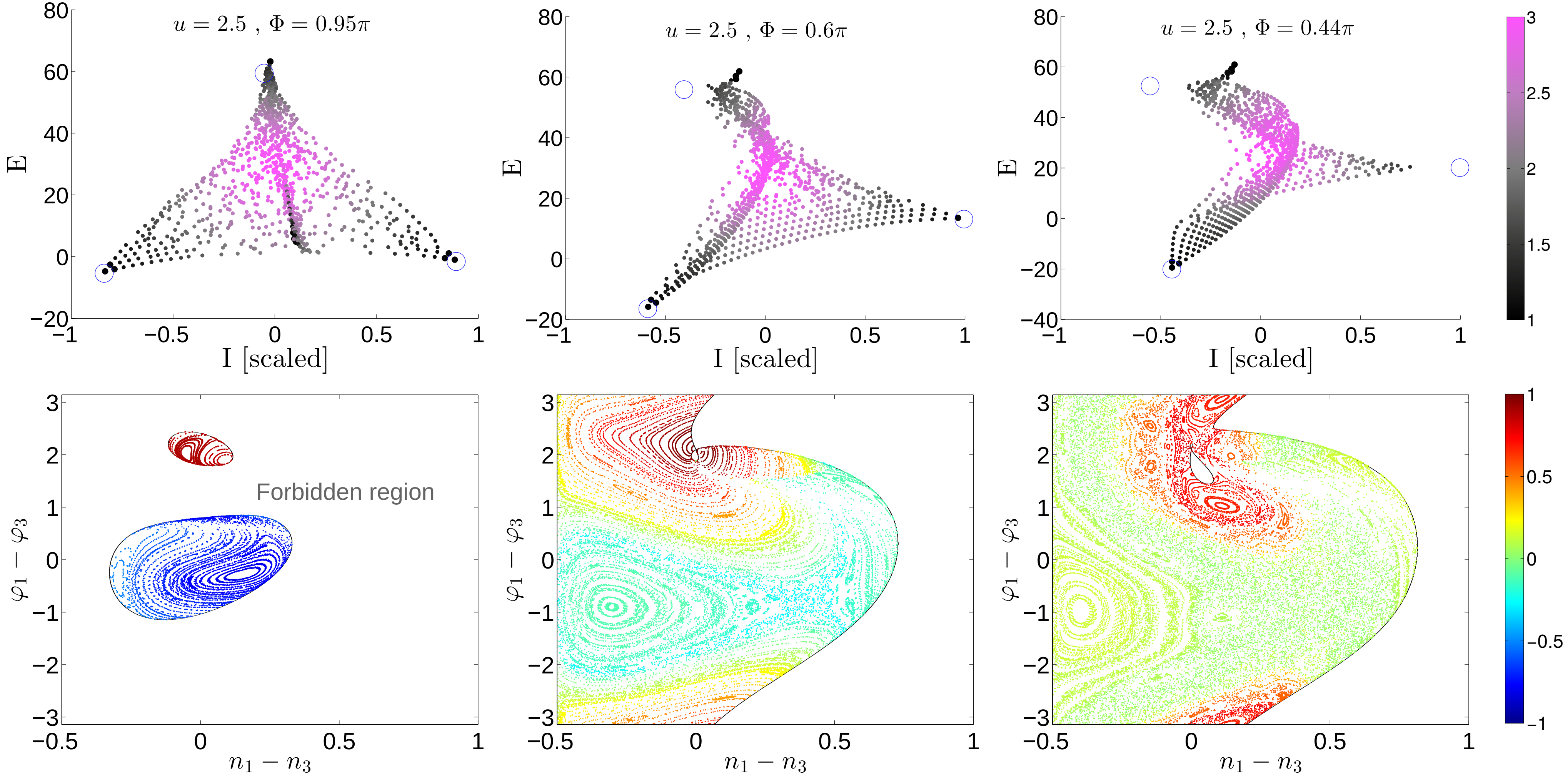
Swap transition (dotted line):

$$u \;=\; 18 \sin\left(\frac{\pi}{6} - \frac{\Phi}{3}\right) \qquad \text{(non-linear resonance)}$$

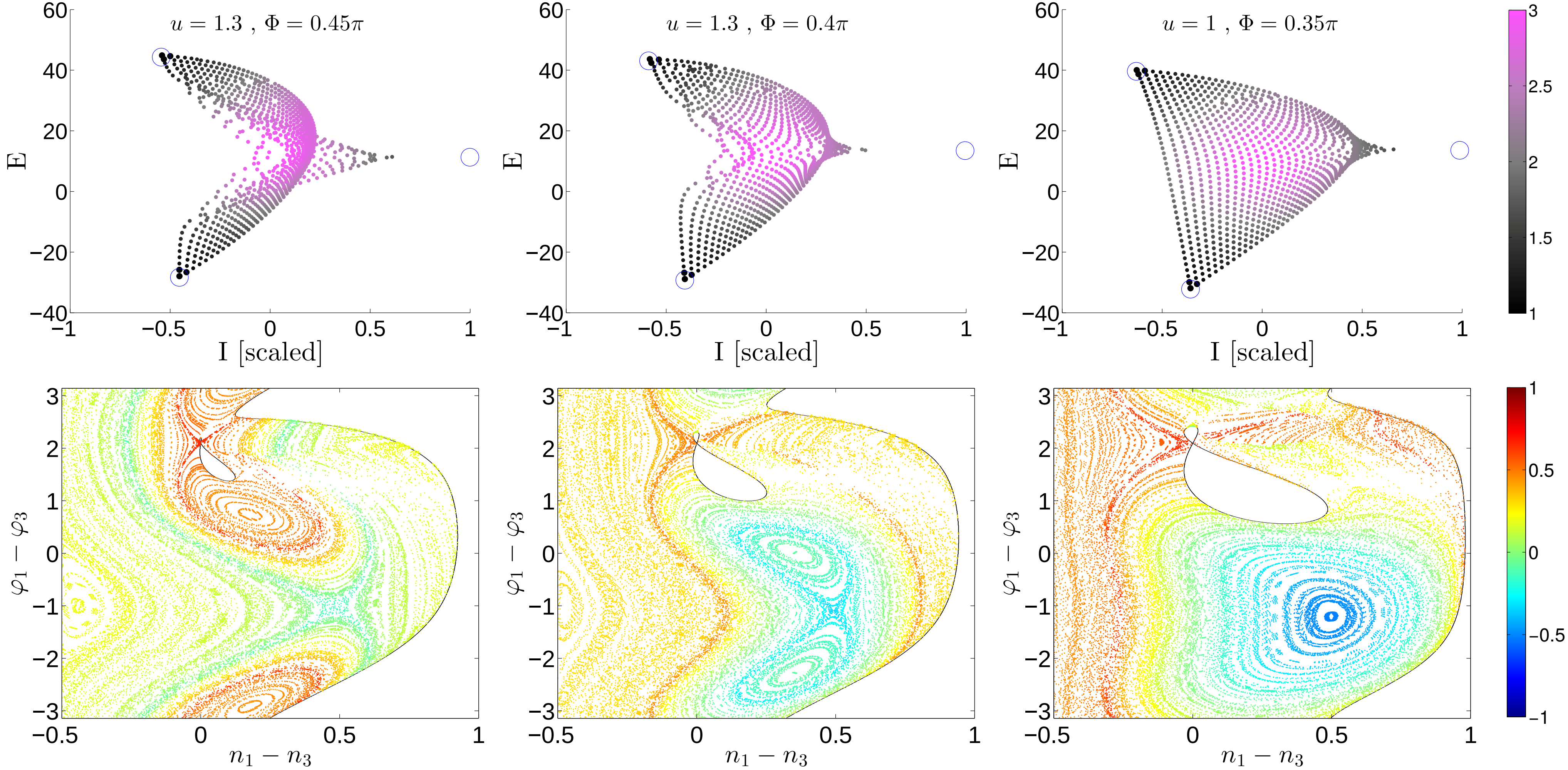




Phase space tomography (I)



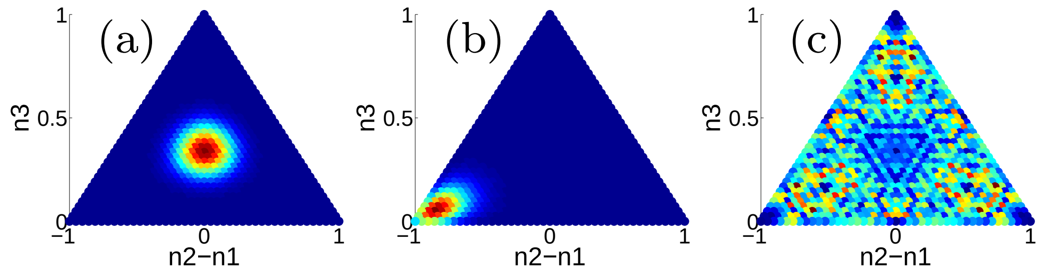
Phase space tomography (II)



Representative Wavefunctions ($M = 3$)

We use standard Fock basis representation.
Images of $|\psi(\boldsymbol{n})|^2 = |\langle \boldsymbol{n}|E_\alpha\rangle|^2$

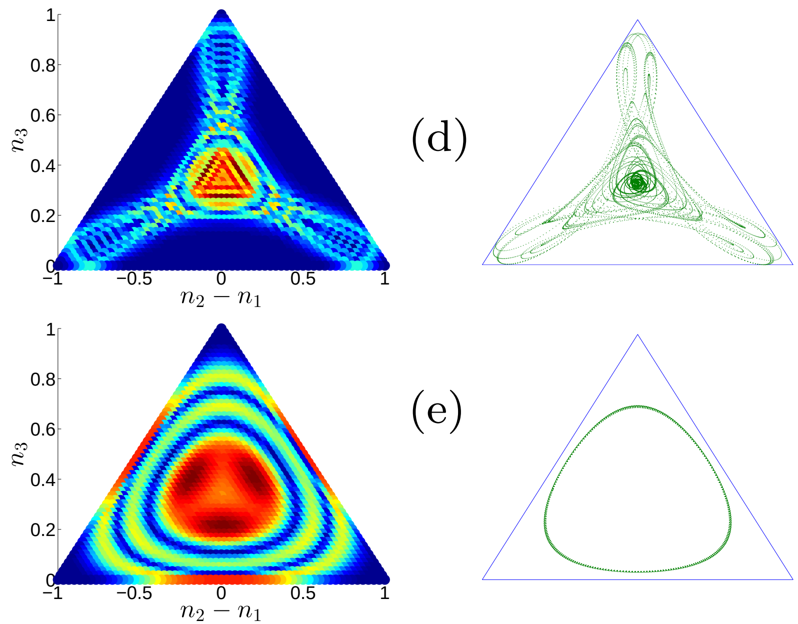
(a) Regular coherent vortex state.
(b) Self-trapped state ("bright soliton").
(c) Typical state in the chaotic sea.



Launching trajectories at the vicinity of the vortex fixed-point we encounter 3 possibilities.

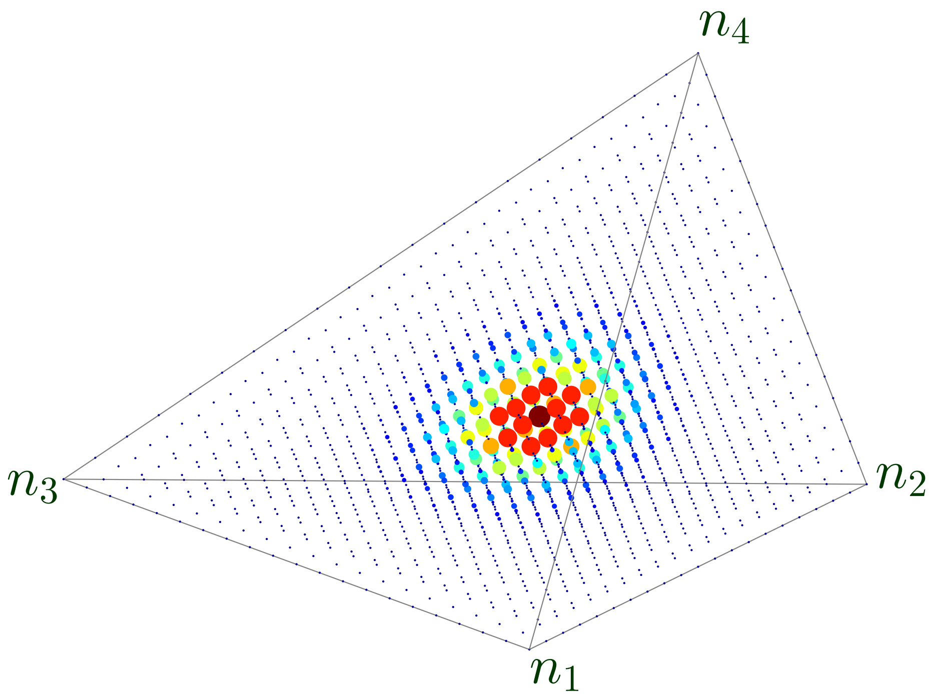
A trajectories might be:

- locked at the vortex fixed point
  (regular vortex state (a))
- chaotic but unidirectional
  (chaotic vortex state (d))
- quasi-periodic in phase-space
  (breathing vortex state (e))

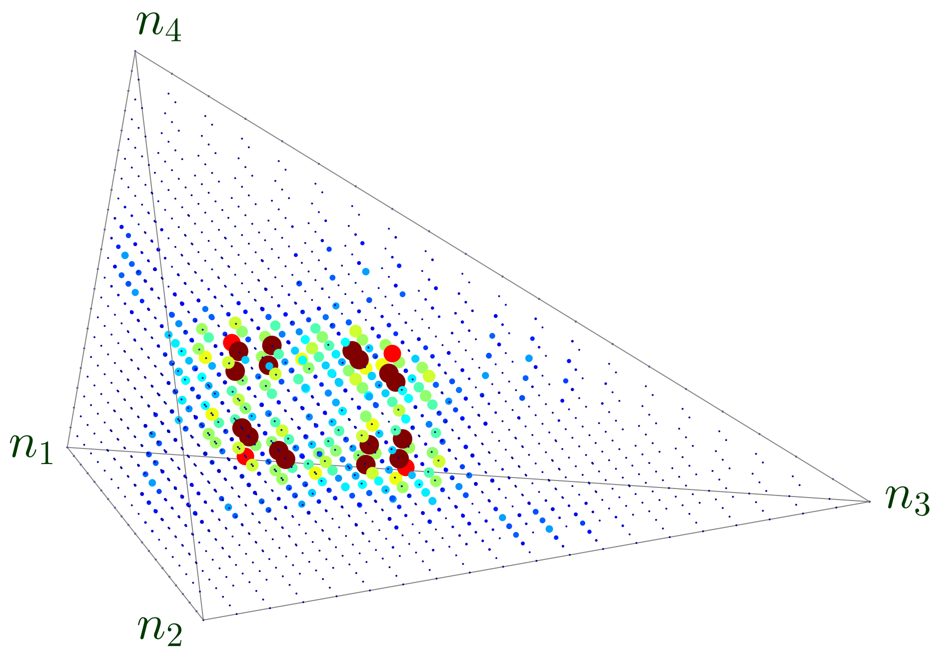


Panels of (d) and (e):
Left: quantum eigenstates.
Right: underlying classical dynamics.

What about $M = 4$?



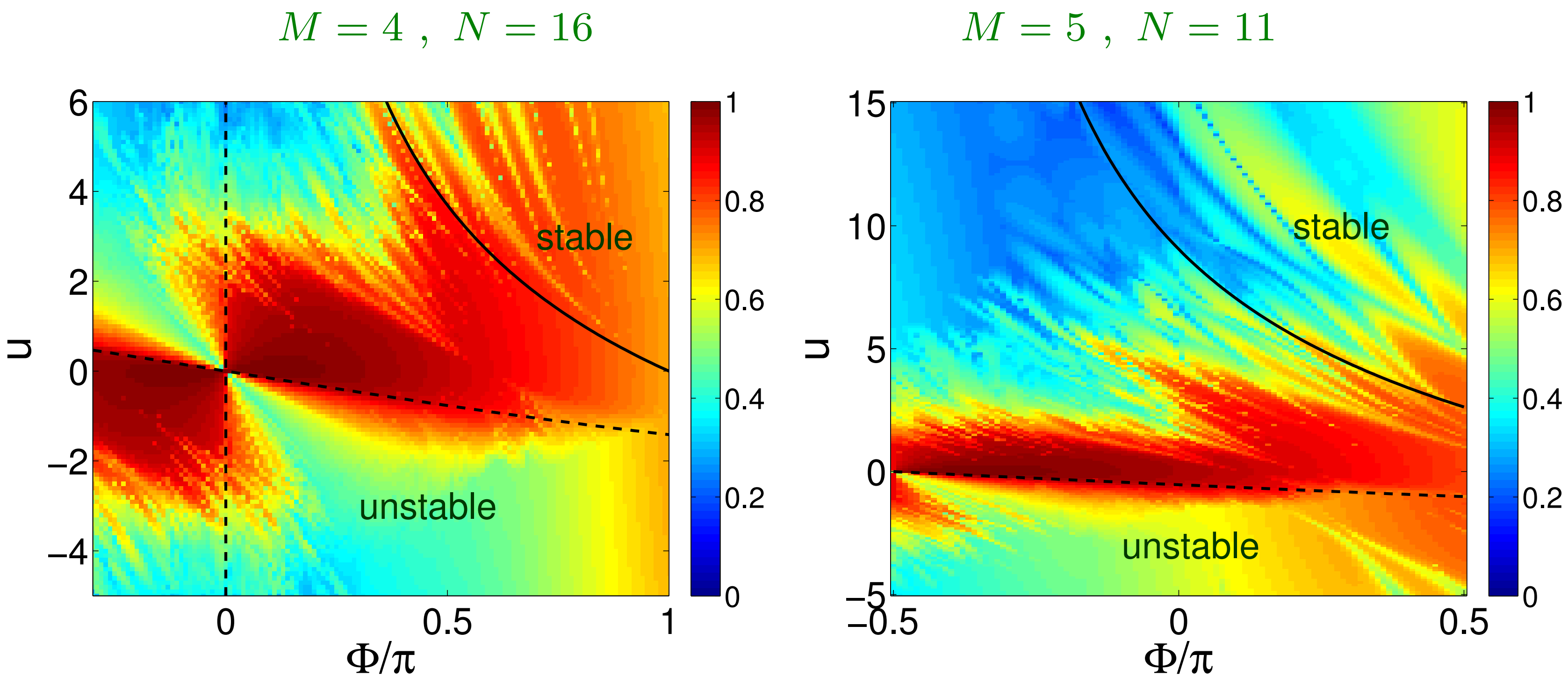
Regular vortex state



Irregular vortex state

But there is a dramatic difference compared to $M = 3$

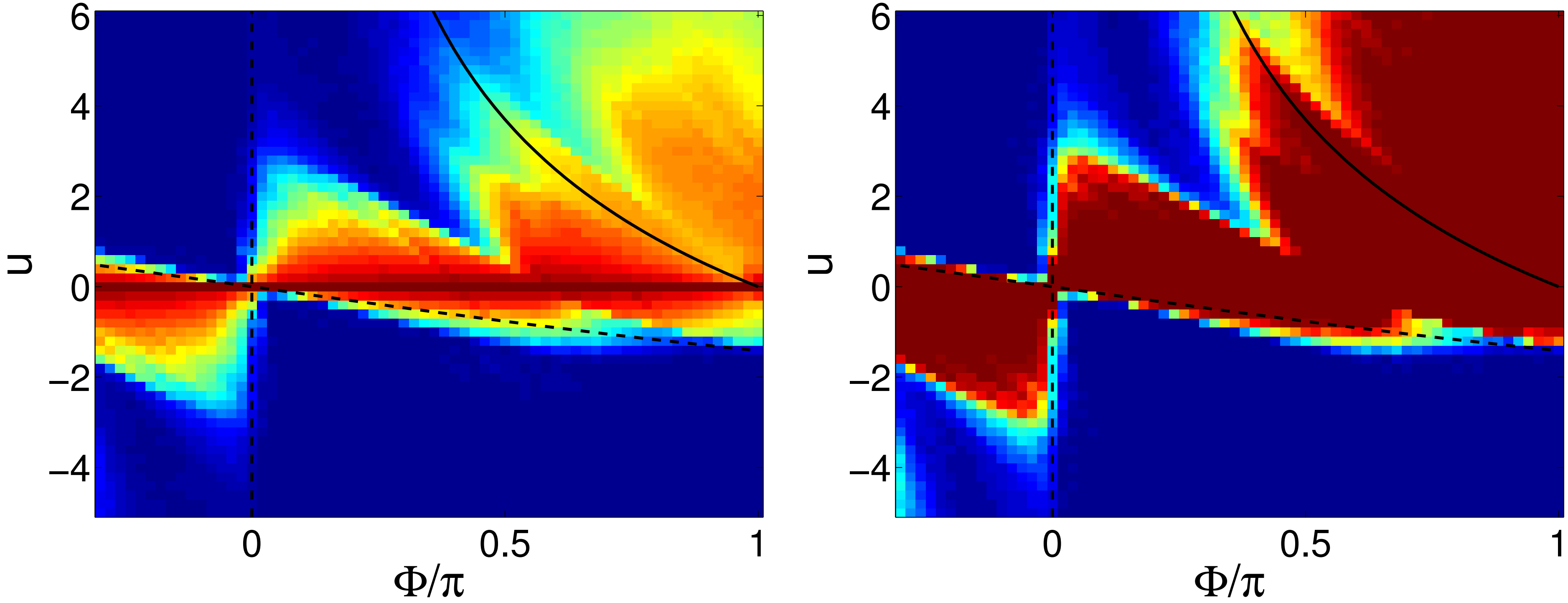
"Large" rings ($M > 3$)



- Energy surface is $2d - 1$ dimensional (reminder: $d = M - 1$)
- KAM tori are d dimensional
- Arnold diffusion: the KAM tori in phase space are not effective in blocking the transport on the energy shell if $d > 2$.
- As u becomes larger this non-linear leakage effect is enhanced, stability of the motion is deteriorated, and the current is diminished.
- Due to the finite uncertainty width of the vortex state superfluidity can diminish even in the spectrally stable region.

Semiclassical reproduction of the regime diagram $M = 4$

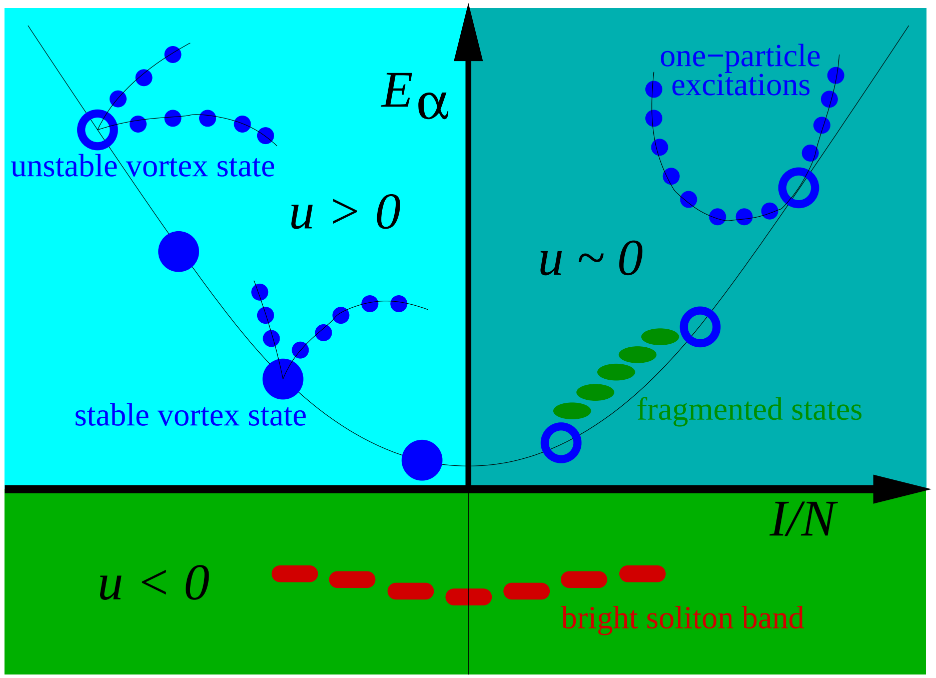
We launch a Gaussian cloud of trajectories that has an uncertainty width that corresponds to N.
Then we calculate the cloud-averaged current $I(t)$.
The criterion for quasi-stability is $I(t_H) \gtrsim (1/2)I(0)$ where $t_H \propto N^d$ (Heisenberg time)
In practice: the fraction of trajectories that escape is used as a measure for the stability.



Results are displayed for clouds that have uncertainty width $\Delta\varphi \sim \pi/2$ (left) and $\Delta\varphi \sim \pi/4$ (right).

Concluding Remarks regarding superfluidity

- The essence of superfluidity is the possibility to witness metastable vortex states ("dissipationless current")
- The standard spectral stability analysis implies that vortex states whose rotation velocity is less than a critical velocity are metastable ("Landau criterion")
- We challenge the application of the traditional BdG analysis to low-dimensional superfluid circuits.
- We have highlighted a novel type of superfluidity that is supported by irregular or chaotic or breathing vortex states.



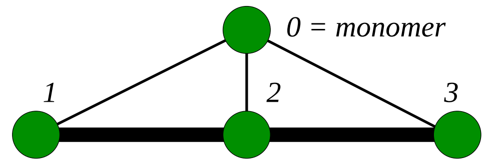
- In a larger perspective we emphasize that the role of chaos should be recognized in the analysis of superfluidity. Furthermore we believe that a global understanding of the mixed phase-space structure is essential in order to analyse dynamical processes such as phase-slips.

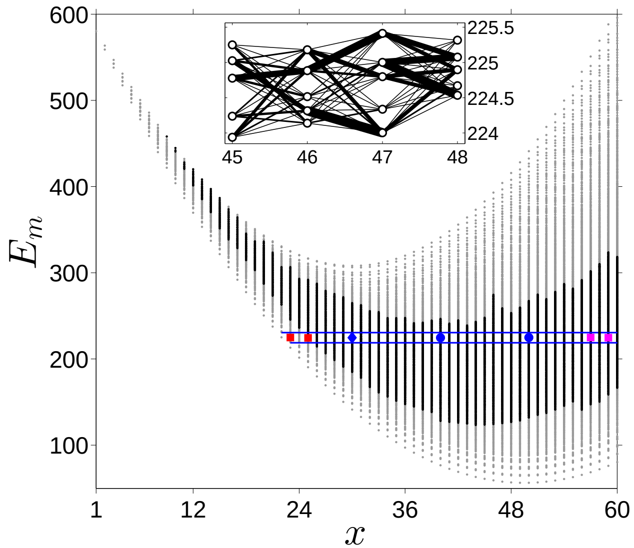
The minimal model for thermalization [CK,AV,DC, NJP (2015)]

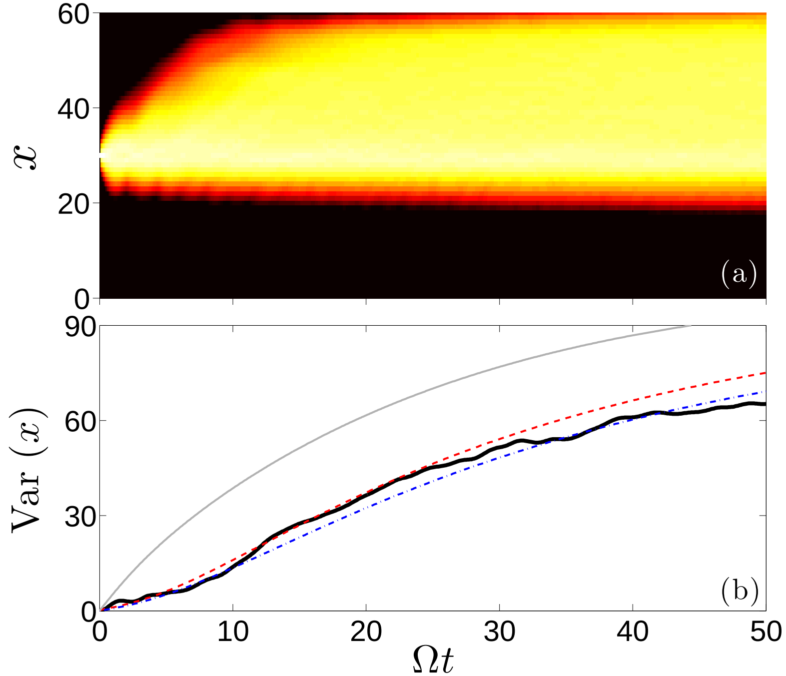
The FPE description makes sense if the sub-systems are chaotic.

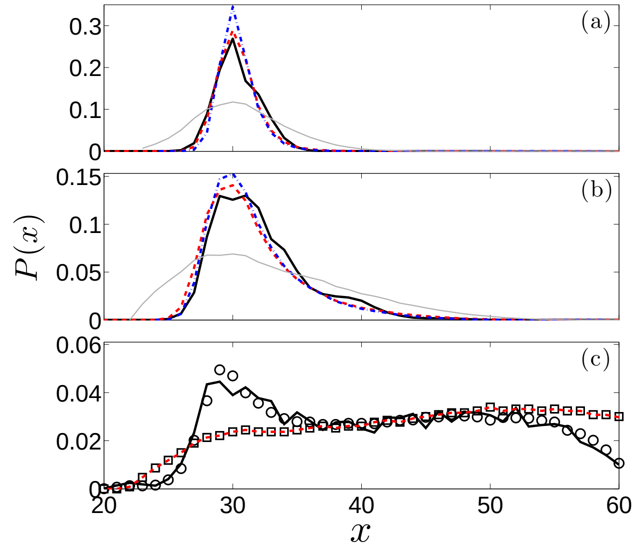
Minimal model for a chaotic sub-system: BHH trimer.

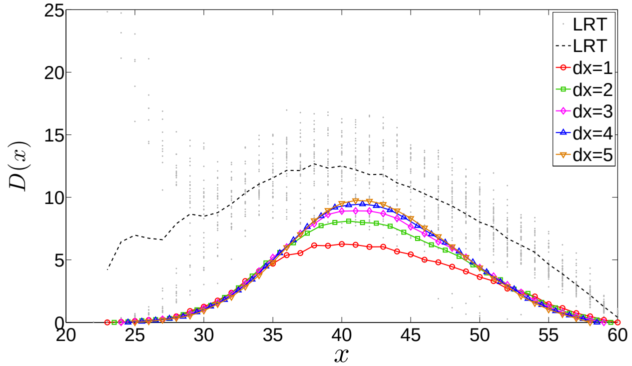
Minimal model for thermalization: BHH trimer + monomer



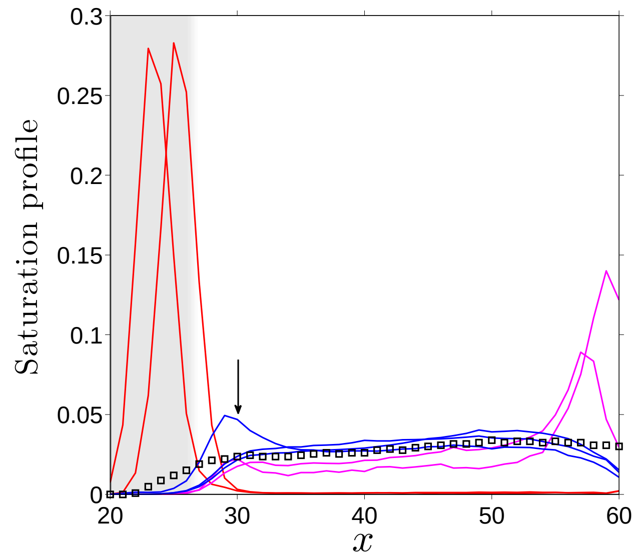








$$\frac{\partial}{\partial t}\rho(x) = \frac{\partial}{\partial x}\left[g(x)D(x)\frac{\partial}{\partial x}\left(\frac{1}{g(x)}\rho(x)\right)\right]$$

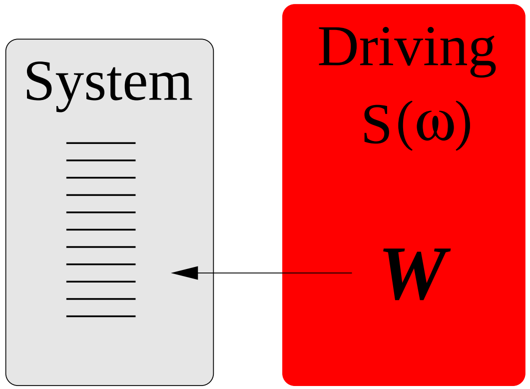


The fluctuation-diffusion-dissipation relation

Rate of energy absorption (work):

$$A(\varepsilon) \;=\; \partial_\varepsilon D_\varepsilon + \beta(\varepsilon) D_\varepsilon, \qquad \dot{W} = \langle A \rangle$$

$$D_\varepsilon \;=\; \int_0^\infty \frac{d\omega}{2\pi}\, \omega^2\, \tilde{C}_\varepsilon(\omega)\, \tilde{S}(\omega)$$



Derivation:

$$\frac{\partial \rho}{\partial t} \;=\; \frac{\partial}{\partial \varepsilon}\left(g(\varepsilon) D(\varepsilon) \frac{\partial}{\partial \varepsilon}\left(\frac{1}{g(\varepsilon)}\rho\right)\right) \;=\; -\frac{\partial}{\partial \varepsilon}\left(A(\varepsilon)\rho - \frac{\partial}{\partial \varepsilon}\left[D(\varepsilon)\rho\right]\right)$$

M. Wilkinson (1988), based on the diffusion picture of **Ott (1979)**

C. Jarzynski (1995) - adding FPE perspective.

D. Cohen (1999) - adding FDT perspective + addressing the quantum case.

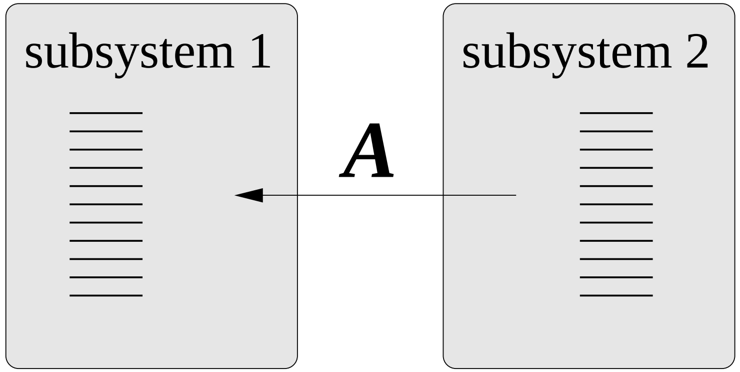
G. Bunin, L. D'Alessio, Y. Kafri, A. Polkovnikov (2011) - adding NFT based derivation.

Thermalization of two subsystems

Rate of energy transfer [FPE version]:

$$A(\varepsilon) \;=\; \partial_\varepsilon D_\varepsilon + (\beta_1 - \beta_2) D_\varepsilon$$

$$D_\varepsilon \;=\; \int_0^\infty \frac{d\omega}{2\pi}\, \omega^2\, \tilde{S}^{(1)}(\omega)\, \tilde{S}^{(2)}(\omega)$$



Derivation: **[Tikhonenkov, Vardi, Anglin, Cohen (PRL 2013)]**

The diffusion is along constant energy lines: $\varepsilon_1 + \varepsilon_2 = \mathcal{E}$

The proper Liouville measure is: $g(\varepsilon) = g_1(\varepsilon) g_2(\mathcal{E} - \varepsilon)$

Note: After canonincal preparation of the two subsystems:

$$\langle A(\varepsilon) \rangle \;=\; \left(\frac{1}{T_1} - \frac{1}{T_2}\right) \langle D_\varepsilon \rangle$$

MEQ version: **Hurowitz, Cohen (EPL 2011)**

NFT version: **Bunin, Kafri (arXiv 2012)**

QCC and quantum anomalies

? Can we trust QCC

? Is the dynamics described by FPE

? Can we use the LRT Kubo estimate for D

Driven integrable system (e.g. "Kicked Rotor") -
quasi-linear behavior shows up only for large driving amplitude.

Driven chaotic system (e.g. "Sinai Billiard" with moving wall) -
Linear response applies; D is a linear functional of $S(\omega)$.
[Instead of "chaos" one can assume system coupled to a bath]

Quantized integrable system $\rightsquigarrow$ no QCC

Quantized chaotic systems $\rightsquigarrow$ restricted QCC **[Cohen (PRL 1999); Cohen, Kottos (PRL 2000)]**

Landau-Zener related anomaly **[Wilkinson and followers]**

Anderson-localization related anomaly **[QKR and follow-up works]**

Sparsity related semi-linear anomaly **[line of study initiated 2005]**

Diffusion in "sparse" networks

Transition rates:

$$w_{nm} \;=\; w_0 \;\exp\left[-\frac{\epsilon_{nm}}{T}\right] \;\exp\left[-\frac{r_{nm}}{\xi}\right]$$

Sparsity parameter ($s \ll 1$ means "sparse"):

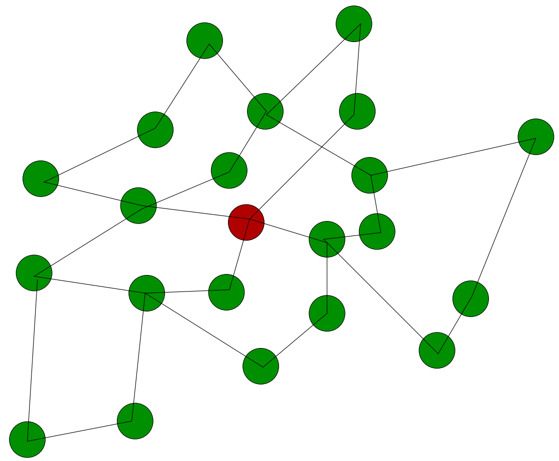
$$s \;\equiv\; \left(\frac{d}{\Omega_d} n_c\right)^{-1/d} \frac{\xi}{r_0}$$

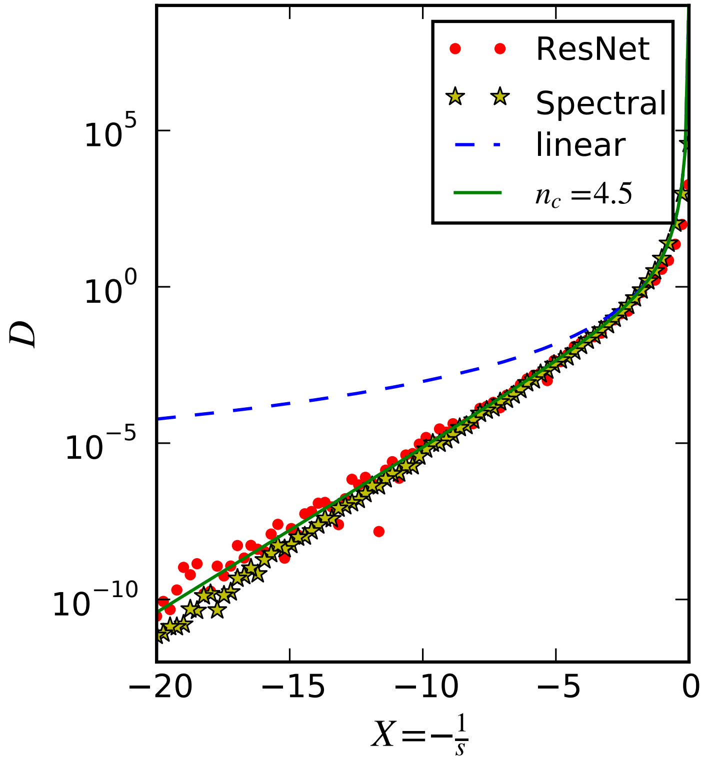
Result for the diffusion coefficient:

$$D \;\approx\; \mathrm{EXP}_{d+2}\left(\frac{1}{s}\right) \; \mathrm{e}^{-1/s} \; D_{\text{linear}}$$

For the non-degenerate Mott hopping model
s depends on the temperature, and use EXP_{d+3}

$$r_0^d = \left(\frac{\Delta_\xi}{T}\right) \xi^d \quad \rightsquigarrow \quad s = \left(\frac{T}{\Delta_\xi}\right)^{1/d}$$





[**Y. de Leeuw, D. Cohen,** *Diffusion in sparse networks*, **PRE (2012)**]

Concluding Remarks regarding thermalization

1. BEC trimers are the minimal building blocks for thermalization
2. The generic package deal: diffusion, LRT and QCC.
3. FPE based FD phenomenology for mesoscopic thermalization
4. Beyond FPE - statistics of dwell times due to sticky dynamics
5. Beyond LRT - sparsity - resistor network picture - semilinear response
6. FD phenomenology for sparse (glassy) systems