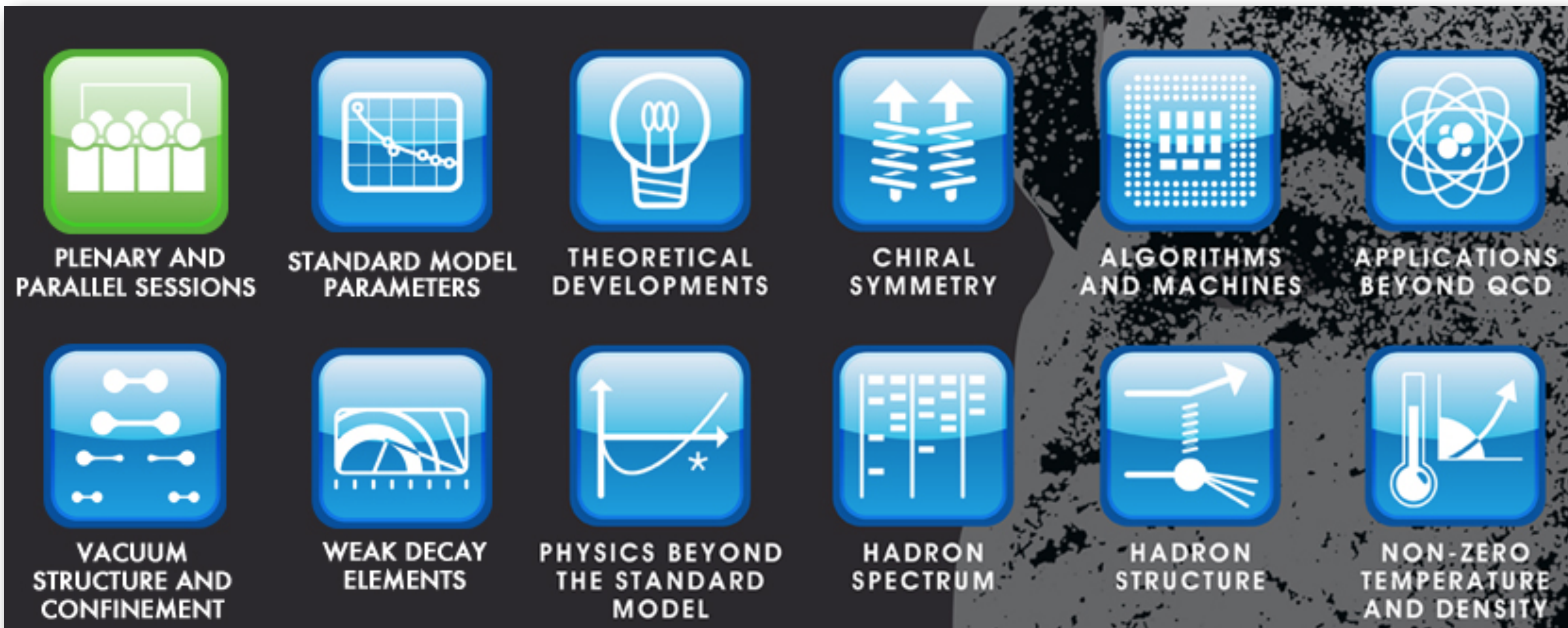
Introduction to Lattice QCD

Carlos Pena



Taller de Altas Energías, Benasque, 15-28 Sep 2013

outline

- motivation: strong interaction(s) and non-perturbative physics

- lattice field theory
  - QFT in Euclidean space
  - matter and gauge fields on a lattice
  - interacting gauge theories on a lattice: QCD

- numerical aspects
  - Monte Carlo techniques for non-perturbative QFT
  - reach of QCD computations
  - anatomy of an example

- overview of physics capabilities
  - FLAG
  - selected lattice QCD results
  - beyond the SM

(some) reference texts

L Lellouch et al. (ed.), *Modern Perspectives in Lattice QCD: Quantum Field Theory and High Performance Computing.*
93rd Session Les Houches International School
Oxford University Press 2011

C Gattringer & CB Lang, *QCD on the Lattice An Introduction for Beginners*
Springer Verlag 2009

T DeGrand & C DeTar, *Lattice Methods for Quantum Chromodynamics*
World Scientific 2006

HJ Rothe, *Lattice Gauge Theories* (3rd ed.)
World Scientific 2005

J Smit, *Introduction to Quantum Fields on a Lattice*
Cambridge University Press 2002

[pioneer] M Creutz, *Quarks, Gluons and Lattices*
Cambridge University Press 1983

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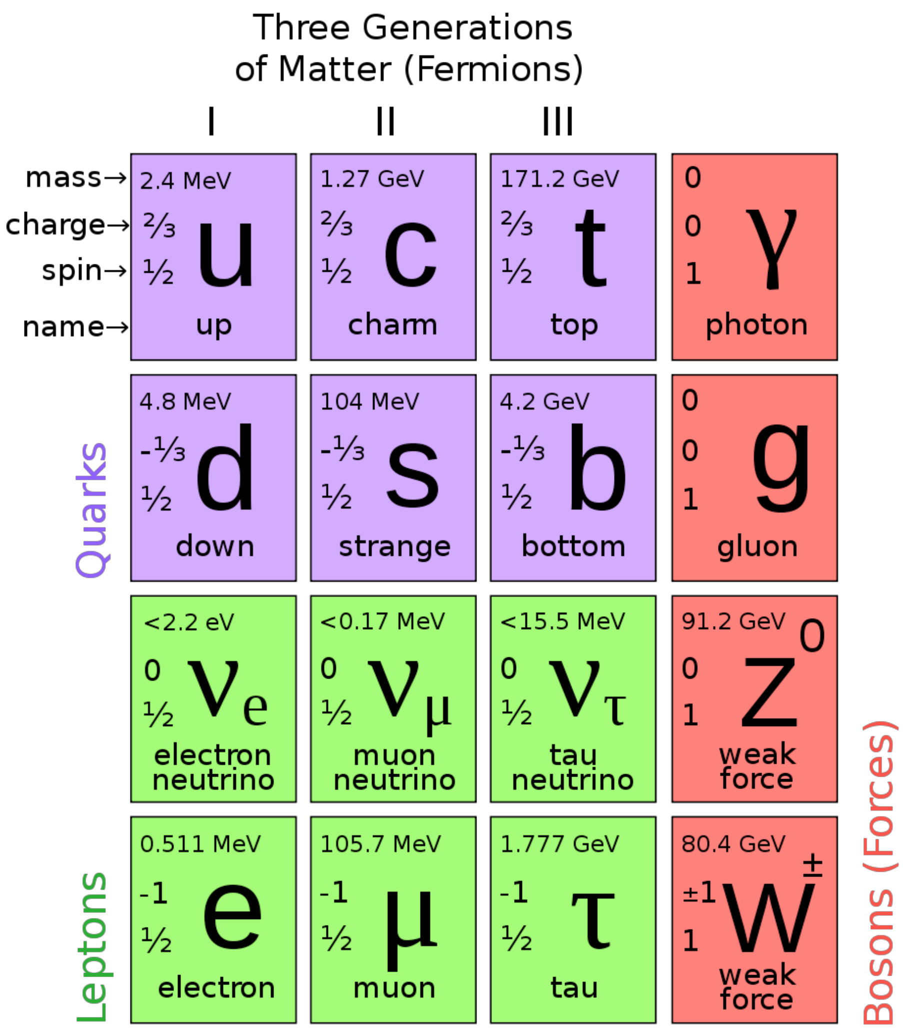
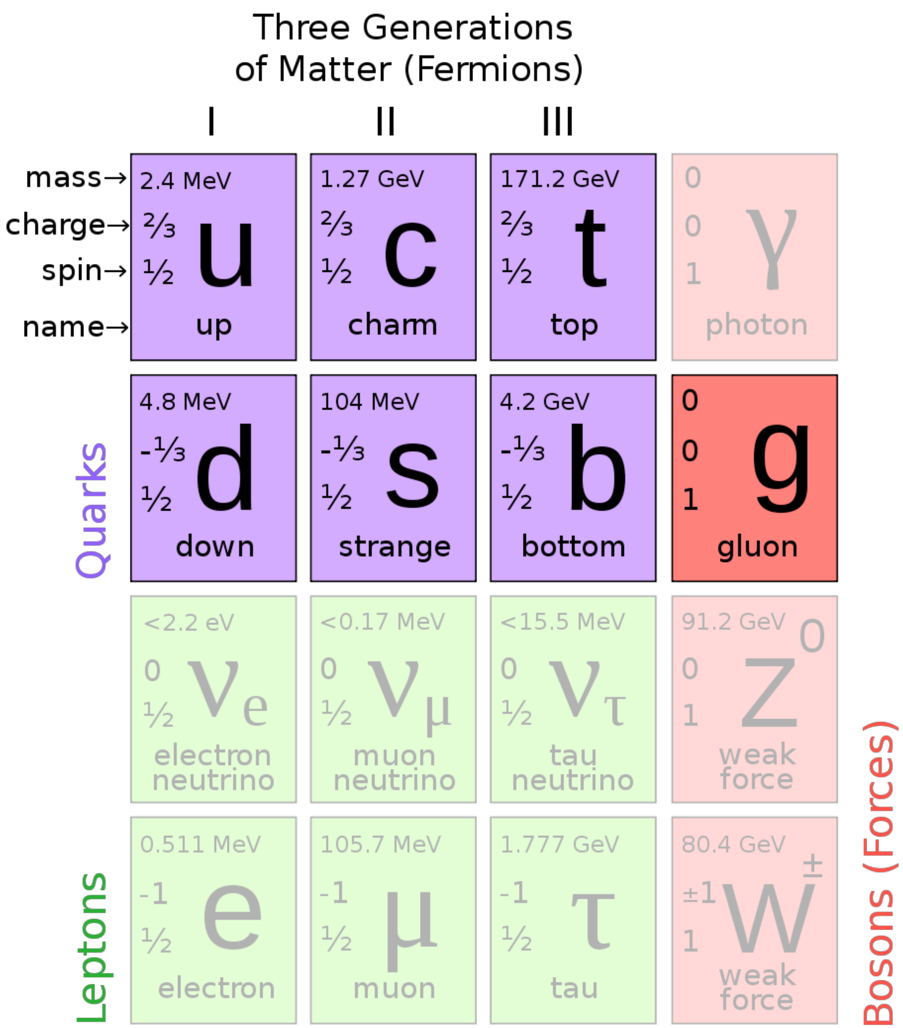
why lattice field theory

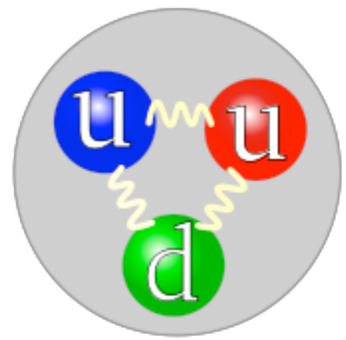
standard model of particle physics

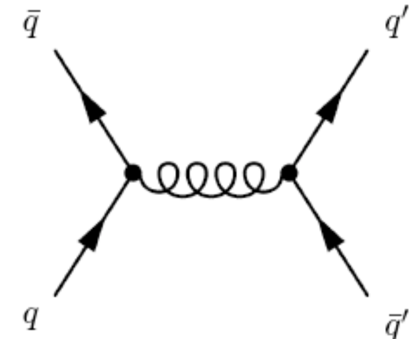


Quantum ChromoDynamics



QFT that describes the strong interaction at a fundamental level



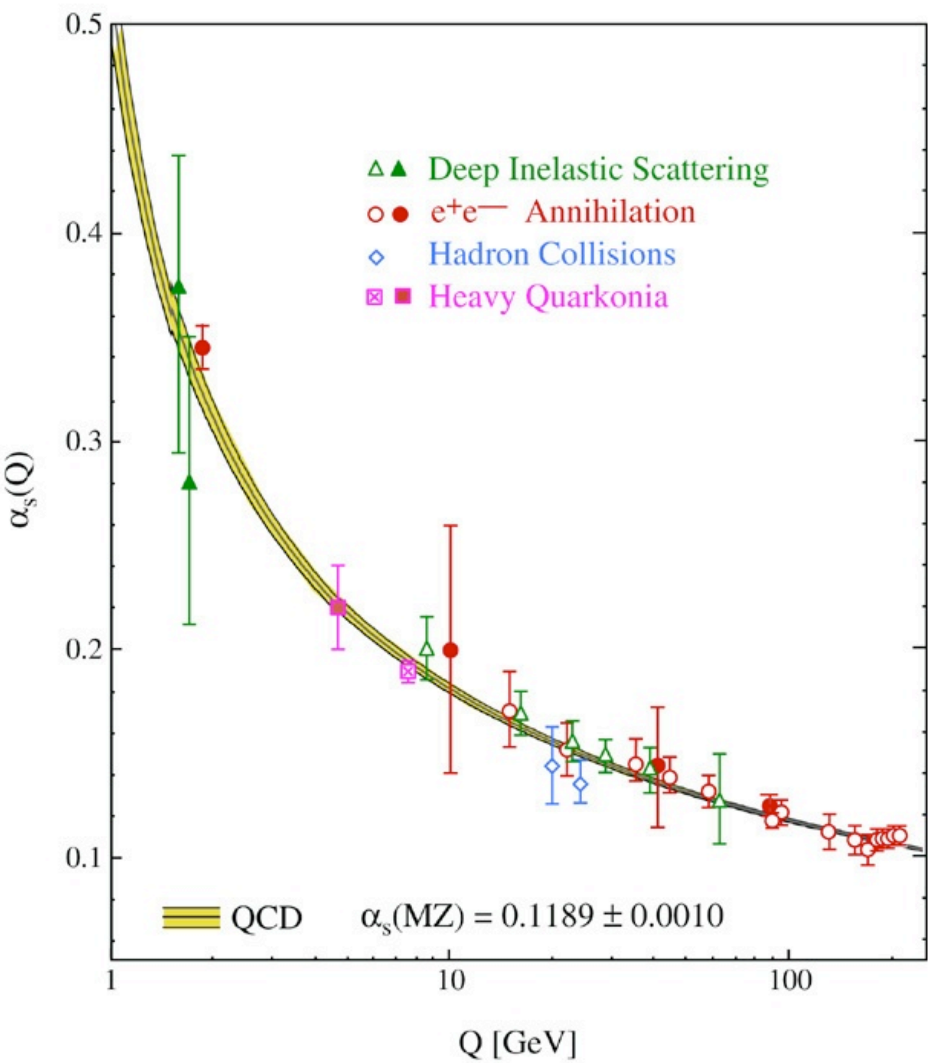


Elementary d.o.f.: gluon exchanges between colour charges (cf. QED: photon exchanges between electrically charged particles).

Distinctive features:

- asymptotic freedom: interaction grows weaker at shorter distances;
- quarks and gluons confined into colourless bound states (hadrons);
- spontaneous symmetry breaking determines low-energy dynamics.

asymptotic freedom



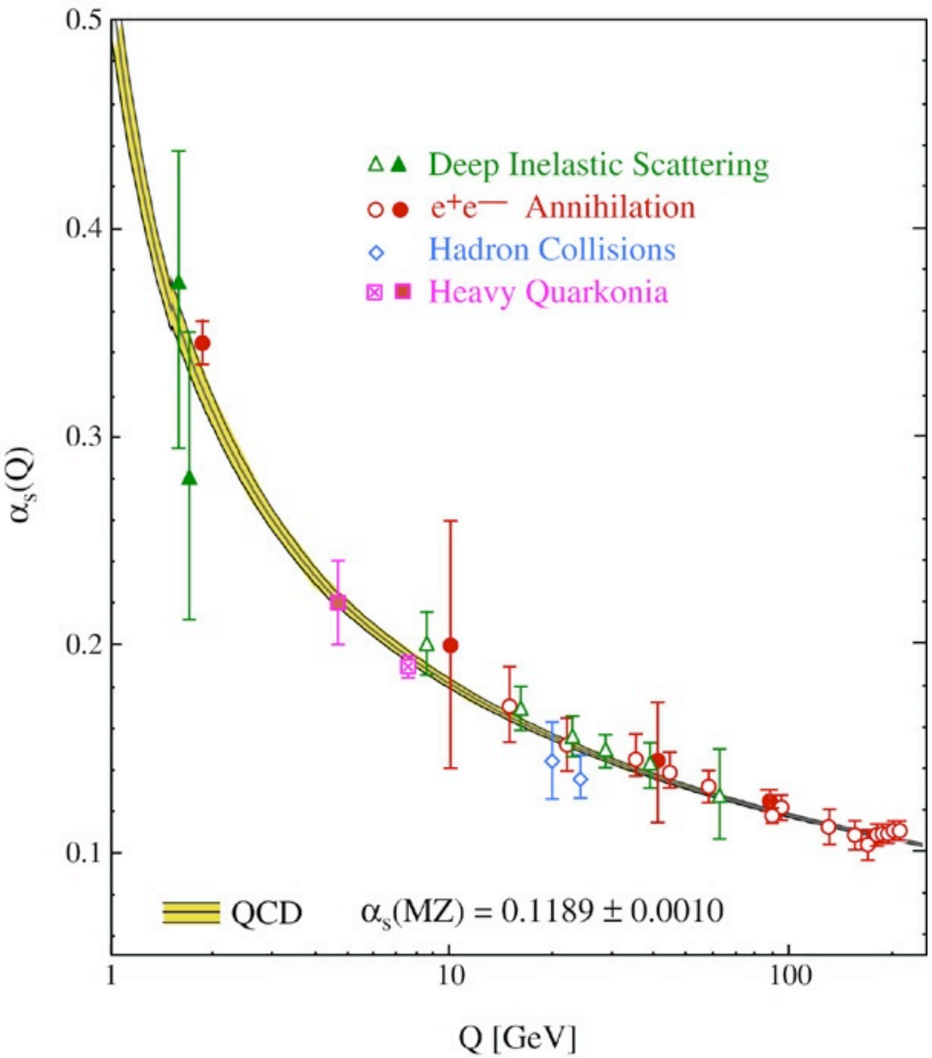
Asymptotic freedom: QCD coupling is weak at short distances (high energies), strong at long distances (low energies).

The Nobel Prize in Physics 2004
David J. Gross, H. David Politzer, Frank Wilczek

David J. Gross | H. David Politzer | Frank Wilczek

The Nobel Prize in Physics 2004 was awarded jointly to David J. Gross, H. David Politzer and Frank Wilczek *"for the discovery of asymptotic freedom in the theory of the strong interaction"*.

asymptotic freedom



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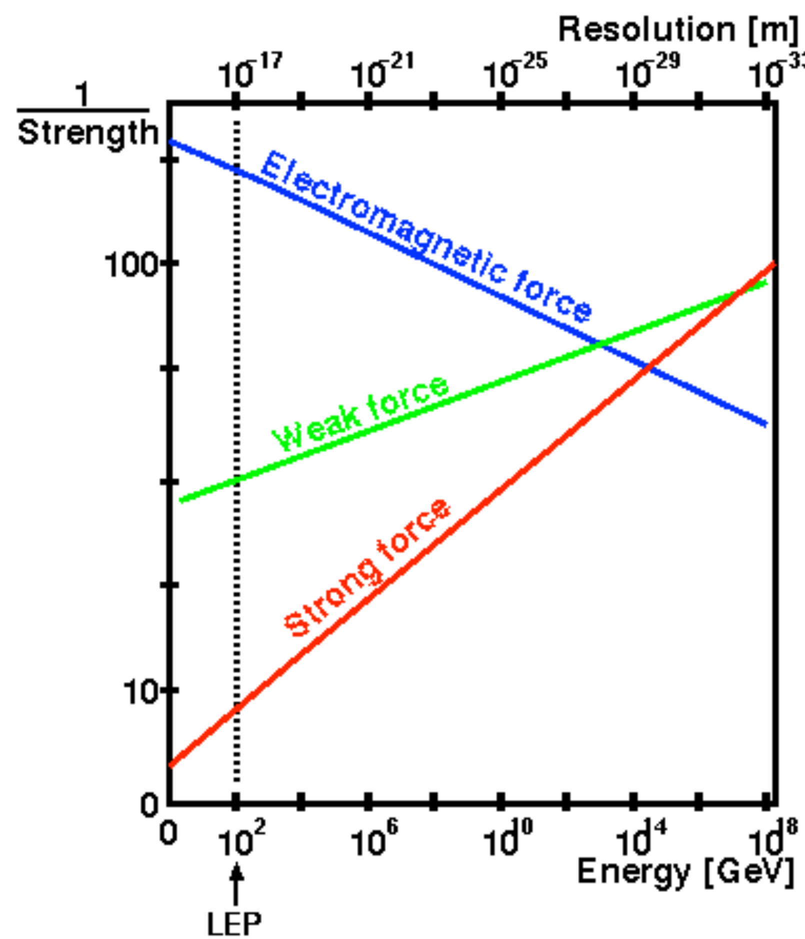
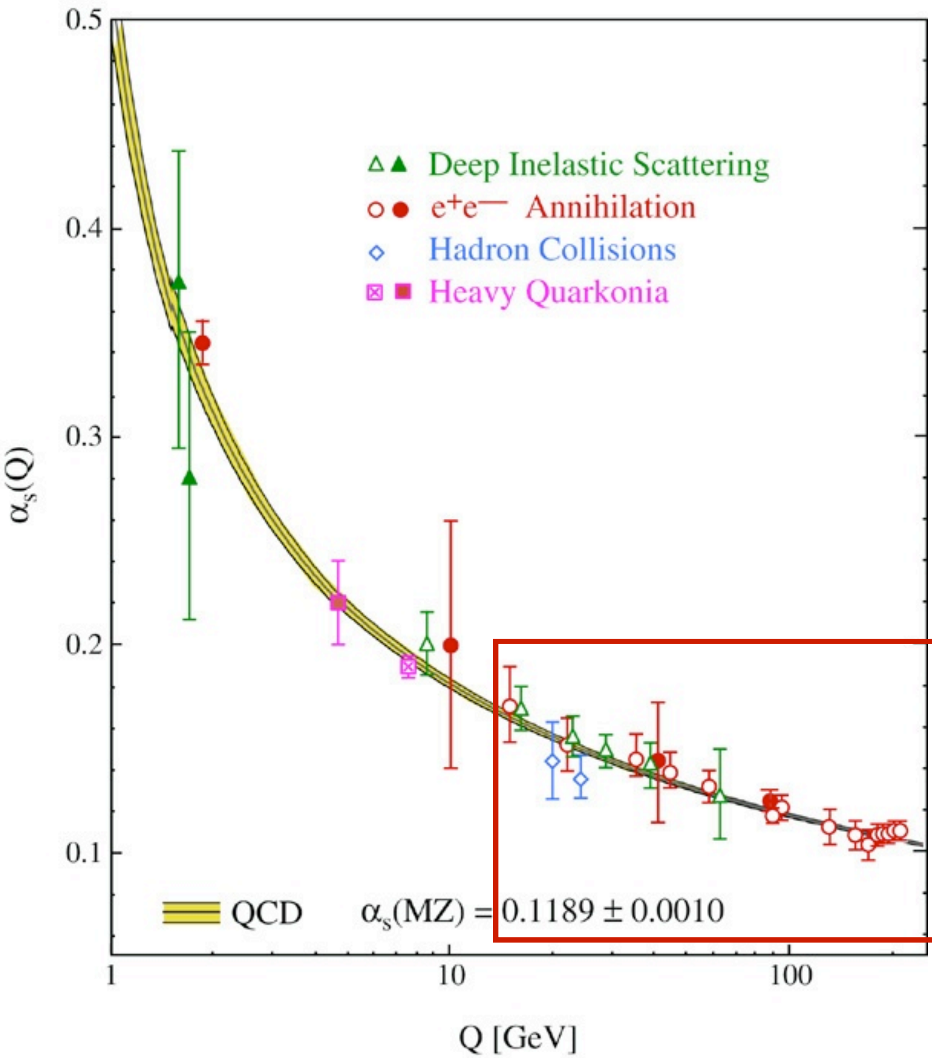


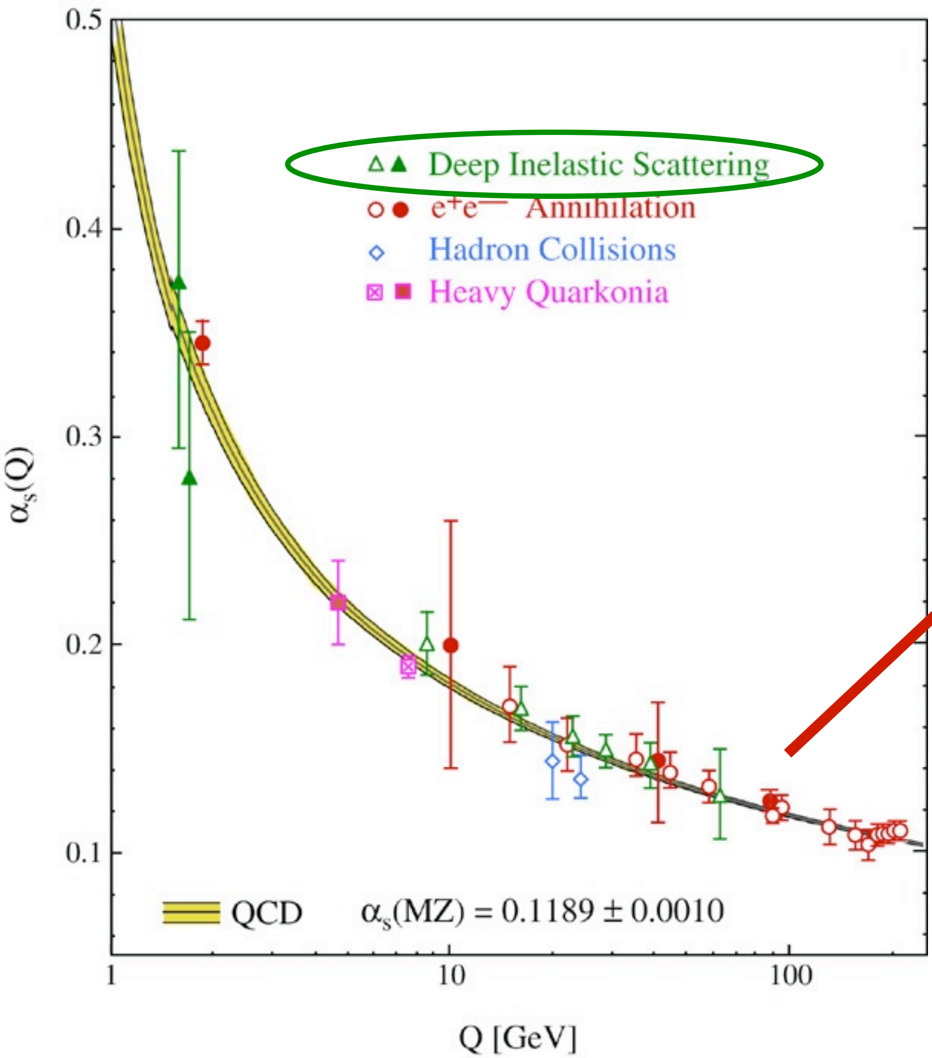
asymptotic freedom

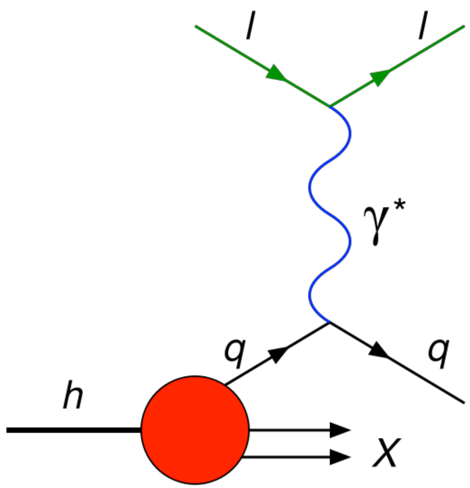


high-energy regime: quarks weakly coupled, "seen" as individual entities by sufficiently energetic probes

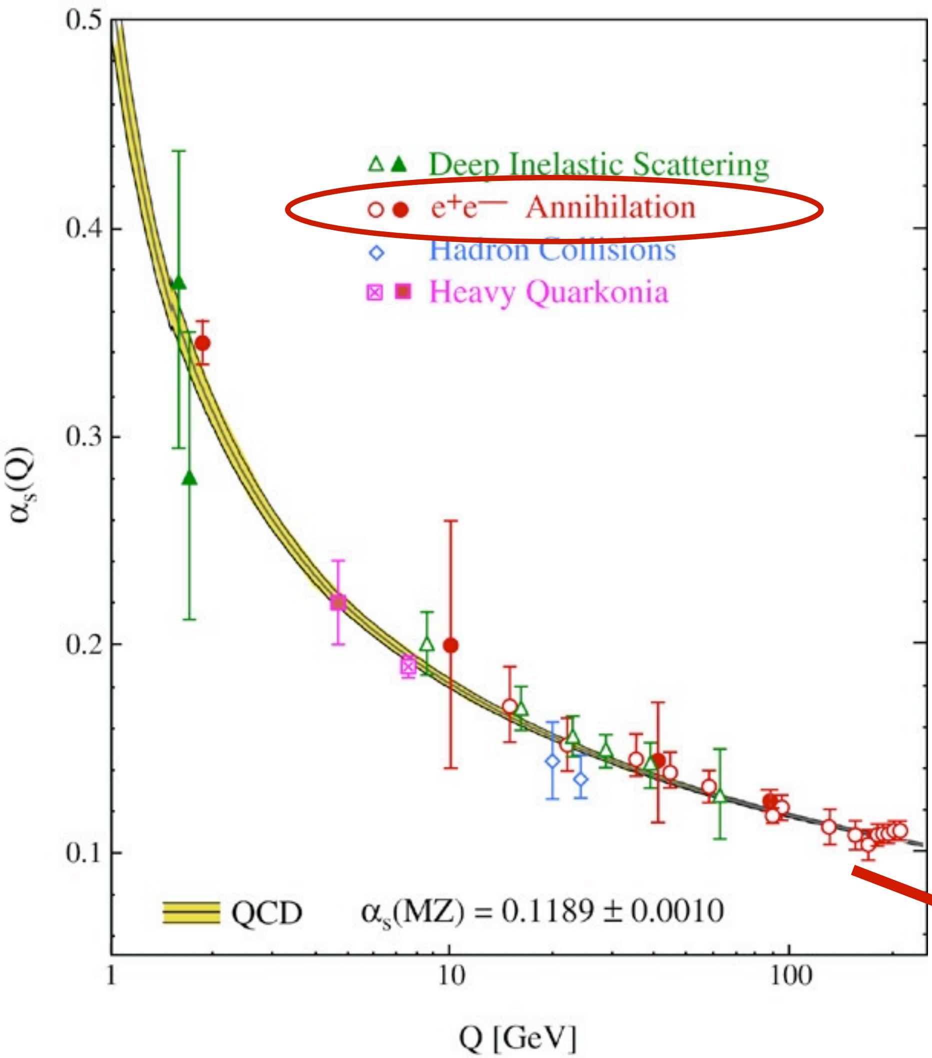
perturbation theory applicable

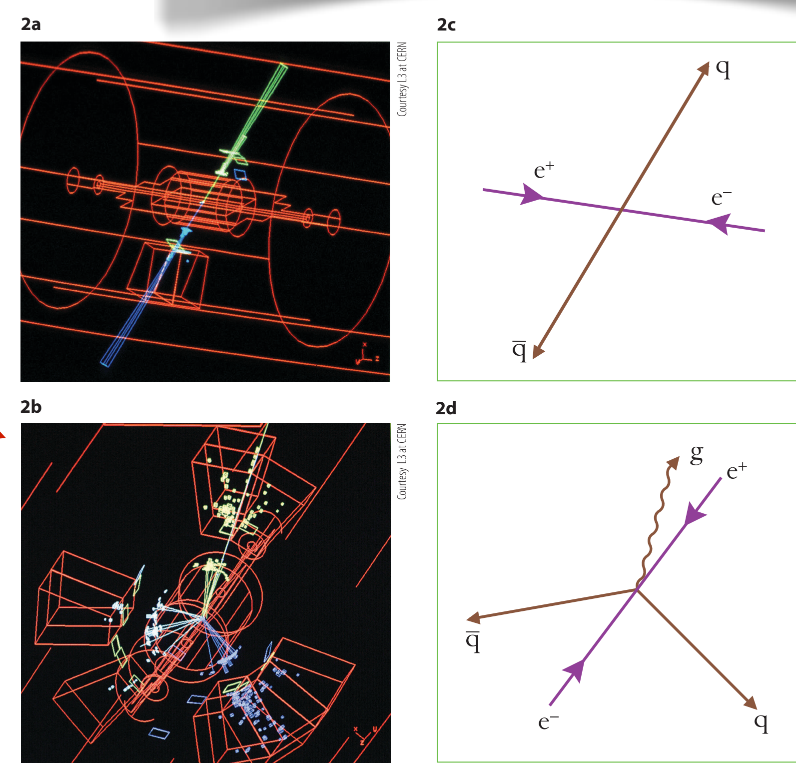
asymptotic freedom



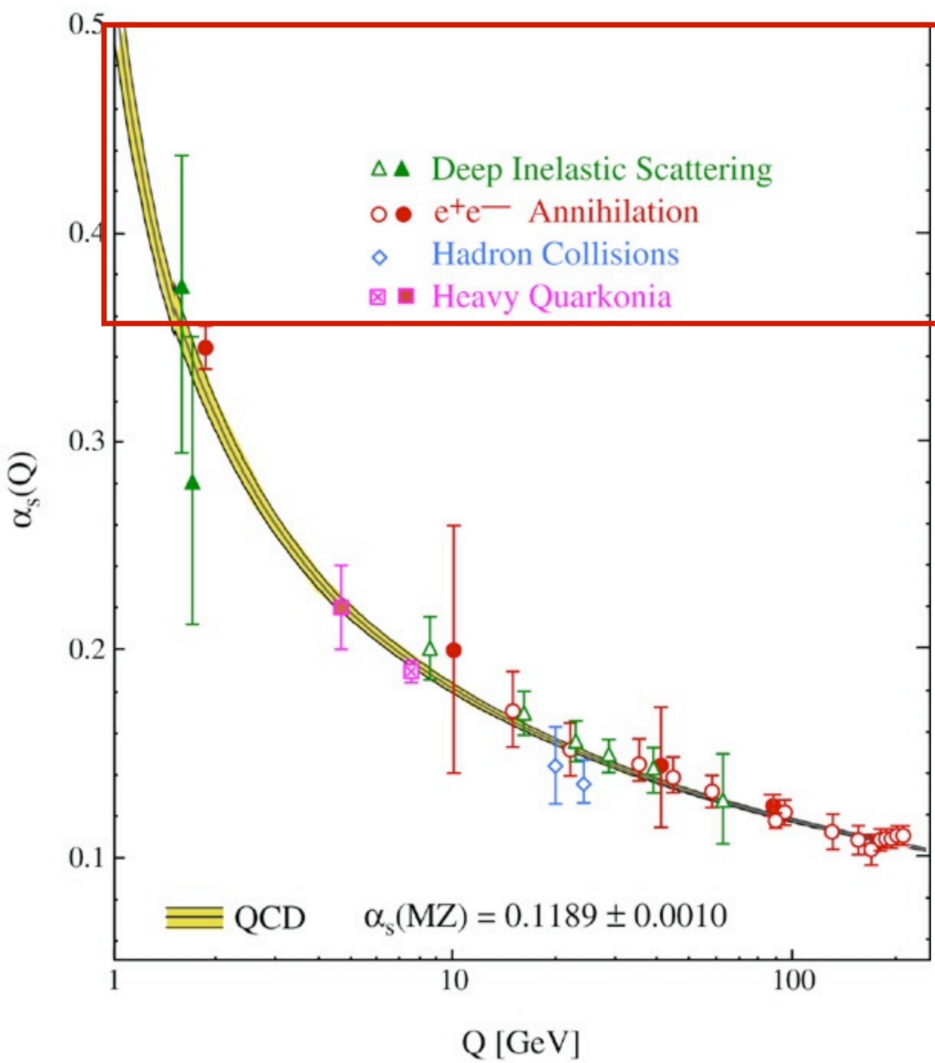


asymptotic freedom

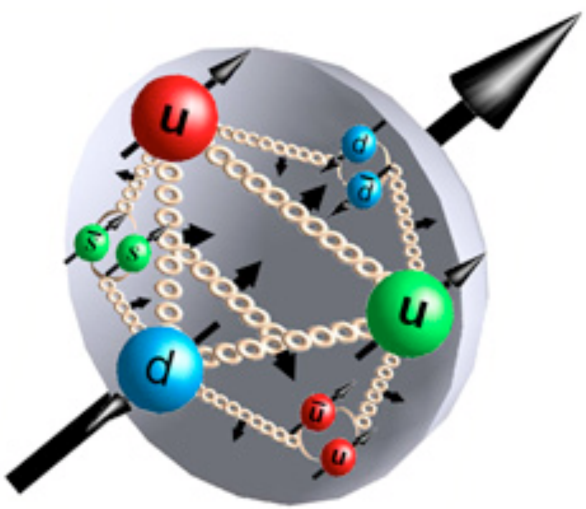




infrared slavery

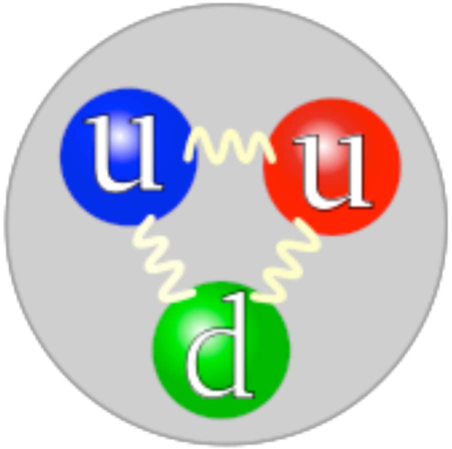


low-energy regime: quarks strongly coupled, relevant degrees of freedom are hadrons



perturbation theory breaks down
need of formulation, computational techniques

how strong?



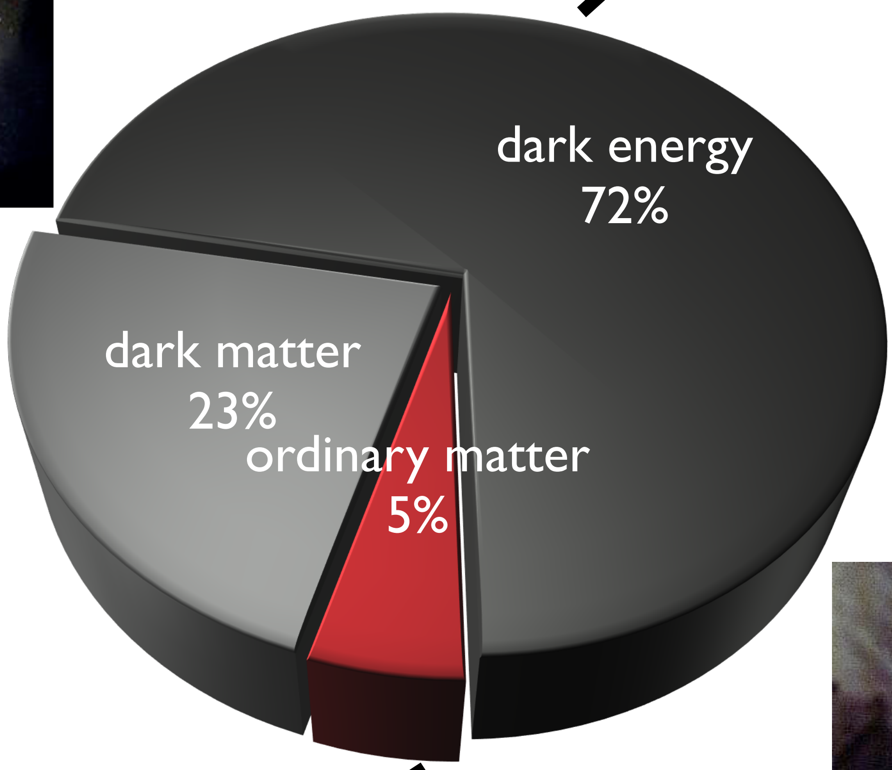
electromagnetism: $$\frac{E_{\text{bind}}(H)}{(m_e + m_p)c^2} \simeq 1.4 \times 10^{-7}$$

strong interaction: $$\frac{E_{\text{bind}}(\text{proton})}{(2m_u + m_d)c^2} \sim 60$$

vast majority of mass of baryonic matter = strong interaction binding energy

how strong?

???

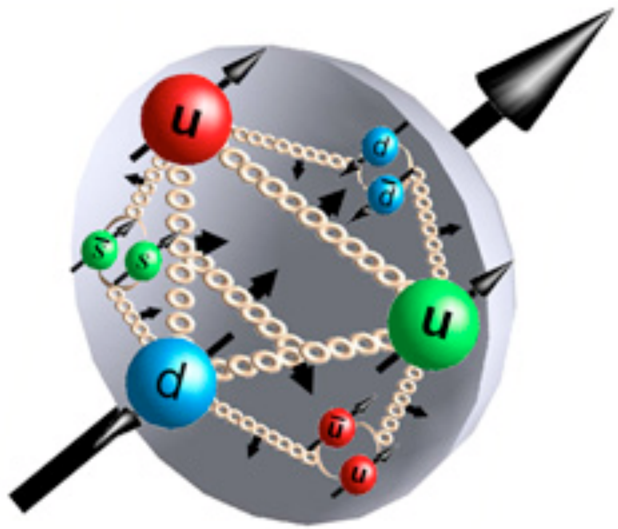


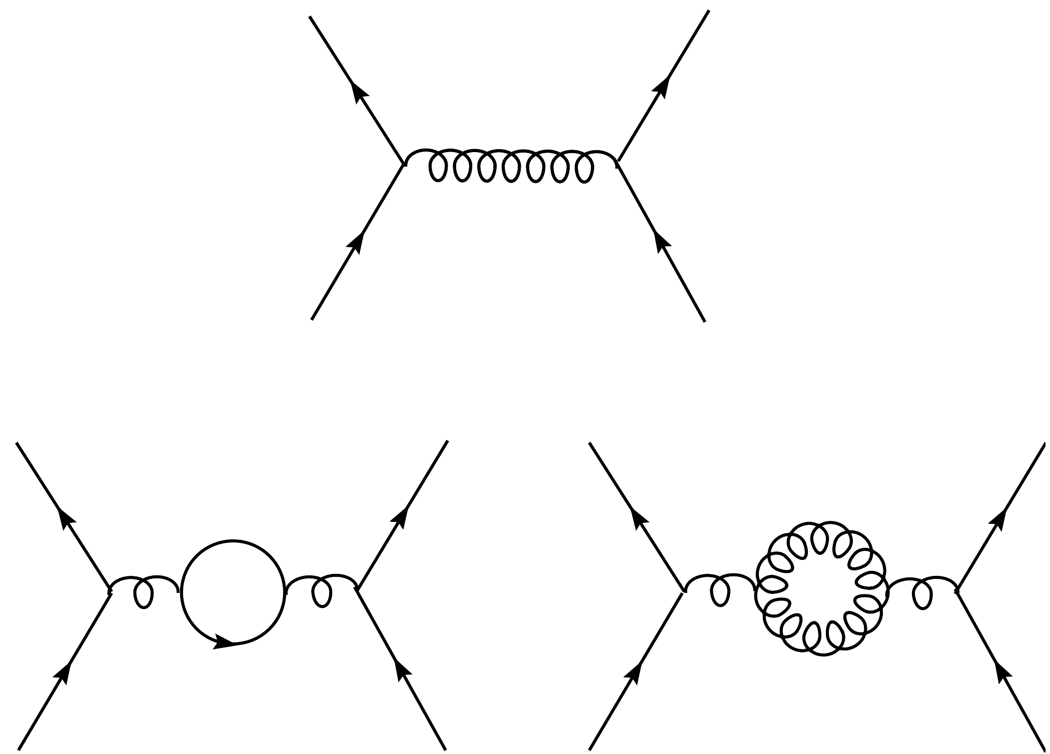
?????

vast majority of mass of baryonic matter = strong interaction binding energy

how strong? confinement and strings

non-trivial vacuum dynamics plays crucial role in hadronic regime





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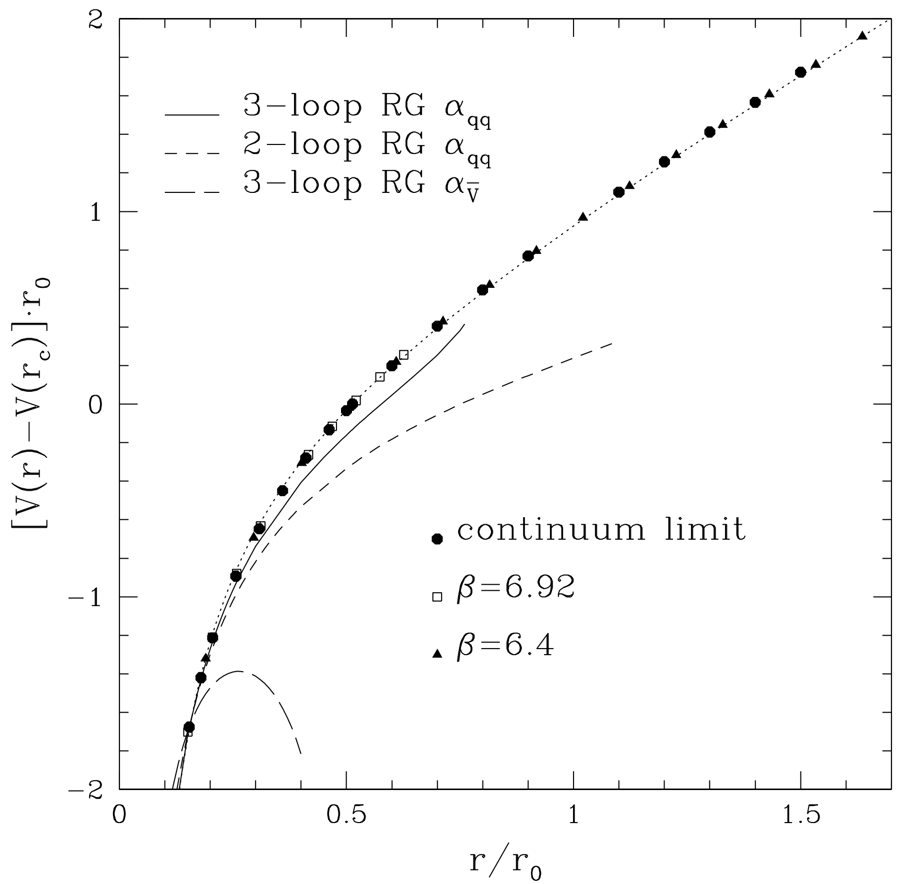
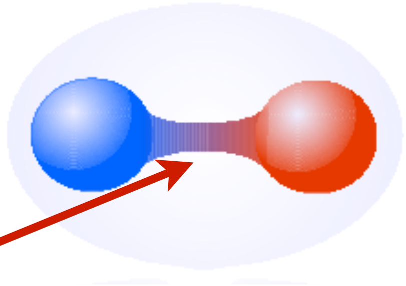
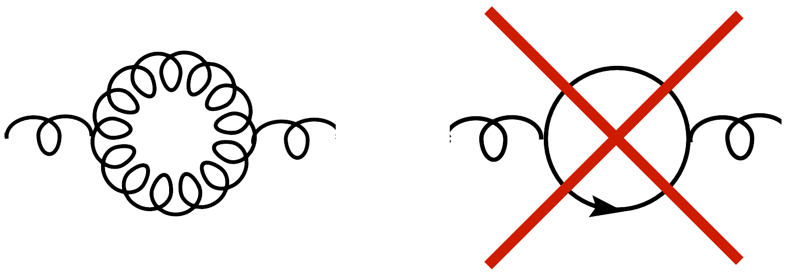


[Necco, Sommer 2001]



$$\sigma \approx (0.4\ \mathrm{GeV})^2$$



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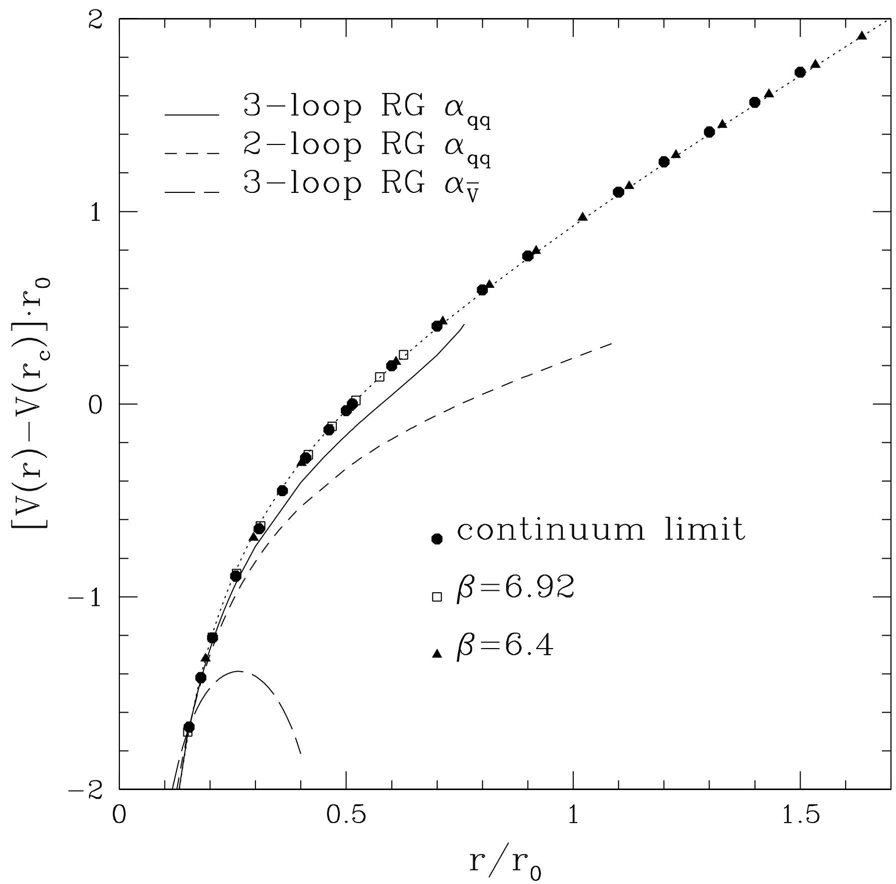

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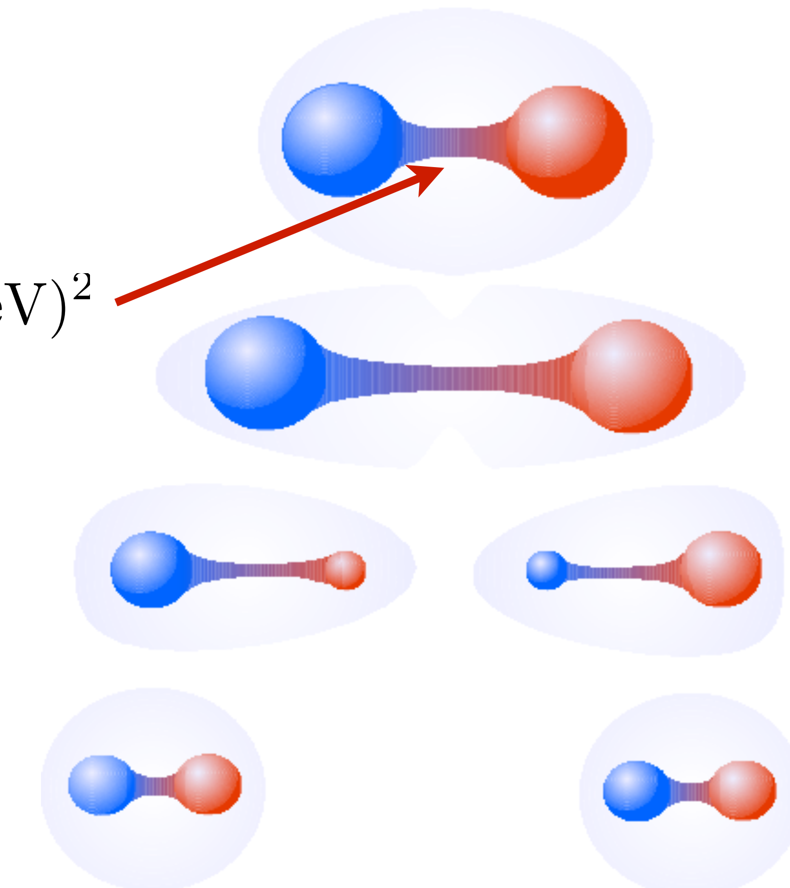

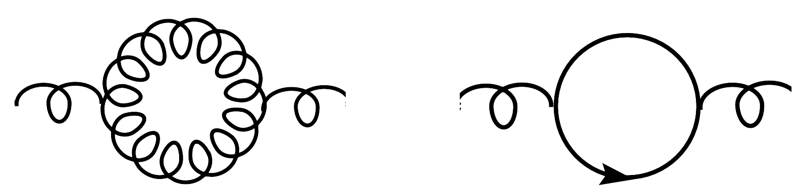

how strong? confinement and strings

non-trivial vacuum dynamics plays crucial role in hadronic regime

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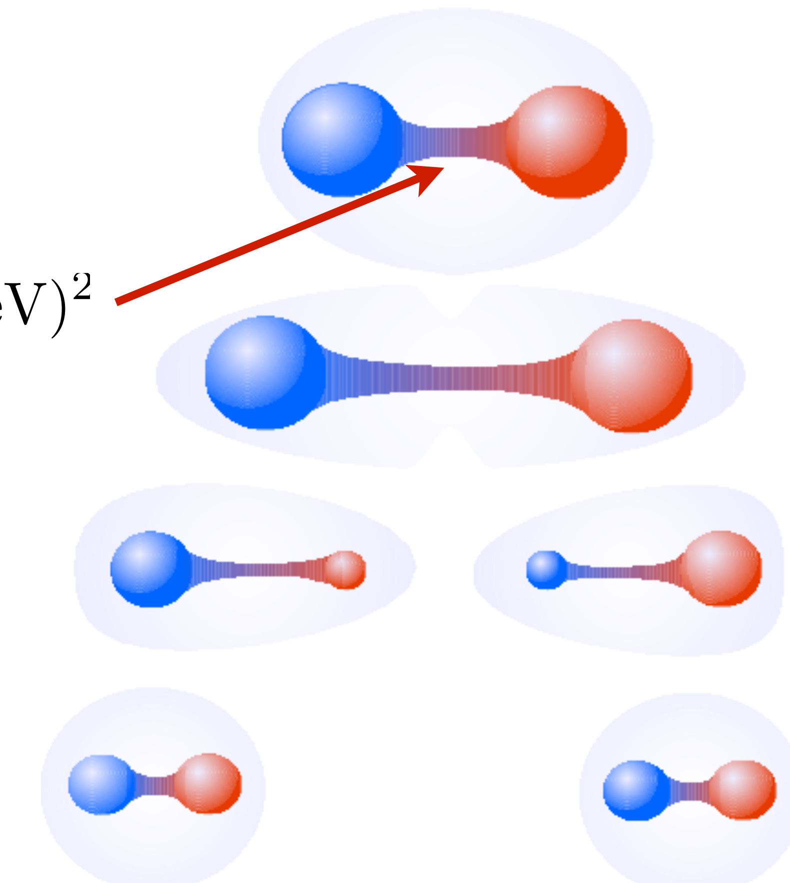
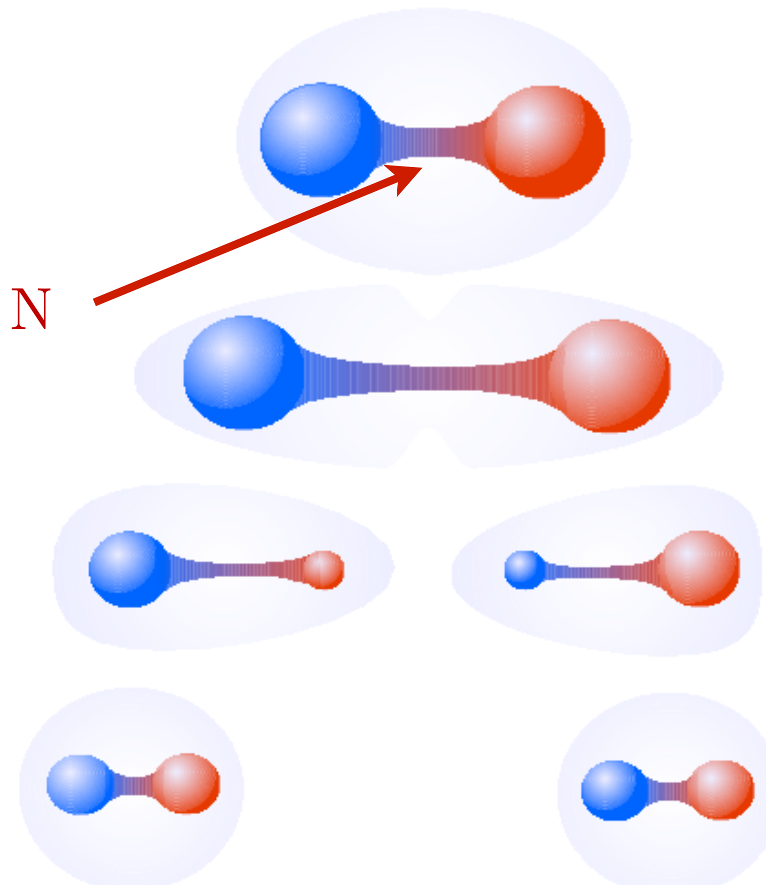


similar to a cm-thick steel cable, but 13 orders of magnitude thinner

how strong? confinement and strings

non-trivial vacuum dynamics plays crucial role in hadronic regime

$F \sim 10^5$ N

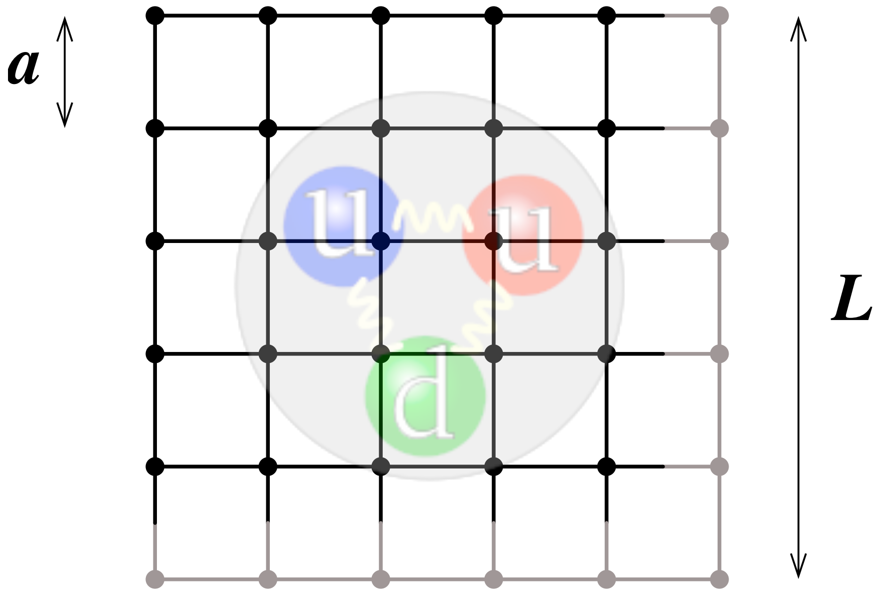


why lattice field theory

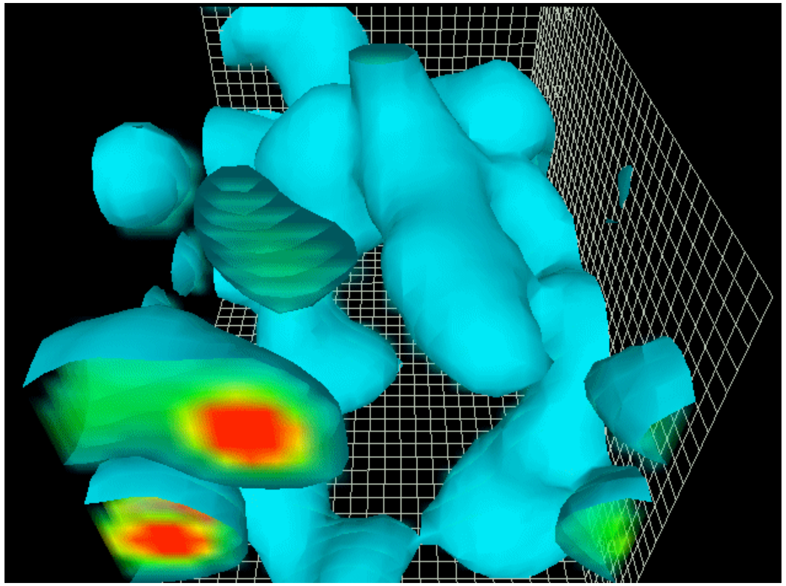
crucial tool to understand physics in a hadronic environment (or: any other strongly coupled dynamics in HEP)

- validate QCD as fundamental theory of strong interaction *at low energies*
- understand confinement
- understand spontaneous chiral symmetry breaking (QCD; EWSB?)
- compute basic hadron properties
- compute electroweak amplitudes involving hadrons
- study exotic states of matter (quark-gluon plasma, ...)
- ....

how lattice field theory



[Wilson 1974]



- define fields on discrete spacetime ⇒ introduce cutoff in a gauge-invariant, nonperturbative way
- [no free lunch: break Poincaré symmetry, face subtleties regarding discrete symmetries]
- quantise using path integral formalism
- remove cutoff non-perturbatively by exploiting renormalisation group

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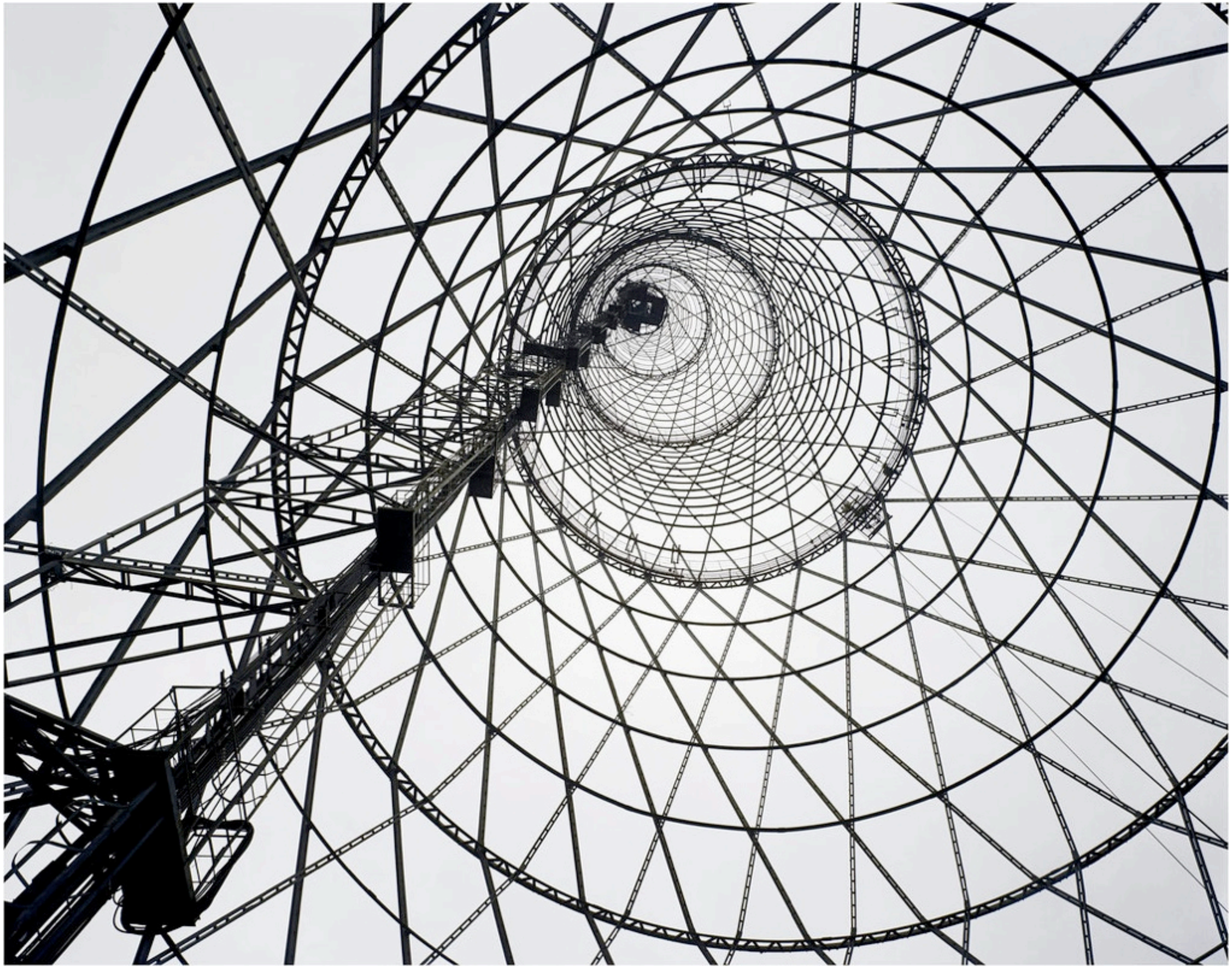


Euclidean correlation functions

Minkowski space:

$$\langle 0|\phi(x)\phi(0)|0\rangle = \langle 0|\phi(0,\mathbf{x})e^{-iHx_0}\phi(0)|0\rangle$$

extend to analytic function for $\operatorname{Im} x_0 < 0$ (n.b.: $H \geq 0$)

$\Rightarrow$ for $x_0 > 0$ we can define:

$$\langle\phi(x)\phi(0)\rangle = \langle 0|\phi(x)\phi(0)|0\rangle_{x_0\to -ix_0} = \langle 0|\phi(0,\mathbf{x})e^{-Hx_0}\phi(0)|0\rangle$$

n-point functions (ordered times):

$$\langle\phi(x_1)\cdots\phi(x_n)\rangle = \langle 0|\phi(0,\mathbf{x}_1)e^{-H(x_1-x_2)_0}\phi(0,\mathbf{x}_2)^{-H(x_{n-1}-x_n)_0}\phi(0,\mathbf{x}_n)|0\rangle$$

theorem: Euclidean n-point functions are real, analytic functions in $x_1,\ldots,x_n$, with power singularities at coinciding points (contact terms).

[Pauli, Jost, Streater & Wightman, ...]

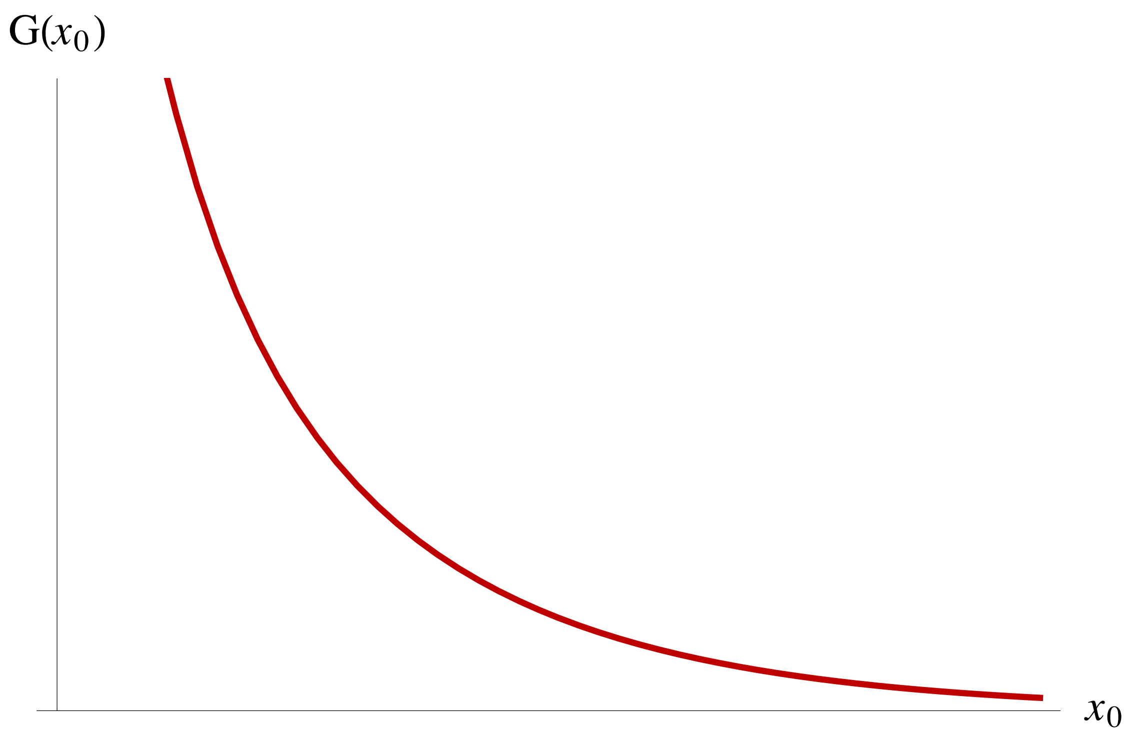
Euclidean correlation functions

example: (charged) pion two-point function

$$G(x_0) = \int \mathrm{d}^3x \, \langle (\overbrace{\bar{u}\gamma_5 d}^{\equiv P(x)})(x)\,(\bar{d}\gamma_5 u)(0)\rangle = -\langle 0|P(0,\mathbf{x})e^{-Hx_0}P(0,\mathbf{0})|0\rangle$$

$$= -\sum_{\mathrm{PS}} \langle 0|P(0,\mathbf{x})e^{-Hx_0}|\mathrm{PS}\rangle\langle \mathrm{PS}|P(0,\mathbf{0})|0\rangle$$

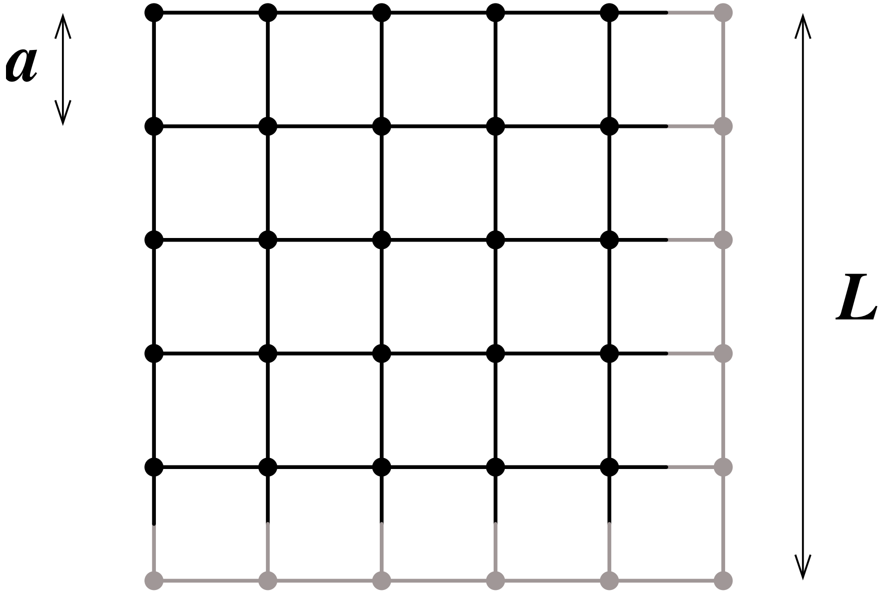
$$= -e^{-M_\pi x_0}|\langle 0|P(x)|\pi\rangle|^2 + \mathcal{O}(e^{-3M_\pi x_0})$$



computation of hadron masses, simple hadronic matrix elements, ... does not require analytic continuation back to Minkowski space

free matter fields on a lattice

replace Euclidean spacetime by 4-dimensional hypercubic lattice



$x = a\,(n_0, n_1, n_2, n_3)\,, \qquad n_\mu \in \mathbb{Z}$

$a \equiv$ lattice spacing

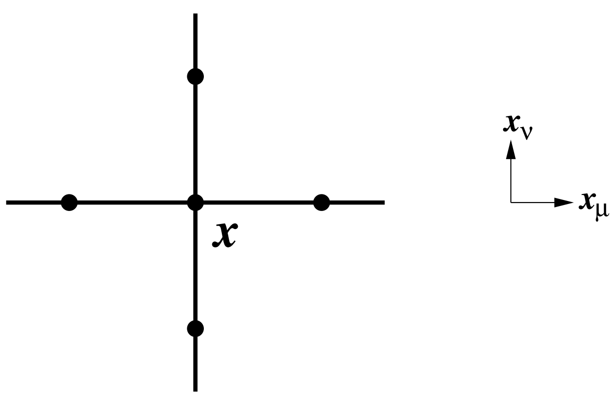
$L/a \equiv$ lattice size

$L \leq \infty$, possibly $T \neq L$

lattice "derivatives" (difference operators):

$$\partial_\mu f(x) = \frac{1}{a}\{f(x + a\hat{\mu}) - f(x)\}$$

$$\partial^*_\mu f(x) = \frac{1}{a}\{f(x) - f(x - a\hat{\mu})\}$$



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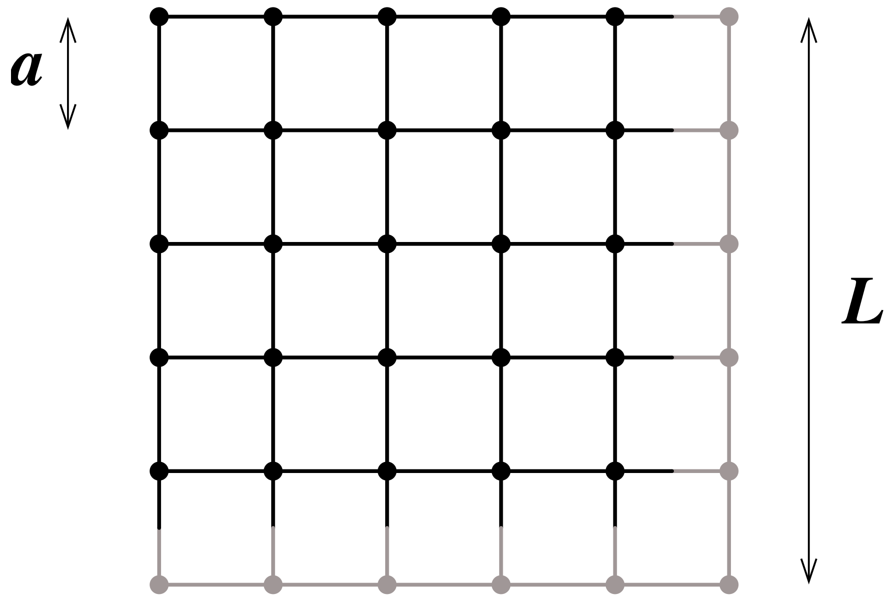

$$x = a\,(n_0, n_1, n_2, n_3)\,, \qquad n_\mu \in \mathbb{Z}$$

$a \equiv$ lattice spacing

$L/a \equiv$ lattice size

$L \leq \infty$, possibly $T \neq L$

Fourier transform:

momentum cutoff

$$\tilde{f}(p) = a^4 \sum_x e^{-ipx} f(x) \quad \Leftrightarrow \quad f(x) = \int_{-\pi/a}^{+\pi/a} \frac{\mathrm{d}^4 p}{(2\pi)^4}\, e^{ipx} \tilde{f}(p)$$

$$\frac{1}{2}(\partial_\mu^* + \partial_\mu) \to \frac{i}{a}\sin(ap_\mu) \equiv i\mathring{p}_\mu\,; \qquad \partial_\mu^*\partial_\mu \to -\hat{p}_\mu\hat{p}_\mu\,, \quad \hat{p}_\mu \equiv \frac{2}{a}\sin\left(\frac{ap_\mu}{2}\right)$$

free matter fields on a lattice

action for a free real scalar field:

$$S_{\mathrm{cont}} = \int \mathrm{d}^4x \left\{ \frac{1}{2} (\partial_\mu \phi)^2 + \frac{m^2}{2} \phi^2 \right\}$$

$$\downarrow$$

$$S_{\mathrm{latt}} = a^4 \sum_x \left\{ \frac{1}{2} \left[\tfrac{1}{2}(\partial_\mu^* + \partial_\mu)\phi \right]^2 + \frac{m^2}{2} \phi^2 \right\}$$

$$= a^4 \sum_{x,\mu} \frac{1}{4a^2} \left[\phi(x + a\hat{\mu}) - \phi(x - a\hat{\mu}) \right]^2 + a^4 \sum_x \frac{m^2}{2} \phi(x)^2$$

free matter fields on a lattice

action for a free (Dirac) fermion field:

$$S_{\mathrm{cont}} = \int \mathrm{d}^4x\,\bar{\psi}(x)\,[\gamma_\mu \partial_\mu + m]\,\psi(x)\,; \qquad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}\,, \quad \gamma_\mu^\dagger = \gamma_\mu$$

$$\downarrow$$

$$S_{\mathrm{latt}} = a^4 \sum_x \bar{\psi}(x)\,\{\tfrac{1}{2}\,[\gamma_\mu(\partial_\mu^* + \partial_\mu) \qquad\qquad] + m\}\,\psi(x)$$

free matter fields on a lattice

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$$\downarrow$$

$$S_{\rm latt} = a^4 \sum_x \bar{\psi}(x)\, \{\tfrac{1}{2}\, [\gamma_\mu(\partial_\mu^* + \partial_\mu) - a\partial_\mu^*\partial_\mu] + m\}\, \psi(x)$$

$$D_{\rm w} = \sum_\mu \frac{1}{2}\, \{\gamma_\mu(\partial_\mu^* + \partial_\mu) - a\partial_\mu^*\partial_\mu\}$$

Wilson-Dirac operator

$$\to i\gamma_\mu \mathring{p}_\mu + \tfrac{1}{2} a\hat{p}^2$$

(n.b. it is a very large, sparse matrix!)

free matter fields on a lattice

quark field two-point function — continuum:

$$\langle\psi(x)\bar{\psi}(0)\rangle = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\,\frac{e^{ipx}}{i\gamma_\mu p_\mu + m} \quad\longleftrightarrow\quad (\gamma_\mu\partial_\mu + m)\langle\psi(x)\bar{\psi}(0)\rangle = \delta(x)$$

quark field two-point function — lattice:

$$\langle\psi(x)\bar{\psi}(0)\rangle = \int_{-\frac{\pi}{a}}^{+\frac{\pi}{a}} \frac{\mathrm{d}^4p}{(2\pi)^4}\,\frac{e^{ipx}}{i\gamma_\mu \mathring{p}_\mu + \frac{1}{2}a\hat{p}^2 + m} \quad\longleftrightarrow\quad (D_{\mathrm{w}} + m)\langle\psi(x)\bar{\psi}(0)\rangle = a^{-4}\delta_{x0}$$

[Källén-Lehmann representation]

$$= \int_{-\frac{\pi}{a}}^{+\frac{\pi}{a}} \frac{\mathrm{d}^3p}{(2\pi)^3}\, e^{-\epsilon_{\mathbf{p}}x_0 + i\mathbf{p}\cdot\mathbf{x}}\,\rho_{\mathbf{p}}$$

$$\epsilon_{\mathbf{p}} = \frac{2}{a}\,\mathrm{asinh}\left\{\frac{a}{2}\sqrt{\frac{\mathring{\mathbf{p}}^2 + m_{\mathbf{p}}^2}{1 + am_{\mathbf{p}}}}\right\}$$

$m_{\mathbf{p}} \equiv m + \frac{1}{2}a\hat{\mathbf{p}}^2\,,\quad \rho_{\mathbf{p}} =$ spectral density

free matter fields on a lattice

Taking the (naive) continuum limit:

- the lattice spacing simply sets the scale ("standard ruler"):

$$\Phi(a, m, p, \ldots) = a^{d_\Phi}\, \Phi(1, am, ap, \ldots)$$

- therefore, the CL is obtained by setting all physical scales far away from the lattice spacing,

$$m \ll a^{-1}\,, \quad |p| \ll a^{-1}\,, \quad |x| \gg a\,, \quad \ldots$$

- one can check e.g. that the Wilson-Dirac propagator has the correct CL:

$$\epsilon_{\mathbf{p}} = \sqrt{m^2 + \mathbf{p}^2} + \mathcal{O}(am, a\mathbf{p})$$

$$\rho_{\mathbf{p}} = \left.\frac{i\gamma_\mu p_\mu - m}{2ip_0}\right|_{p_0 = i\sqrt{m^2+\mathbf{p}^2}} + \mathcal{O}(am, a\mathbf{p})$$

free matter fields on a lattice

Taking the (naive) continuum limit:

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- therefore, the CL is obtained by setting all physical scales far away from the lattice spacing,

$$m \ll a^{-1}\,, \quad |p| \ll a^{-1}\,, \quad |x| \gg a\,, \quad \ldots$$

taking the CL in an interacting theory will be much more complicated:

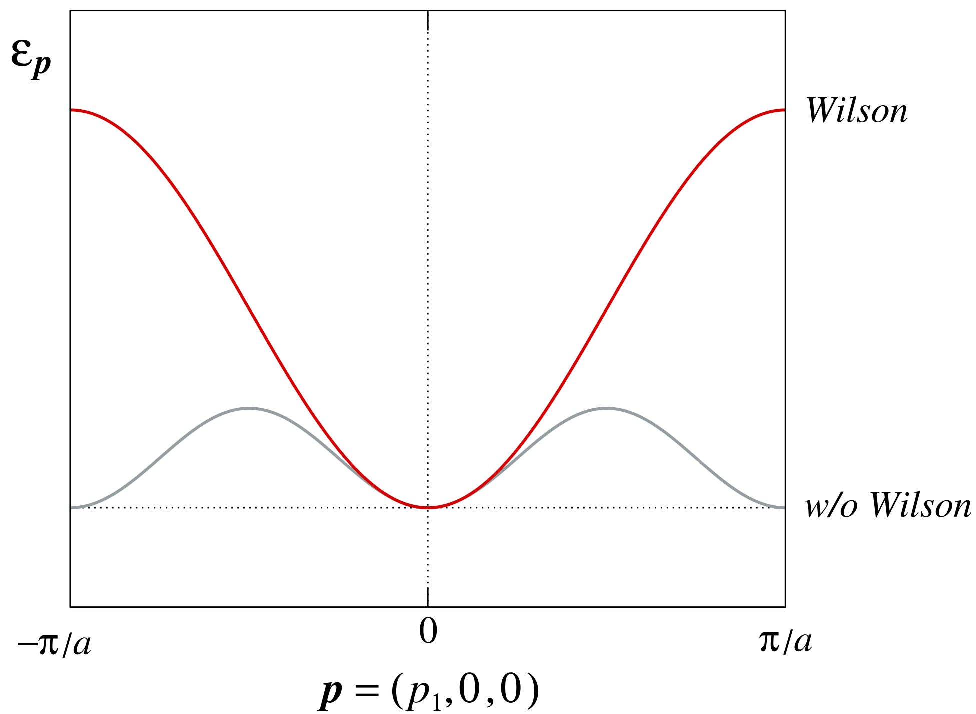
- the lattice spacing will depend on dynamical quantities (e.g. in a gauge theory it will be related to the gauge coupling)
- in the presence of interactions, all couplings and correlation functions will require renormalisation (unless protected by symmetries)

free matter fields on a lattice

so, why did we introduce the Wilson term?

$$D_\mathrm{w} = \sum_\mu \frac{1}{2} \left\{ \gamma_\mu (\partial_\mu^* + \partial_\mu) - a \partial_\mu^* \partial_\mu \right\}$$

- irrelevant in CL (disappears as $a \to 0$) ...
- ... but breaks chiral symmetry at $a \neq 0$!



wrong continuum limit without Wilson term: additional massless states ("doublers") with energy $\ll \pi/a$!

free matter fields on a lattice

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- ... but breaks chiral symmetry at $a \neq 0$!

N.B.: several other actions for lattice fermions exist, that treat the doubling/chiral symmetry breaking problem in various ways — including the exact preservation of (a generalised form of) chiral symmetry — which in turn provides insight into the very nature of the latter.

- staggered (Kogut-Susskind) fermions
- Wilson twisted-mass QCD
- Ginsparg-Wilson fermions (Neuberger, domain wall, fixed-point ...)
- ...

gauge fields on a lattice

gauge transformations and covariant derivatives in continuum theory:

$$\psi(x) \;\rightarrow\; \Lambda(x)\psi(x)\,, \quad \Lambda(x) \in \mathrm{SU}(N)$$

$$D_\mu \psi = (\partial_\mu - iA_\mu)\psi\,, \qquad A_\mu \;\rightarrow\; \Lambda A_\mu \Lambda^\dagger + i\Lambda\partial_\mu \Lambda^\dagger$$

the gauge potential provides a connection between colour spaces at infinitesimally separated points in spacetime

gauge fields on a lattice

on the lattice:

$$\partial_\mu f(x) = \frac{1}{a}\{f(x+a\hat{\mu}) - f(x)\} \;\rightarrow\; \frac{1}{a}\{\Lambda(x+a\hat{\mu})f(x+a\hat{\mu}) - \Lambda(x)f(x)\}$$

$$U_\mu(x) \;\rightarrow\; \Lambda(x)U_\mu(x)\Lambda(x+a\hat{\mu})^\dagger\,, \qquad U_\mu(x) \in \mathrm{SU}(3)$$ colour transport

$$\nabla_\mu f(x) \equiv \frac{1}{a}\{U_\mu(x)f(x+a\hat{\mu}) - f(x)\} \;\rightarrow\; \Lambda(x)\nabla_\mu f(x)$$ covariant diff. op.

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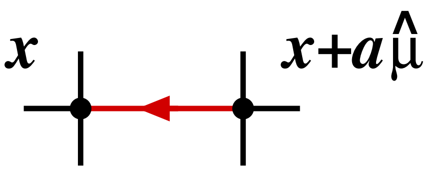
similarly, define a covariant backward derivative and a covariant Wilson-Dirac operator:

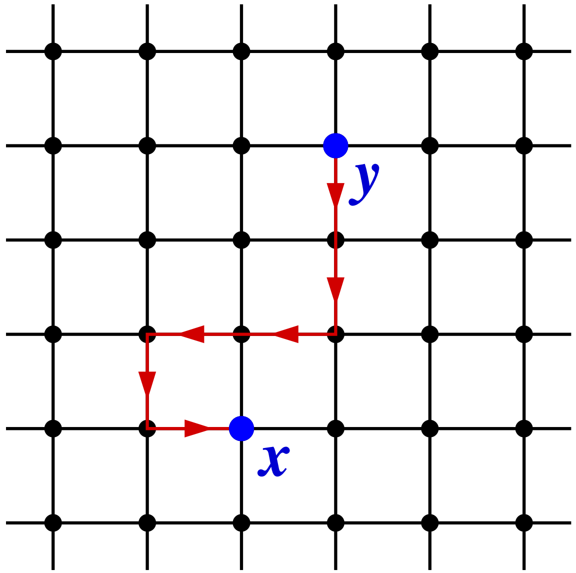
$$\nabla^*_\mu f(x) \equiv \frac{1}{a}\{f(x)U_\mu(x-a\hat{\mu})^\dagger f(x-a\hat{\mu})\}$$

$$D_\mathrm{w} = \frac{1}{2}\sum_\mu \{\gamma_\mu(\nabla^*_\mu + \nabla_\mu) - a\nabla^*_\mu\nabla_\mu\}$$

an $\mathrm{SU}(N)$ lattice gauge field is an assignment of an $\mathrm{SU}(N)$ matrix $U_\mu(x)$ to every link on the lattice



gauge fields on a lattice



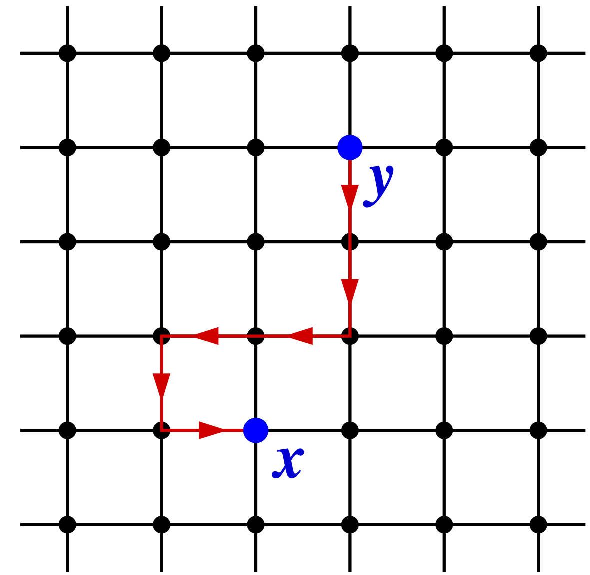
Wilson lines: any path-ordered product of gauge links is gauge covariant

$$U(x,y;\mathcal{P}) \;\rightarrow\; \Lambda(x)U(x,y;\mathcal{P})\Lambda(y)^\dagger$$

Wilson loops: the trace of a closed loop is gauge invariant

$$W(\mathcal{P}) = \mathrm{tr}[U(x,x;\mathcal{P})] \;\rightarrow\; W(\mathcal{P})$$

gauge fields on a lattice



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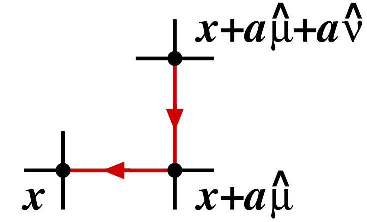
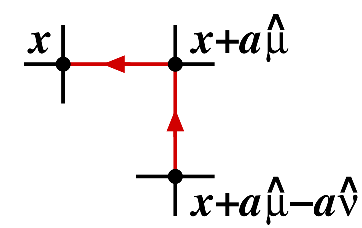
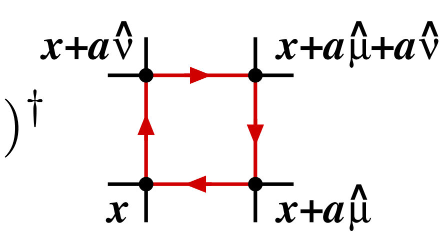
$$U_\mu(x)U_\nu(x+a\hat{\mu})$$



$$U_\mu(x)U_\nu(x+a\hat{\mu}-a\hat{\nu})^\dagger$$



plaquette loop: $U_\mu(x)U_\nu(x+a\hat{\mu})U_\mu(x+a\hat{\nu})^\dagger U_\nu(x)^\dagger$



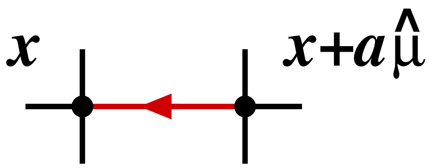
gauge fields on a lattice

classical continuum limit: how can we ...

- connect link variables to continuum gauge potential?
- construct an action that reduces to the correct classical Yang-Mills theory in the continuum limit?

links = continuum Wilson lines (parallel transport) along corresponding paths

$$U_\mu(x) = \mathrm{P}\exp\left\{ia\int_0^1 \mathrm{d}t\, A_\mu(x+(1-t)a\hat{\mu})\right\}$$
$$= \mathbf{1} + iaA_\mu(x) + \mathcal{O}(a^2)$$



mapping $A \rightarrow U$ uniquely defined, gauge covariant

gauge fields on a lattice

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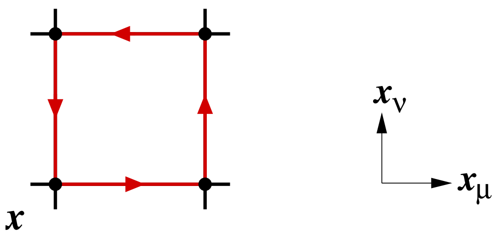
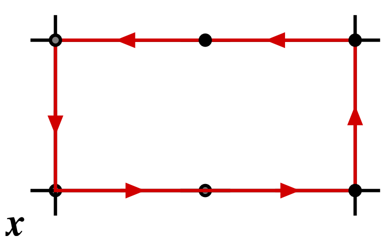


$$P_{\mu\nu}(x) = \mathrm{Re}\,\mathrm{tr}\{\mathbf{1} - U(x,x;\square)\}$$
$$= -\tfrac{1}{2}a^4\mathrm{tr}\{F_{\mu\nu}(x)F_{\mu\nu}(x)\} + \mathcal{O}(a^5)$$

[different!]



$$R_{\mu\nu}(x) = \mathrm{Re}\,\mathrm{tr}\{\mathbf{1} - U(x,x;\boxed{})\}$$
$$= -2a^4\mathrm{tr}\{F_{\mu\nu}(x)F_{\mu\nu}(x)\} + \mathcal{O}(a^5)$$

any gauge-invariant, local continuum field can be represented in the lattice; however, the representation is not unique.

lattice QCD

now we know how to construct gauge-invariant operators involving both fermion and gauge fields on the lattice

$$\bar{\psi}\psi\,, \qquad \bar{\psi}\gamma_5\tau^a\psi\,, \qquad \ldots$$

$$\mathrm{tr}P_{\mu\nu}(x)\,, \qquad \mathrm{tr}R_{\mu\nu}(x)\,, \qquad \ldots$$

$$\bar{\psi}\gamma_\mu(\nabla^*_\mu + \nabla_\mu)\psi\,, \qquad \bar{\psi}\nabla_\mu\nabla_\nu\psi\,, \qquad \ldots$$

classical continuum limit well understood for gauge and free fermion fields: lattice fields can be classified by their leading behaviour in the CL

$$O(x) \underset{a\to 0}{\sim} \sum_{n\geq 0} a^n O_n(x)$$

combine invariants into action that becomes SU(N) gauge theory in classical CL

lattice QCD

Wilson action:

$$S = S_{\rm G} + S_{\rm F}$$

$$S_{\rm G} = \frac{1}{g_0^2} \sum_{x;\mu,\nu} P_{\mu\nu}(x)$$

$P_{\mu\nu}(x) =$ plaquette field

$$S_{\rm F} = a^4 \sum_{x} \bar{\psi}(x)(D_{\rm w} + M)\psi(x)$$

$D_{\rm w} =$ Wilson-Dirac operator (with SU(3) covariant derivatives)

$M =$ quark mass matrix

[Wilson 1974]

- infinitely many lattice actions with correct continuum limit can be written
- extra terms can be tuned to control approach to continuum limit (e.g. depressing the subleading terms in $a \equiv$ cutoff effects)

  $\Rightarrow$ Symanzik improvement programme

quantisation of lattice QCD

employ path integral formalism: Euclidean correlation function of *n* gauge-invariant fields given by

$$\langle \phi_1(x_1)\cdots\phi_n(x_n)\rangle = \frac{1}{\mathcal{Z}}\int D[U]\int D[\psi]D[\bar\psi]e^{-S[U,\bar\psi,\psi]}\,\phi_1(x_1)\cdots\phi_n(x_n)$$

$\psi, \bar\psi$ indep. variables in Euclidean qft

LQCD action

$$\mathcal{Z} = \int D[U]\int D[\psi]D[\bar\psi]e^{-S[U,\bar\psi,\psi]}$$

- the functional integral is the definition of the quantum theory
- the integration measures $D[U], D[\psi]D[\bar\psi]$ are local and mostly determined by symmetry
- provided basic properties (locality, gauge symmetry, ...) are respected, we expect good behaviour as $a \to 0$

quantisation of lattice QCD

employ path integral formalism: Euclidean correlation function of *n* gauge-invariant fields given by

$$\langle \phi_1(x_1)\cdots\phi_n(x_n)\rangle = \frac{1}{\mathcal{Z}}\int D[U]\int D[\psi]D[\bar{\psi}]e^{-S[U,\bar{\psi},\psi]}\,\phi_1(x_1)\cdots\phi_n(x_n)$$

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LQCD action

crucial: on a lattice, this is a standard integral (over a very large number of variables)

$$D[\phi] = \prod_x \mathrm{d}\phi(x)$$

$$D[U] = \prod_{x,\nu} \mu_{\mathrm{Haar}}[U_\nu(x)]$$

$$D[\psi]D[\bar{\psi}] = \prod_x \mathrm{d}c(x)\mathrm{d}\bar{c}(x)$$

generators of Grassmann algebra (for each fermionic d.o.f.) at spacetime point *x*

quantisation of lattice QCD

integration over fermion fields can be done explicitly, since the action is a bilinear

$$\mathcal{Z}_{\rm F} = \int D[\psi]D[\bar{\psi}]\,\exp\left\{-a^4\sum_x \bar{\psi}(x)[D_{\rm w}+M]\psi(x)\right\}$$

$$= \det(D_{\rm w}+M) = \prod_{q=1}^{N_f}\det(D_{\rm w}+m_q)$$

(up to a power of a)

[n.b.: the Wilson-Dirac operator is a function of the gauge field]

quark propagator and correlation functions involving fermion fields:

$$(D_{\rm w}+M)\,S(x,y;U) = a^{-4}\delta_{xy}$$

$$\langle\psi(x)\bar{\psi}(y)\rangle_{\rm F} = S(x,y;U)$$

$$\langle\psi(x_1)\bar{\psi}(y_1)\psi(x_2)\bar{\psi}(y_2)\rangle_{\rm F} = S(x_1,y_1;U)\,S(x_2,y_2;U)\;-\;[\mathrm{perm}]$$

quantisation of lattice QCD

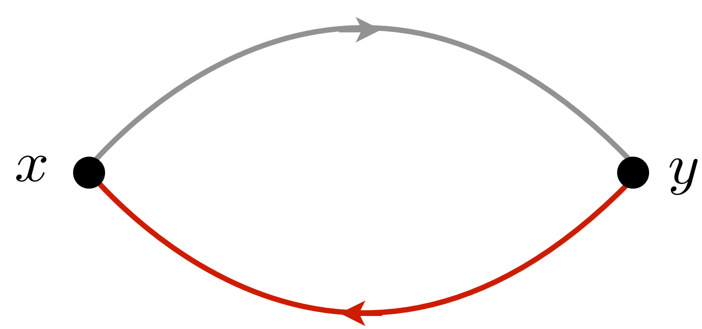
⇒ in QCD functional integral quark fields can be integrated out completely

$$\langle \phi_1(x_1)\cdots\phi_n(x_n)\rangle = \frac{1}{\mathcal{Z}}\int D[U]\langle\phi_1(x_1)\cdots\phi_n(x_n)\rangle_{\mathrm{F}} \times$$

$$\times \prod_{q=1}^{N_f} \det[D_{\mathrm{w}}(U)+m_q]e^{-S_{\mathrm{G}}[U]}$$

for instance, the charged pion propagator can be obtained from a purely bosonic integral:

$$\langle(\bar{u}\gamma_5 d)(x)(\bar{d}\gamma_5 u)(y)\rangle_{\mathrm{F}} = -\mathrm{tr}\,\{\gamma_5\, S(x,y;U)_d\,\gamma_5\, S(y,x;U)_u\}$$



definition of Wilson lattice QCD theory completed by integrating over the gauge field (integration well-defined by considering Haar measure for each link)

lattice QCD: basic properties

1. regularity

in finite volume:

- the space of all gauge fields is compact
- after fermions are integrated out, one is normally left with a well-behaved integrand
- the partition function is positive

$\Rightarrow$ correlation functions are completely well-defined

$\Rightarrow$ lattice QCD provides a non-perturbative regularisation of QCD

lattice QCD: basic properties

1. regularity

2. gauge invariance

for any observable and (regular) gauge function Λ

$$\langle O \rangle = \langle O^{\Lambda} \rangle \,, \qquad O^{\Lambda}[U, \psi, \bar{\psi}] = O[U^{\Lambda}, \bar{\psi}^{\Lambda}, \psi^{\Lambda}]$$

expectation values of non-invariant quantities naturally vanish; e.g.

$$\langle \psi(x) \bar{\psi}(y) \rangle = \Lambda(x) \langle \psi(x) \bar{\psi}(y) \rangle \Lambda(y)^{\dagger} = 0 \quad \text{if} \quad x \neq y$$

gauge invariance is fully respected by the regulator, and there is no need of gauge fixing (although it may be convenient in some computations)

lattice QCD: basic properties

1. regularity

2. gauge invariance

3. spacetime symmetries

Poincaré symmetry is broken: correlation functions are only invariant under a discrete subgroup of the full Poincaré group

- translations by lattice vectors
- rotations in the four-dimensional hypercubic group
- parity, time reversal, and charge conjugation (although some of them are broken/modified by some fermion actions)

lattice QCD: basic properties

1. regularity

2. gauge invariance

3. spacetime symmetries

4. global (flavour) symmetries

   the vector $\mathrm{U}(N_f)$ symmetry works as in the continuum

   the axial symmetry is explicitly broken (Wilson term); it can be recovered in the CL by properly tuning counterterms that restore AWI's, which is feasible but amounts to quite some amount of non-trivial work

   [other fermion regularisations preserve more axial symmetry; trade breakings between vector and axial symmetries; or preserve chiral symmetry altogether; always at the price of other complications (no-free-lunch theorem)]

lattice QCD: basic properties

1. regularity

2. gauge invariance

3. spacetime symmetries

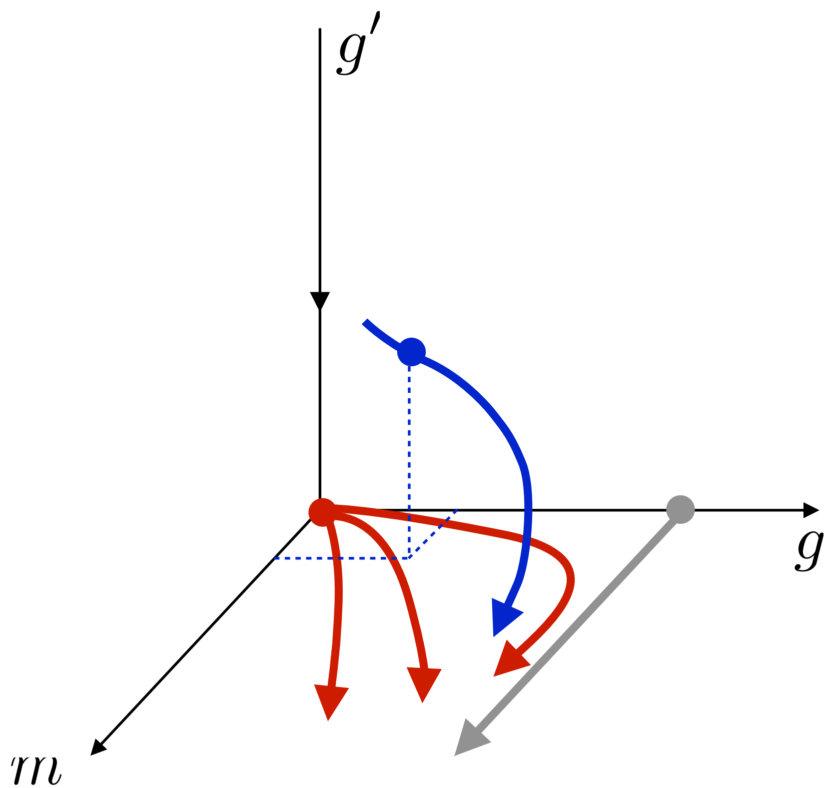
4. global (flavour) symmetries

5. unitarity

can be shown to hold rigorously in the Wilson theory; more sophisticated lattice regularisations typically involve harmless short-distance violations

the continuum limit

in order to obtain fully physical results, the cutoff has to be removed: this is accomplished by taking the CL $a \to 0$ in the interacting quantum theory



once QCD has been regularised on a lattice, the result is a statistical mechanical system: the UV divergences expected as the cutoff is removed, which will require renormalisation, adopt the form of critical behaviour.

the continuum renormalised quantum theory appears as a 2nd-order phase transition in the CL

how does the lattice spacing relate to physical scales?

the continuum limit

how is the lattice spacing fixed?

consider two-flavour QCD in the isospin limit:

$$M = \begin{pmatrix} m_0 & 0 \\ 0 & m_0 \end{pmatrix} \; ; \qquad m_0 = \text{bare mass of } u,d \text{ quarks}$$

the parameters in the lattice action are $g_0, \; am_0$ and a— which disappears if quark fields are rescaled as $\psi \to a^{-3/2}\psi \, , \; \bar{\psi} \to a^{-3/2}\bar{\psi}$

fixing the two parameters in the action thus requires computing two physical observables — e.g. the pion and proton masses

$$O_\pi(x) = \bar{u}(x)\gamma_5 d(x) \quad \to \quad a^3 \sum_{\mathbf{x}} \langle O_\pi(x) O_\pi(0) \rangle \underset{x_0 \to \infty}{\sim} e^{-M_\pi x_0}$$

$$O_p(x) = \epsilon_{\alpha\beta\gamma}(d_a^T C\gamma_5 u_\beta)u_\gamma \quad \to \quad a^3 \sum_{\mathbf{x}} \langle O_p(x) O_p(0) \rangle \underset{x_0 \to \infty}{\sim} e^{-M_p x_0}$$

$x_0 = na \, , \; n = 0, 1, 2, \ldots \quad \Rightarrow \quad aM_\pi, aM_p$ are obtained

the continuum limit

one thus has

$$aM_\pi = f_\pi(g_0, am_0)$$

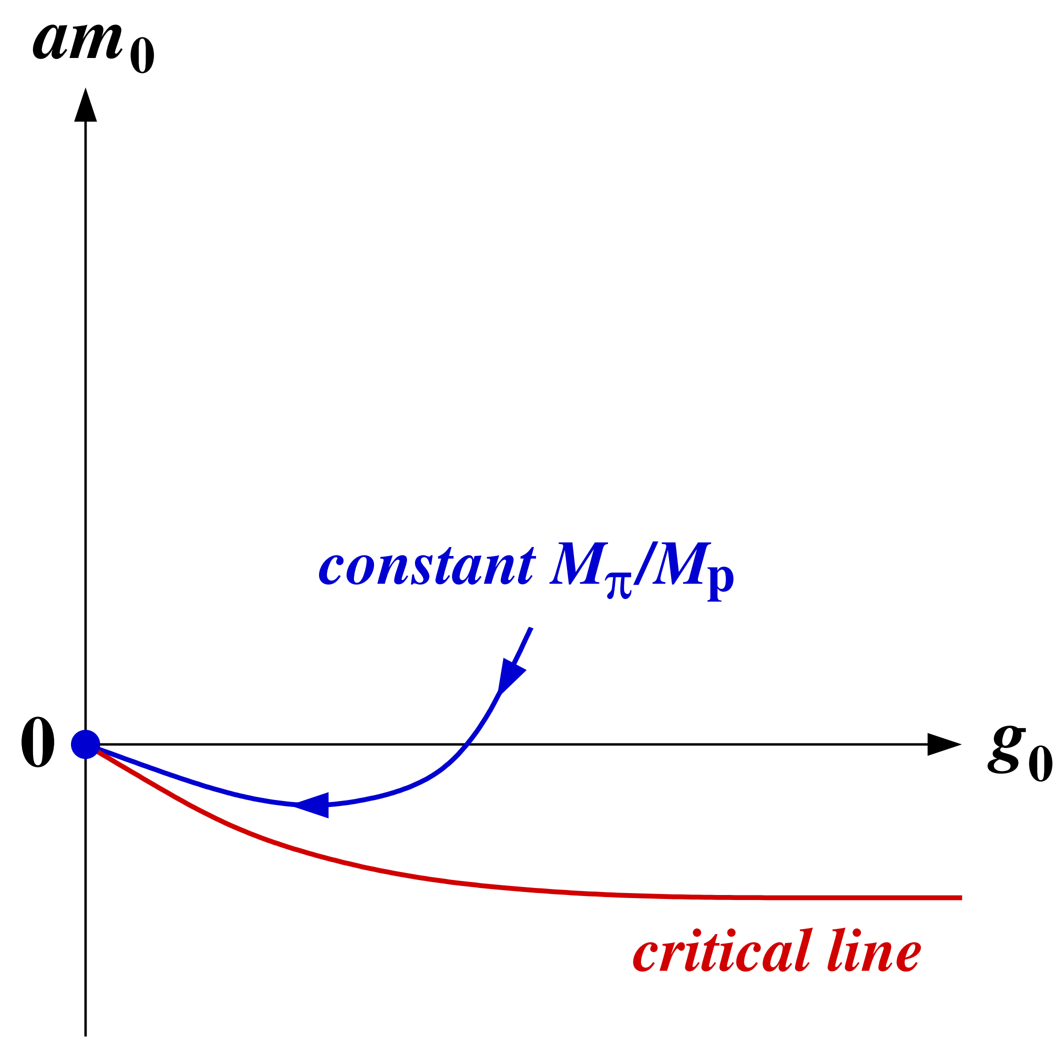
$$aM_p = f_p(g_0, am_0)$$

taking the continuum limit means

$$M_\pi, M_p \ll a^{-1}$$

keeping the ratio M_π/M_p fixed
(constant physics!)

equivalent to taking $g_0, am_0 \to 0$
asymptotic freedom!



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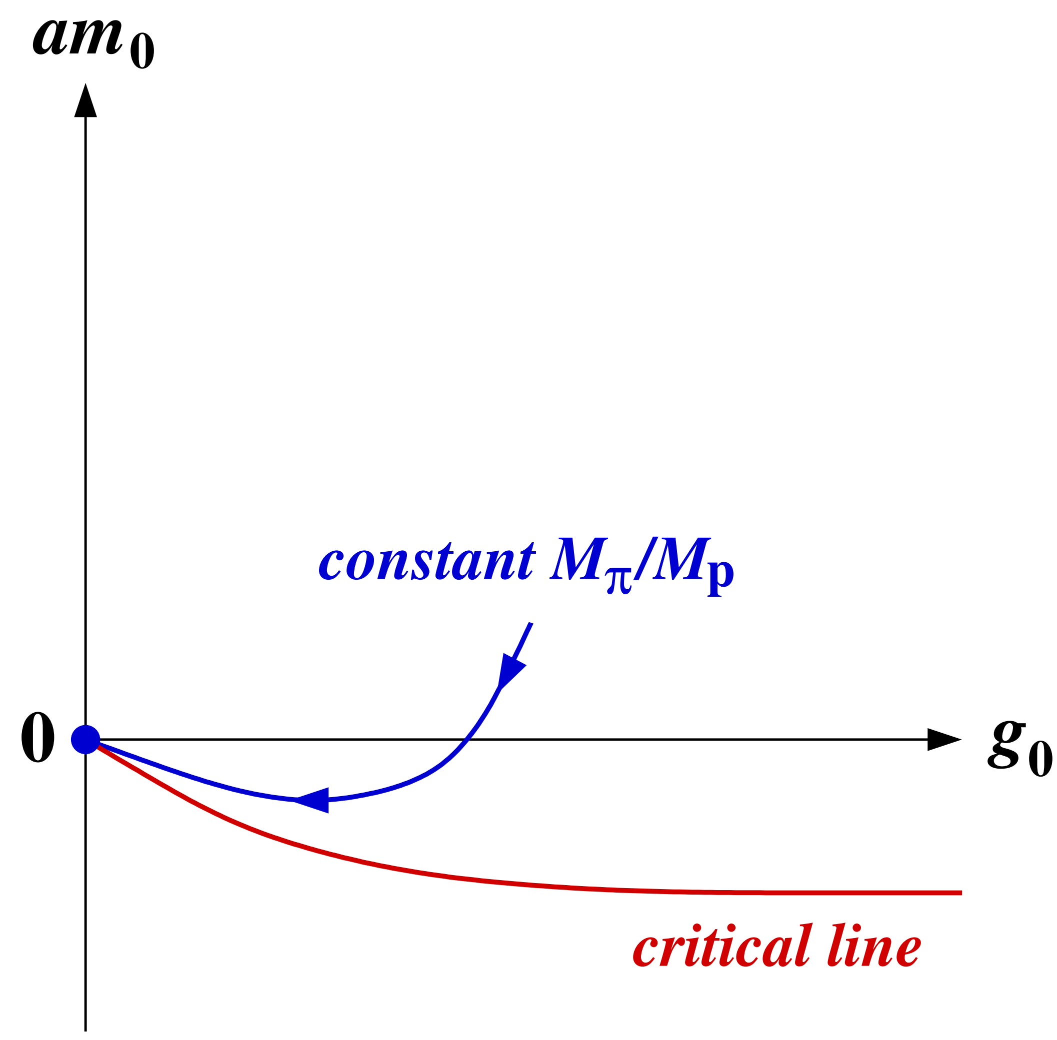

- setting $M_p = 938\ \mathrm{MeV}$ gives $a = \dfrac{aM_p}{M_p} = 0.21 \times aM_p\ \mathrm{fm}$ along trajectory
- other physical scales can be used: $a[\mathrm{fm}]$ slightly convention-dependent

the continuum limit

one thus has

$$aM_\pi = f_\pi(g_0, am_0)$$

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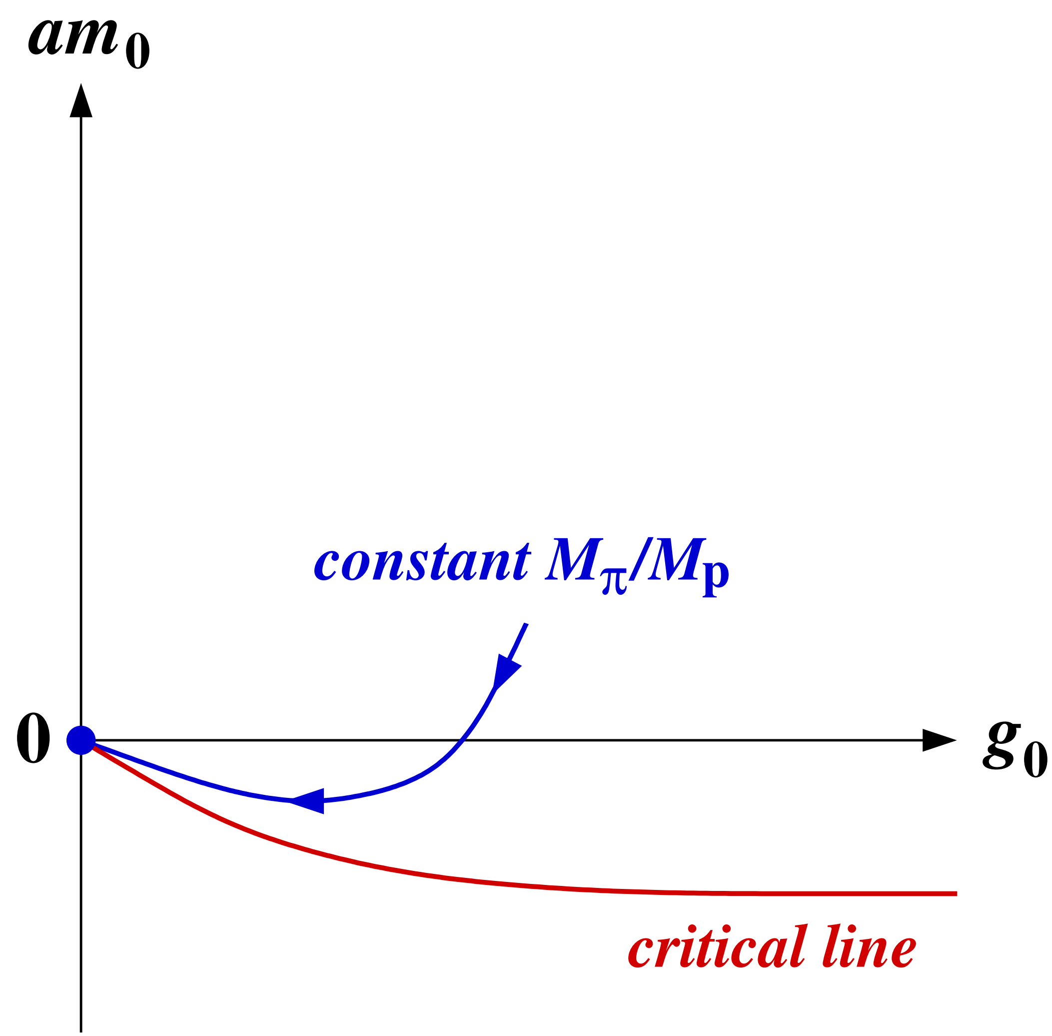


renormalisation of couplings and other divergent quantities still required

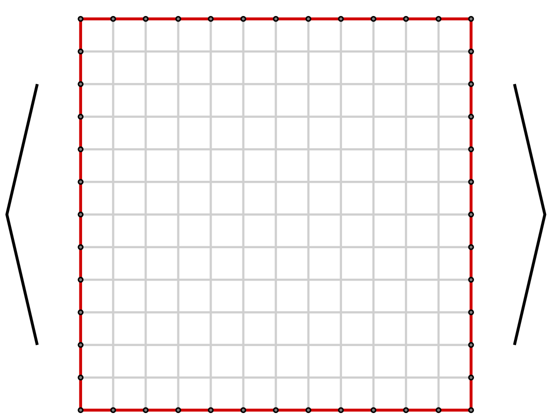
analytical tools: strong coupling expansion

$$\psi \to a^{-2} m_0^{-1/2} \psi \,, \qquad \bar\psi \to a^{-2} m_0^{-1/2} \bar\psi$$

$$S = \sum_{\mathbf{x}} \left\{ \bar\psi(x)\psi(x) \;+\; \frac{1}{m_0}\,\bar\psi(x) D_{\mathrm{w}} \psi(x) \;+\; \frac{1}{g_0^2} \sum_{\mu\nu} P_{\mu\nu}(x) \right\}$$

⇒ simple expansion around $\dfrac{1}{m_0}\,,\,\dfrac{1}{g_0^2} \to 0$

⇒ simple picture of confinement at $m_0 \to \infty$



$$\sim (1/g_0^2)^{N_{\mathrm{plaq}}} = \exp\{-\sigma \,\times\, \mathrm{area}\}$$

N.B.: not really physical — recall continuum theory is realised as $g_0 \to 0$

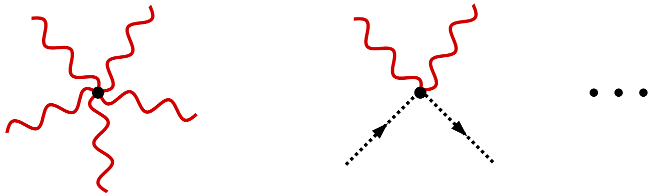
analytical tools: weak coupling expansion

perturbation theory can be defined as usual, by expanding in g_0 around free fields

$$\langle \phi_1(x_1) \cdots \phi_n(x_n) \rangle \sim \sum_{k=0}^{\infty} g_0^{2k}\, C_k(x_1, \ldots, x_n)$$

(sum over Feynman diagrams)

Lorentz symmetry breaking generates new vertices proportional to $a^{n \geq 1}$



useful to

- make contact with other regularisations (needed e.g. to match high-energy observables)
- study approach to continuum limit: recall the latter is realised at $g_0 \to 0$
- obtain formal results (e.g. Reisz's theorem: lattice QCD rigorously proven to be renormalisable at all orders in perturbation theory)

renormalisation

renormalisation of couplings and other divergent quantities still required

easy to study in perturbation theory

$$a\,\frac{\partial g_0}{\partial a} = \beta(g_0(a)) \approx -g_0^3(b_0 + b_1 g_0^2 + \ldots)\,, \qquad b_0 = -\frac{1}{(4\pi)^2}\{11 - \tfrac{2}{3}N_f\}$$

$$\Rightarrow \quad g_0^2 \underset{a\to 0}{\sim} \frac{1}{b_0 \ln(a\mu)} + \ldots$$

RG equations do however hold beyond perturbation theory, and frameworks exist to work out the renormalisation of QCD (and other strongly coupled gauge theories) non-perturbatively

- Schrödinger Functional
- RI/MOM
- ...

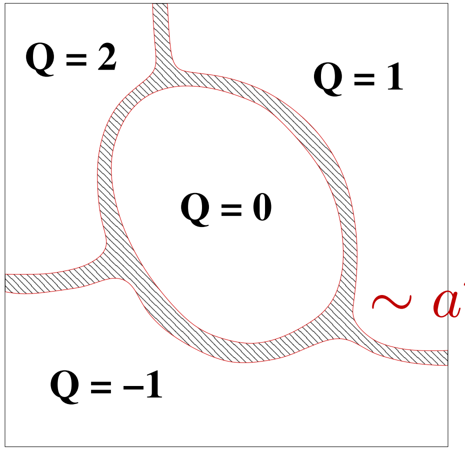
finite volume

lattice field theories are usually formulated in a finite volume, which raises subtleties:

- boundary conditions: note that even periodic boundary conditions are non-trivial — fields with gauge d.o.f. periodic only up to gauge transformations that satisfy cocycle condition.

$$A_\mu(x+L\hat{\nu}) = \Omega_\nu(x)A_\mu(x)\Omega_\nu(x)^\dagger + i\Omega_\nu(x)\partial_\mu\Omega_\nu(x)^\dagger$$

- in the continuum gauge fields have topological structure, and can be classified by their topological charge (instanton number) — but on the lattice gauge topological sectors become connected.



$\sim a^4$

- finite volume ⇒ periodicity structure, finite volume effects in Euclidean correlators ⇒ difficulties (some severe) to get V=∞ physics.

$$|h_1\rangle \rightarrow |h_1' \ldots h_n'\rangle \;???$$

- ...

outline

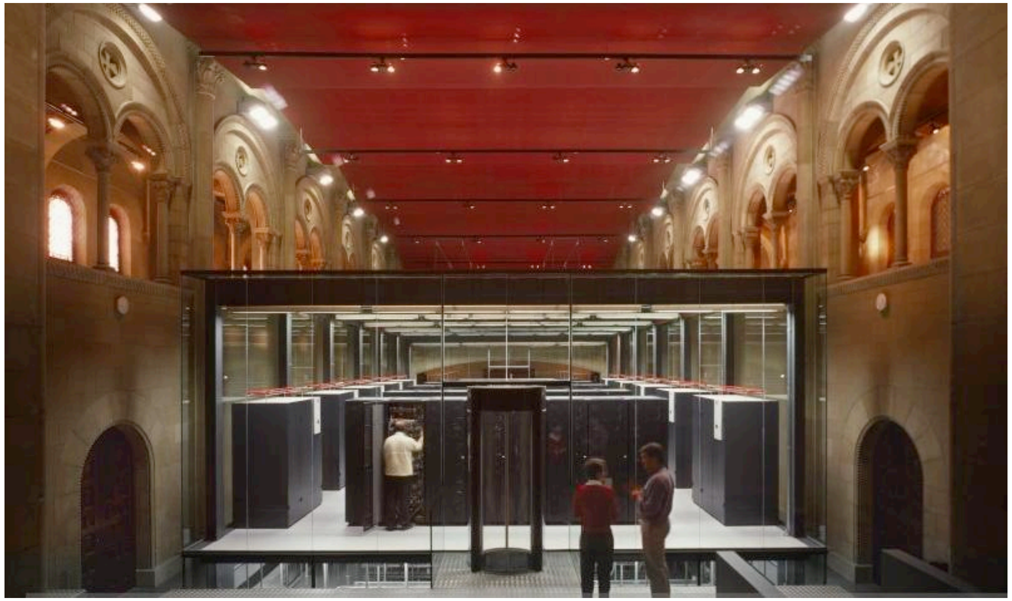
- motivation: strong interaction(s) and non-perturbative physics

- lattice field theory
  - QFT in Euclidean space
  - matter and gauge fields on a lattice
  - interacting gauge theories on a lattice: QCD

- numerical aspects
  - Monte Carlo techniques for non-perturbative QFT
  - reach of QCD computations
  - anatomy of an example

- overview of physics capabilities
  - FLAG
  - selected lattice QCD results
  - beyond the SM

numerical simulations



numerical simulations

employ path integral formalism: Euclidean correlation function of *n* gauge-invariant fields given by

$$\langle \phi_1(x_1)\cdots\phi_n(x_n)\rangle = \frac{1}{\mathcal{Z}}\int D[U]\int D[\psi]D[\bar{\psi}]e^{-S[U,\bar{\psi},\psi]}\,\phi_1(x_1)\cdots\phi_n(x_n)$$

$\psi, \bar{\psi}$ indep. variables in Euclidean qft

crucial: on a lattice, this is a standard integral (over a very large number of variables)

$$D[\phi] = \prod_x \mathrm{d}\phi(x)$$

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generators of Grassmann algebra (for each fermionic d.o.f.) at spacetime point *x*

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$\psi, \bar{\psi}$ indep. variables in Euclidean qft

crucial: on a lattice, this is a standard integral (over a very large number of variables)

$$N_f = 2 + 1 + 1\,, \quad (L/a)^3 \times (T/a) = 64^3 \times 128$$

$$\Rightarrow \; D_{\mathrm{w}} = (1.6 \times 10^9)^2 \;\; \text{complex matrix}$$

untractable analytically: use numerical techniques to compute correlation functions

Monte Carlo integration

$\int_{\alpha}^{\beta} \mathrm{d}\,x\, f(x)$ Riemann integrability:

$$[x_0 = \alpha, x_1], [x_1, x_2], \ldots, [x_{N-1}, x_N = \beta]$$

$$\varepsilon(N) = \frac{\beta - \alpha}{N} \sum_{i=0}^{N-1} \left[\max_{x \in [x_i, x_{i+1}]} \{f(x)\} - \min_{x \in [x_i, x_{i+1}]} \{f(x)\} \right] \stackrel{N \to \infty}{\to} 0$$

yields approximation method for the integral:

$$\int_{\alpha}^{\beta} \mathrm{d}x\, f(x) = \sum_{i=0}^{N-1} \frac{\beta - \alpha}{N} f(x_i) + \mathcal{O}(f'/N^2)$$

Monte Carlo integration

$\int_\alpha^\beta \mathrm{d}\,x\, f(x)$ Riemann integrability:

$$[x_0 = \alpha, x_1], [x_1, x_2], \ldots, [x_{N-1}, x_N = \beta]$$

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Monte Carlo algorithm:

1. generate a set of N random points $\{x^{[i]}\}$uniformly distributed in the integration interval (domain)
2. compute $f(x^{[i]}) \ \forall\, i$
3. compute the average $I(N) = \frac{\beta - \alpha}{N} \sum_{i=0}^{N-1} f(x^{[i]})$

f is Riemann integrable $\Rightarrow \quad I(N) \overset{N \to \infty}{\to} \int_\alpha^\beta \mathrm{d}\,x\, f(x)$

Monte Carlo integration

basic MC technique easily generalisable to arbitrary number of variables, but path integrals are more complicated:

$$\langle \phi_1(x_1) \cdots \phi_n(x_n) \rangle = \int \frac{D\phi\, e^{-S[\phi]}}{\mathcal{Z}} \phi_1(x_1) \cdots \phi_n(x_n)$$

Monte Carlo integration

basic MC technique easily generalisable to arbitrary number of variables, but path integrals are more complicated:

$$\langle \phi_1(x_1)\cdots\phi_n(x_n)\rangle = \int \boxed{\frac{D\phi\, e^{-S[\phi]}}{\mathcal{Z}}}\,\phi_1(x_1)\cdots\phi_n(x_n)$$

integration measure

$$I = \int_0^1 \mathrm{d}x_0 \int_0^1 \mathrm{d}x_1 \cdots \int_0^1 \mathrm{d}x_{K-1}\, P(\mathbf{x})\, f(\mathbf{x})$$

points for MC cannot be uniformly distributed: they are distributed with weights given by measure

Monte Carlo integration

$$I = \int_0^1 \mathrm{d}x_0 \int_0^1 \mathrm{d}x_1 \cdots \int_0^1 \mathrm{d}x_{K-1}\, P(\mathbf{x})\, f(\mathbf{x})$$

1. generate a set of N random points $\{\mathbf{x}^{[i]}\}$ distributed with $P(\mathbf{x})$ in the integration domain $\mathcal{D}$
2. compute $f(\mathbf{x}^{[i]})\ \ \forall\, i$
3. compute the average $I(N) = \dfrac{\mathrm{Vol}(\mathcal{D})}{N} \displaystyle\sum_{i=0}^{N-1} f(\mathbf{x}^{[i]})$

convergence guaranteed under certain conditions by Central Limit theorem; in general, the convergence rate is $1/\sqrt{N}$

Monte Carlo integration

$$I = \int_0^1 \mathrm{d}x_0 \int_0^1 \mathrm{d}x_1 \cdots \int_0^1 \mathrm{d}x_{K-1}\, P(\mathbf{x})\, f(\mathbf{x})$$

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how to distribute points properly: choose them to be a Markov chain

sequence of random variables: $X_1, X_2, X_3, \ldots$

$$P(X_{n+1} = x | X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = P(X_{n+1} = x | X_n = x_n)$$

each step in the chain "knows" only of the immediately previous step

several standard algorithms available and optimised for lattice QCD computations

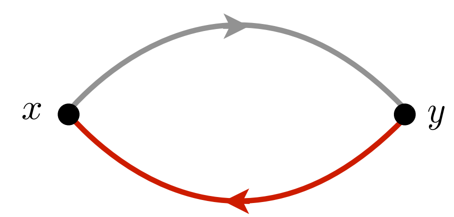
Monte Carlo integration

in practice: computational needs determined by inversion of lattice Dirac operator

$$\langle \phi_1(x_1) \cdots \phi_n(x_n) \rangle = \frac{1}{\mathcal{Z}} \int D[U] \langle \phi_1(x_1) \cdots \phi_n(x_n) \rangle_{\mathrm{F}} \times$$

$$\times \prod_{q=1}^{N_f} \det[D_{\mathrm{w}}(U) + m_q] e^{-S_{\mathrm{G}}[U]}$$

$$\langle (\bar{u}\gamma_5 d)(x)(\bar{d}\gamma_5 u)(y) \rangle_{\mathrm{F}} = -\mathrm{tr}\,\{\gamma_5\, S(x,y;U)_d\, \gamma_5\, S(y,x;U)_u\}$$



- quark propagators
- computation of the determinant

cost of computation $\longleftrightarrow$ condition number of lattice Dirac operator

Monte Carlo integration

reaching physical quarks masses difficult because of spontaneous chiral symmetry breaking

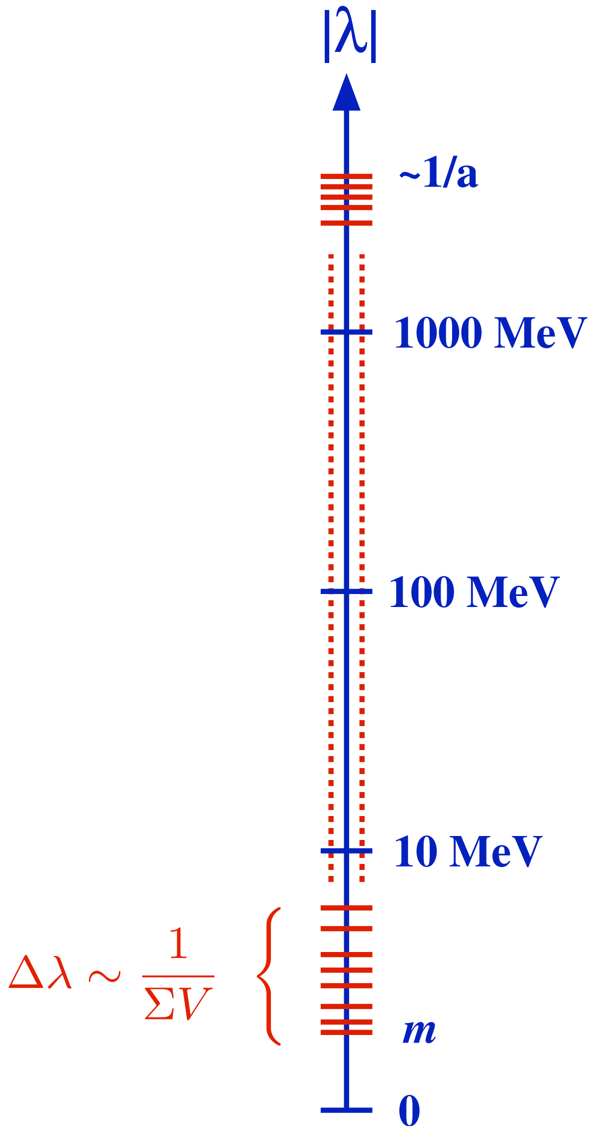
hierarchy of scales ↔ SSB: $m \ll M_\pi \ll 4\pi F_\pi$

[Leutwyler, Smilga 1992]

- condition number $\lambda_{\max}/\lambda_{\min}$ large.
- computation of $D^{-1}\phi$ expensive.

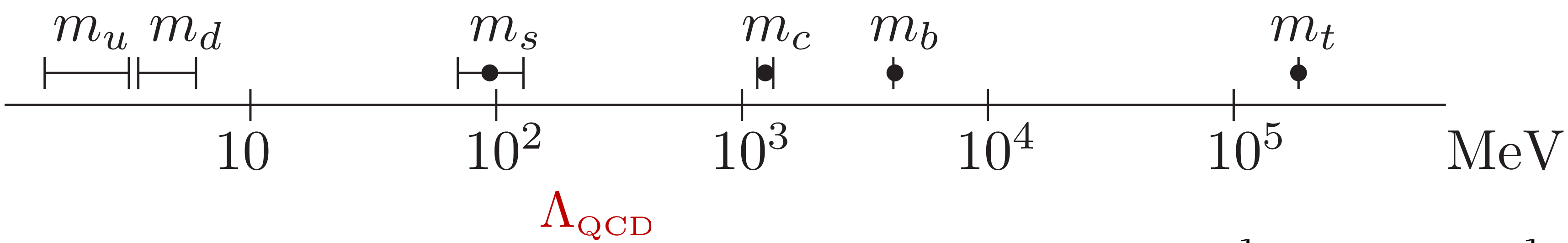
teach the physics to the algorithms!

[Sexton-Weingarten 1990s; Hasenbusch, Lüscher 2000s]



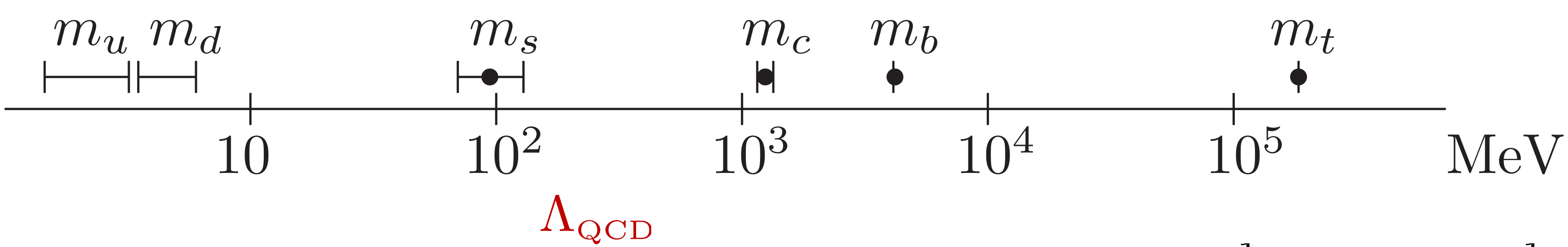
mass preconditioning/domain decomposition, deflation ⇒ mild mass dependence

reach of modern lattice QCD computations



$$L^{-1} \ll \mu \ll a^{-1}$$

reach of modern lattice QCD computations



$$L^{-1} \ll \mu \ll a^{-1}$$

main cost factor: reiterated inversion of lattice Dirac operator on fixed gauge field

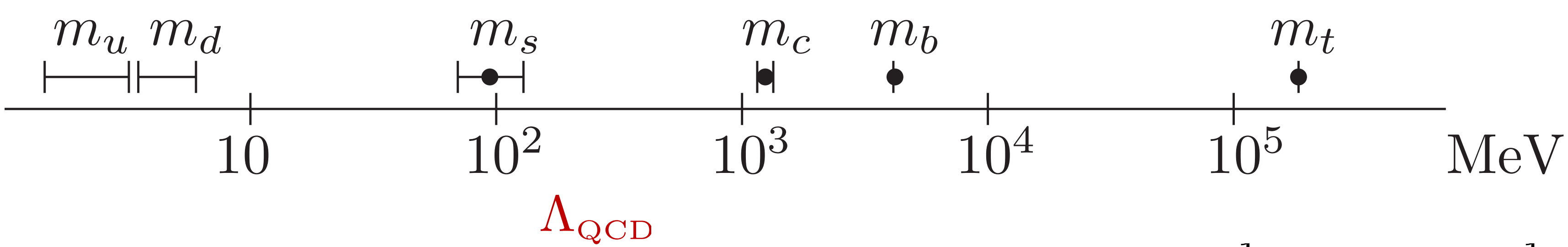
$$\text{cost} \;=\; N \left(\frac{20\ \text{MeV}}{m}\right)^{\alpha} \left(\frac{L}{3\ \text{fm}}\right)^{\beta} \left(\frac{0.1\ \text{fm}}{a}\right)^{\gamma}$$

overall cost (⇒ cpu power)

physics reach

for a long time: serious difficulties in reaching light dynamical quark masses

reach of modern lattice QCD computations



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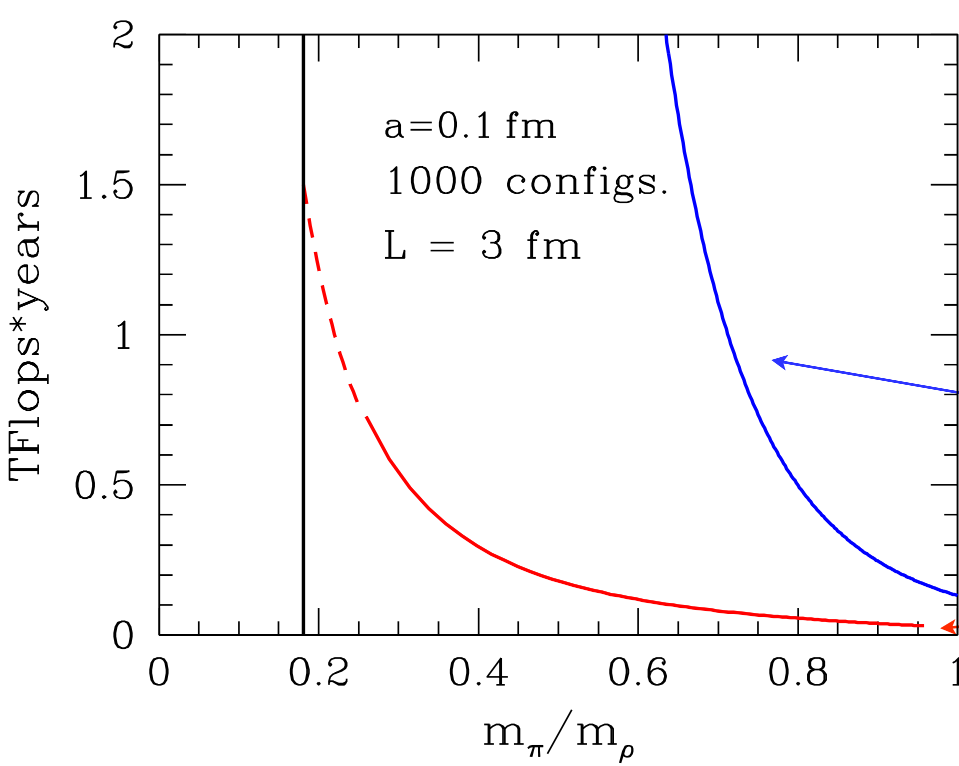
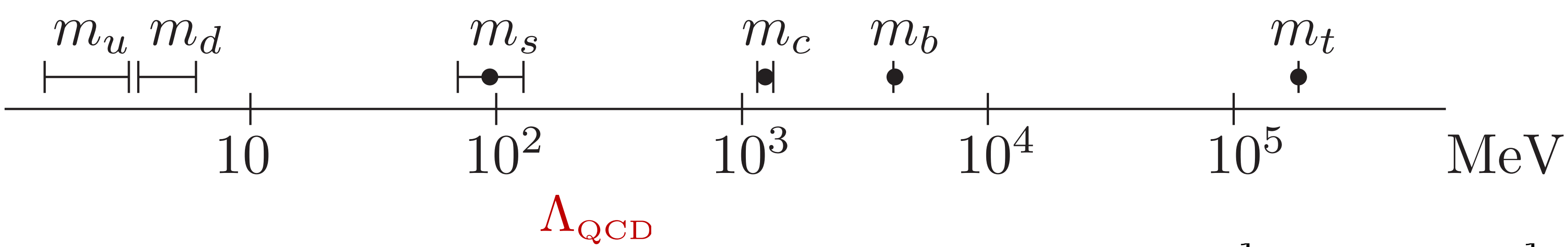
main cost factor: reiterated inversion of lattice Dirac operator on fixed gauge field



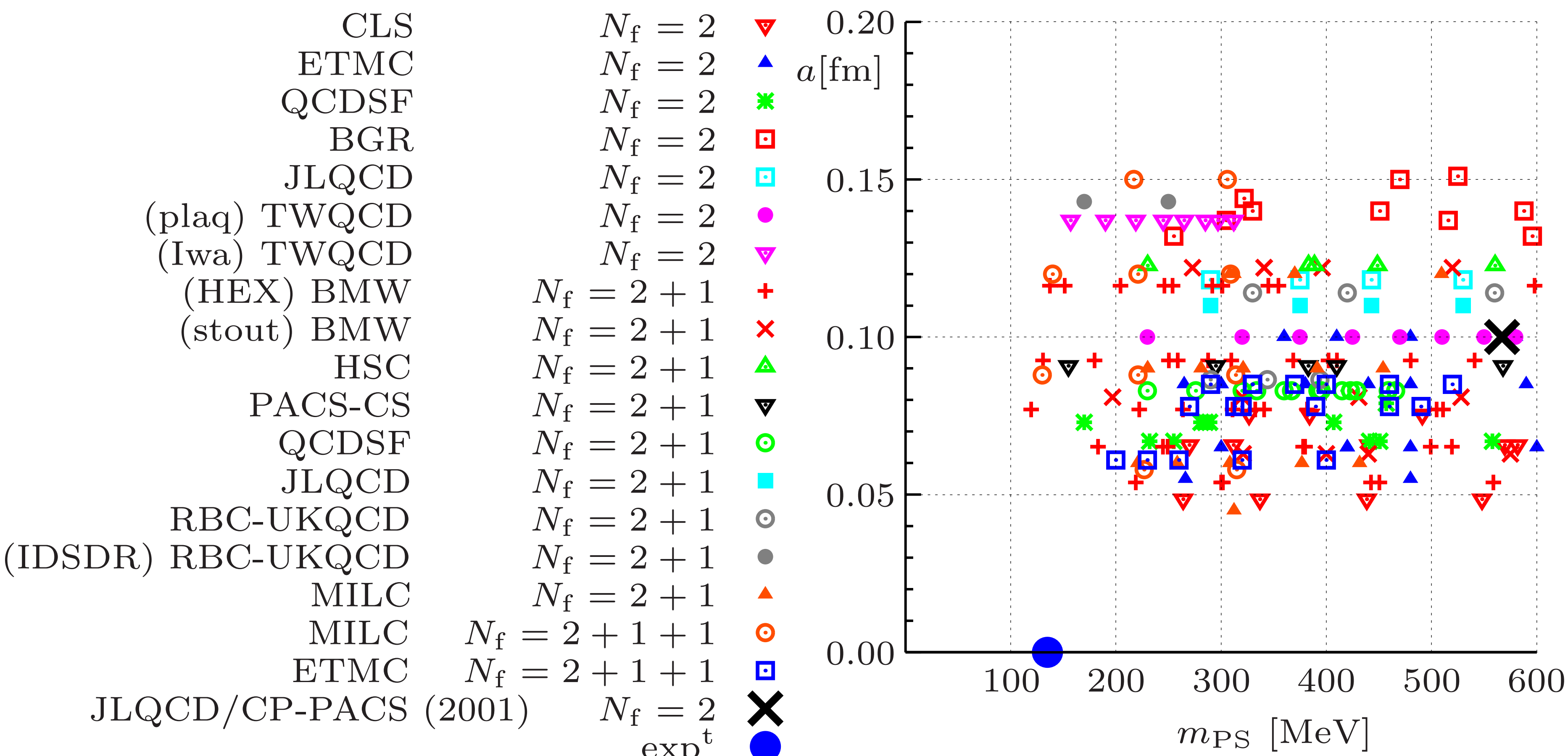
$$5\left[\frac{20\ \mathrm{MeV}}{m}\right]^3\left[\frac{L}{3\ \mathrm{fm}}\right]^5\left[\frac{0.1\ \mathrm{fm}}{a}\right]^7$$ [Ukawa 2001]

$$0.05\left[\frac{20\ \mathrm{MeV}}{m}\right]^1\left[\frac{L}{3\ \mathrm{fm}}\right]^5\left[\frac{0.1\ \mathrm{fm}}{a}\right]^6$$ [Giusti 2006]

lattice QCD reach: simulation landscape

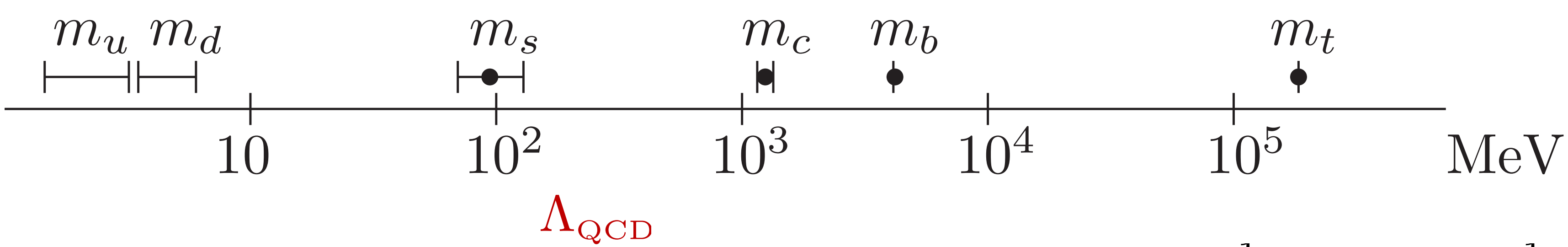


$L^{-1} \ll \mu \ll a^{-1}$

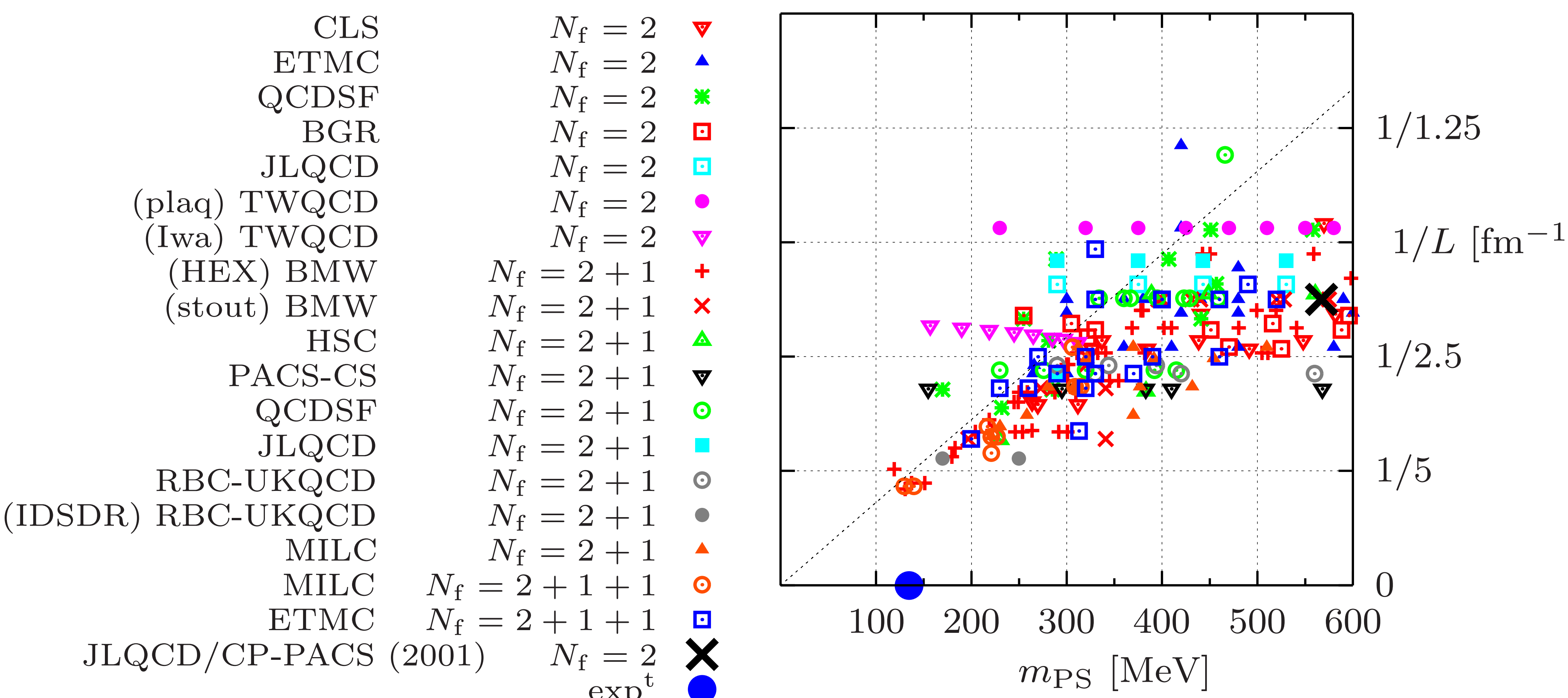


[plot courtesy of G. Herdoíza]

lattice QCD reach: simulation landscape

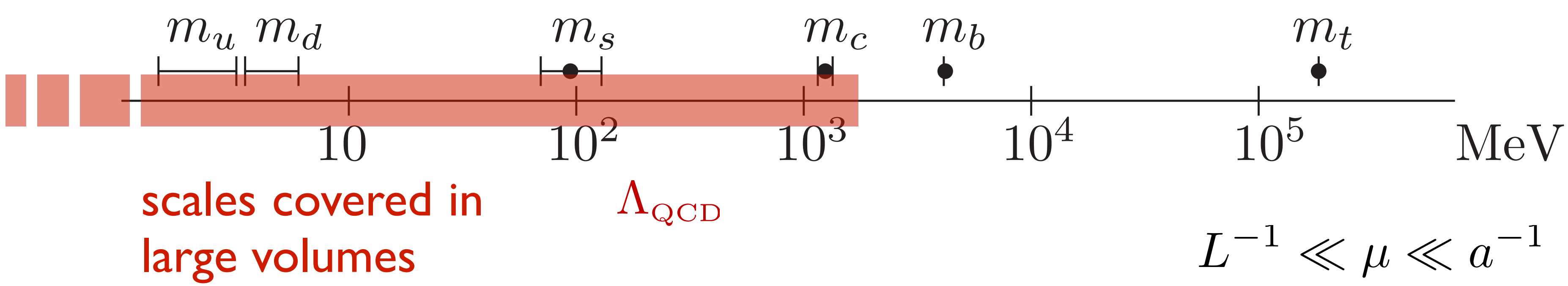


$$L^{-1} \ll \mu \ll a^{-1}$$

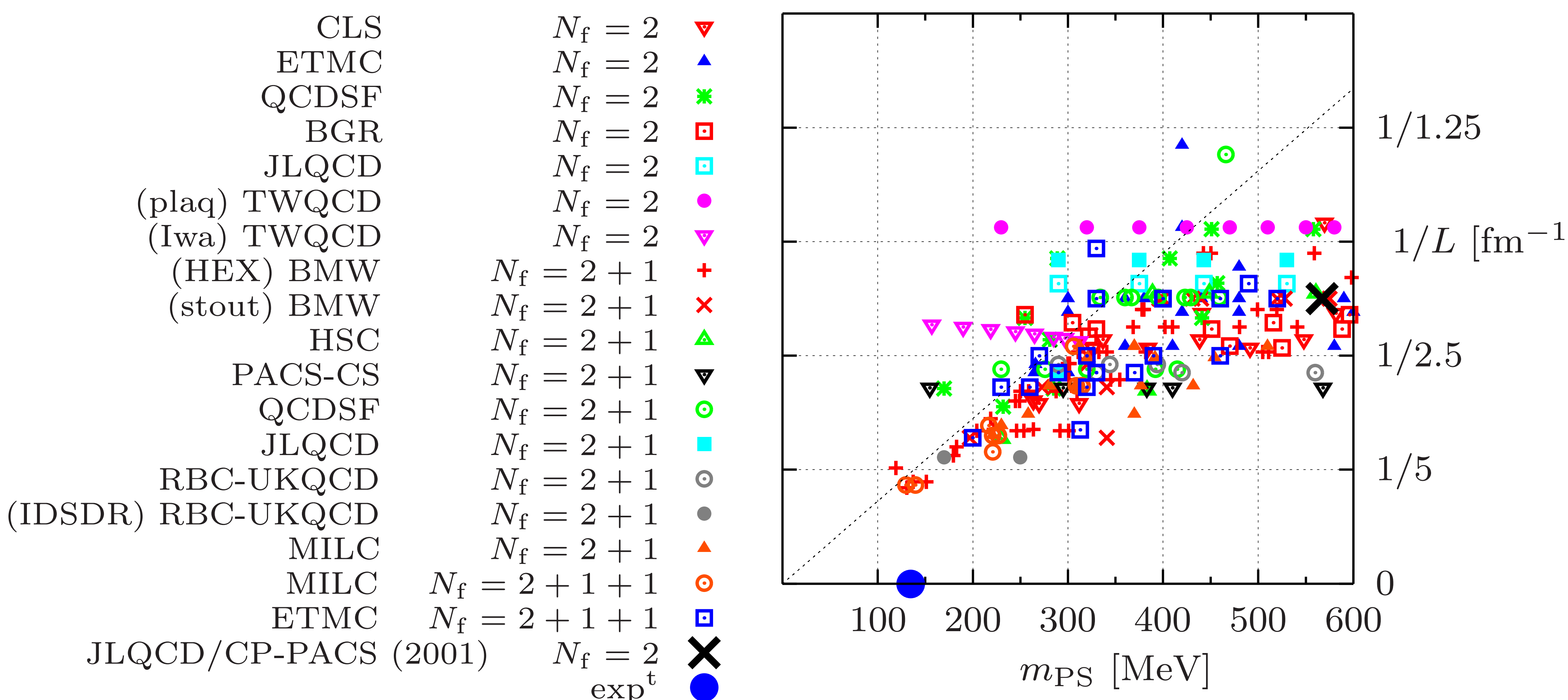


[plot courtesy of G. Herdoíza]

lattice QCD reach: simulation landscape



$L^{-1} \ll \mu \ll a^{-1}$



[plot courtesy of G. Herdoíza]

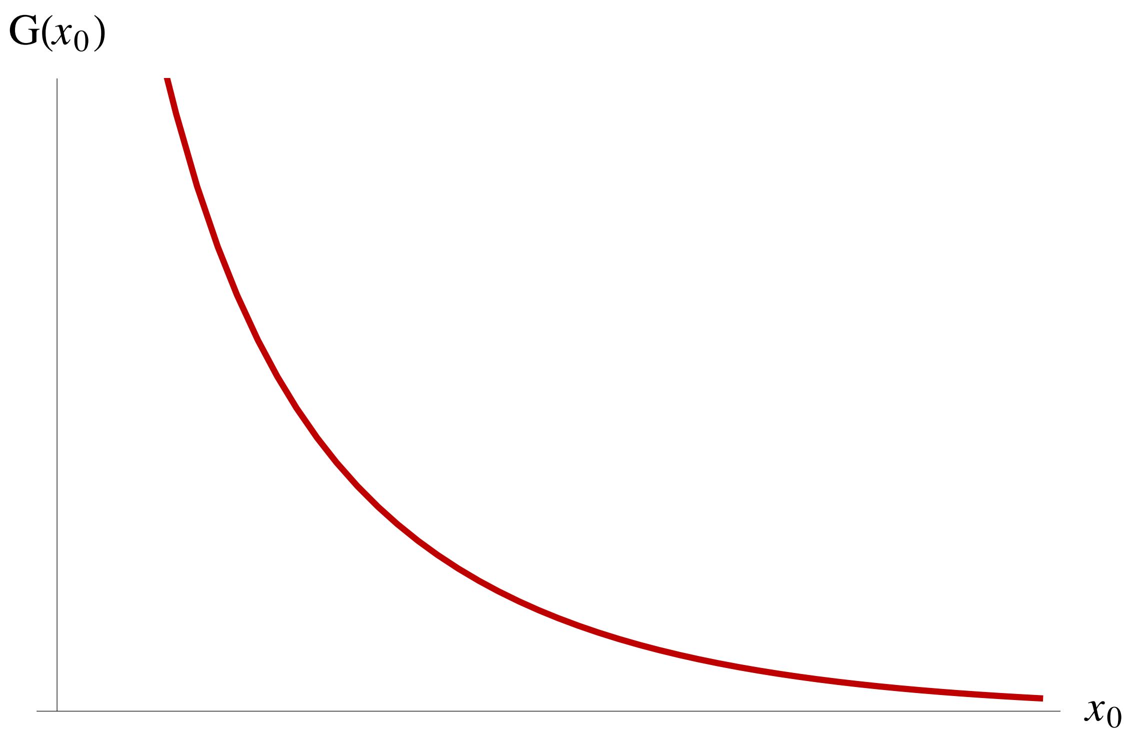
example computation: pion mass and decay constant

(charged) pion two-point function

$$G(x_0) = \int \mathrm{d}^3x \, \langle (\overbrace{\bar{u}\gamma_5 d}^{\equiv P(x)})(x) \, (\bar{d}\gamma_5 u)(0) \rangle = -\langle 0|P(0,\mathbf{x})e^{-Hx_0}P(0,\mathbf{0})|0\rangle$$

$$= -\sum_{\mathrm{PS}} \langle 0|P(0,\mathbf{x})e^{-Hx_0}|\mathrm{PS}\rangle\langle \mathrm{PS}|P(0,\mathbf{0})|0\rangle$$

$$= -e^{-M_\pi x_0}|\langle 0|P(x)|\pi\rangle|^2 + \mathcal{O}(e^{-3M_\pi x_0})$$



computation of hadron masses, simple hadronic matrix elements, ... does not require analytic continuation back to Minkowski space

example computation: pion mass and decay constant

(charged) pion two-point function

$$G(x_0) = \int \mathrm{d}^3x \, \langle (\overbrace{\bar{u}\gamma_5 d}^{\equiv P(x)})(x)\, (\bar{d}\gamma_5 u)(0) \rangle = -\langle 0|P(0,\mathbf{x})e^{-Hx_0}P(0,\mathbf{0})|0\rangle$$

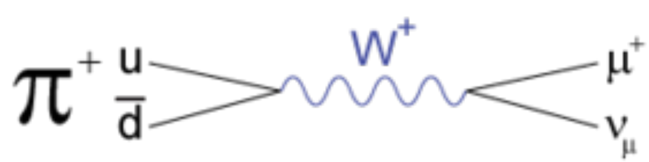
$$= -\sum_{\mathrm{PS}} \langle 0|P(0,\mathbf{x})e^{-Hx_0}|\mathrm{PS}\rangle\langle \mathrm{PS}|P(0,\mathbf{0})|0\rangle$$

$$= -e^{-M_\pi x_0}|\langle 0|P(x)|\pi\rangle|^2 + \mathcal{O}(e^{-3M_\pi x_0})$$

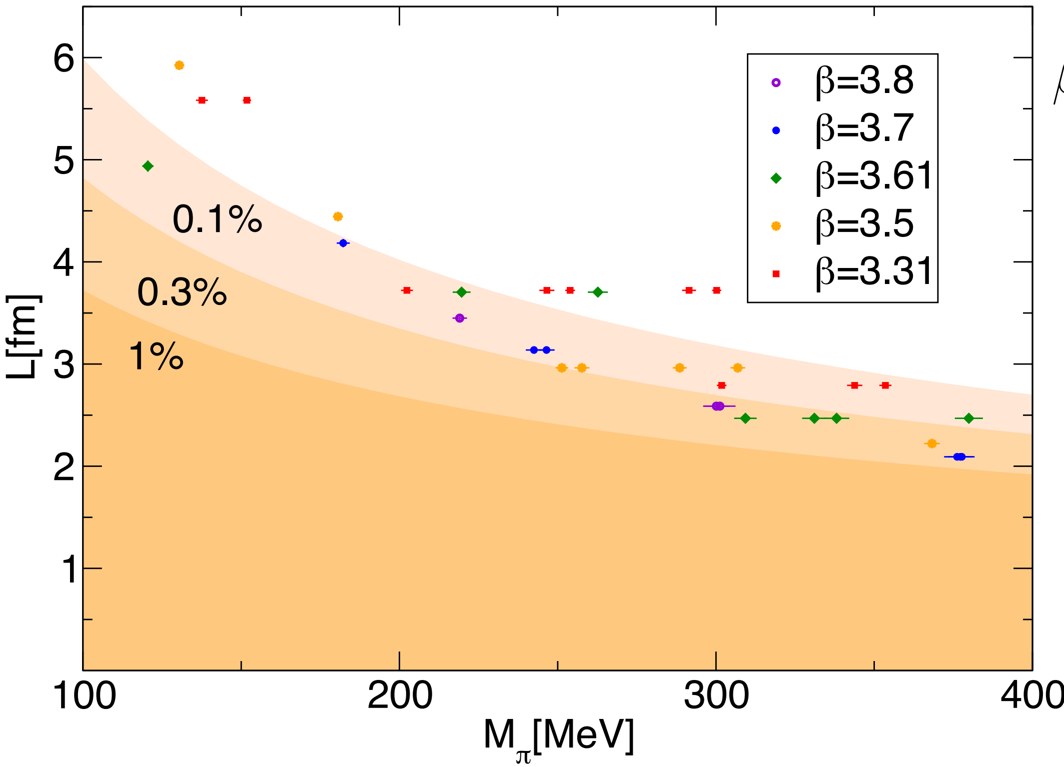
decay constant: combine with

$$G_A(x_0) = \int \mathrm{d}^3x \langle (\bar{u}\gamma_0\gamma_5 d)(x)(\bar{d}\gamma_5 u)(0)\rangle$$

$$\propto F_\pi e^{-M_\pi x_0}$$

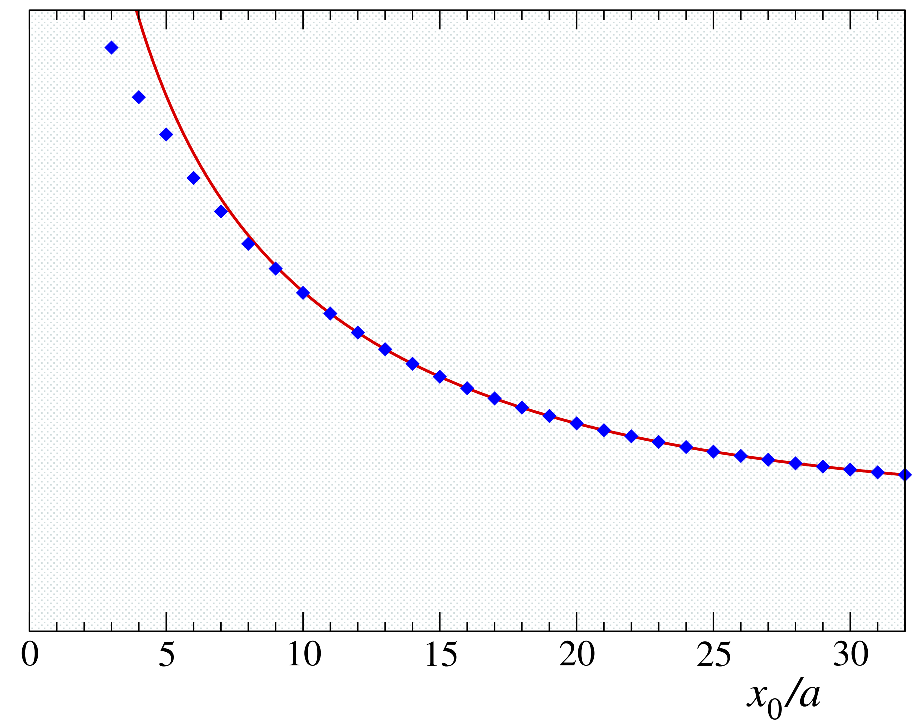


example computation: pion mass and decay constant



$\beta = 6/g_0^2$

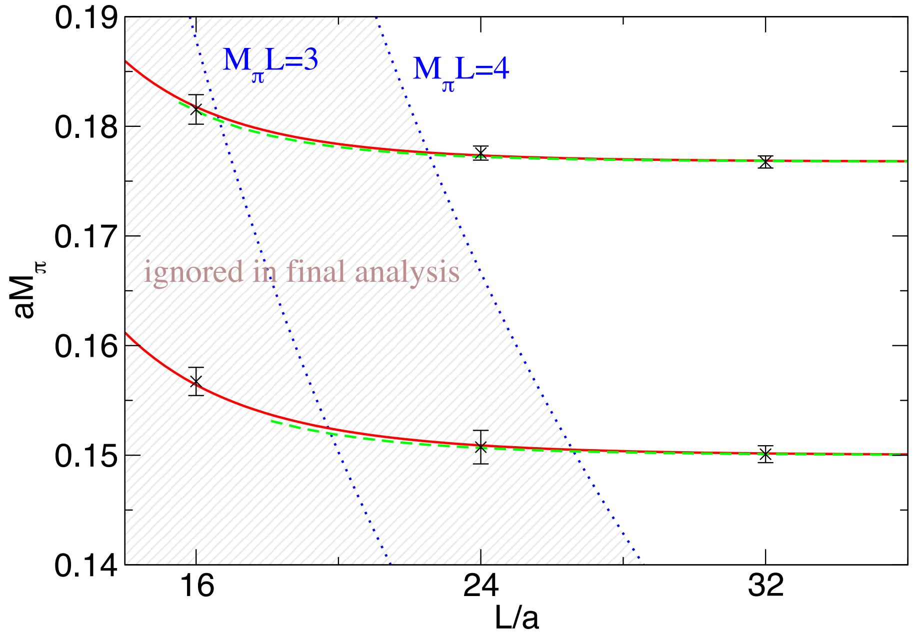
for each simulation point, extract $M_\pi\,, F_\pi$

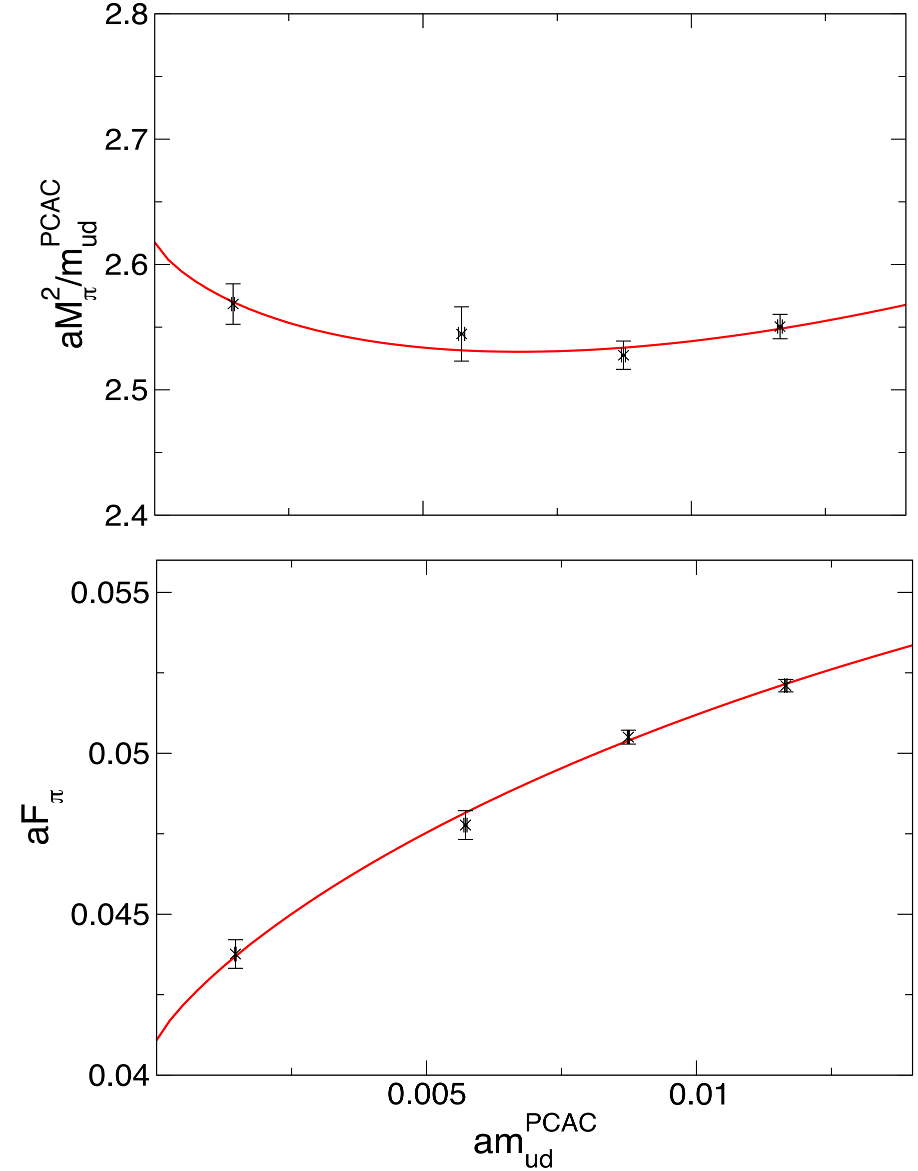


[BMW Collaboration, 2010]

example computation: pion mass and decay constant

determine dependence with quark mass, volume, lattice spacing and extra/interpolate to physical point





[BMW Collaboration, 2010]

outline

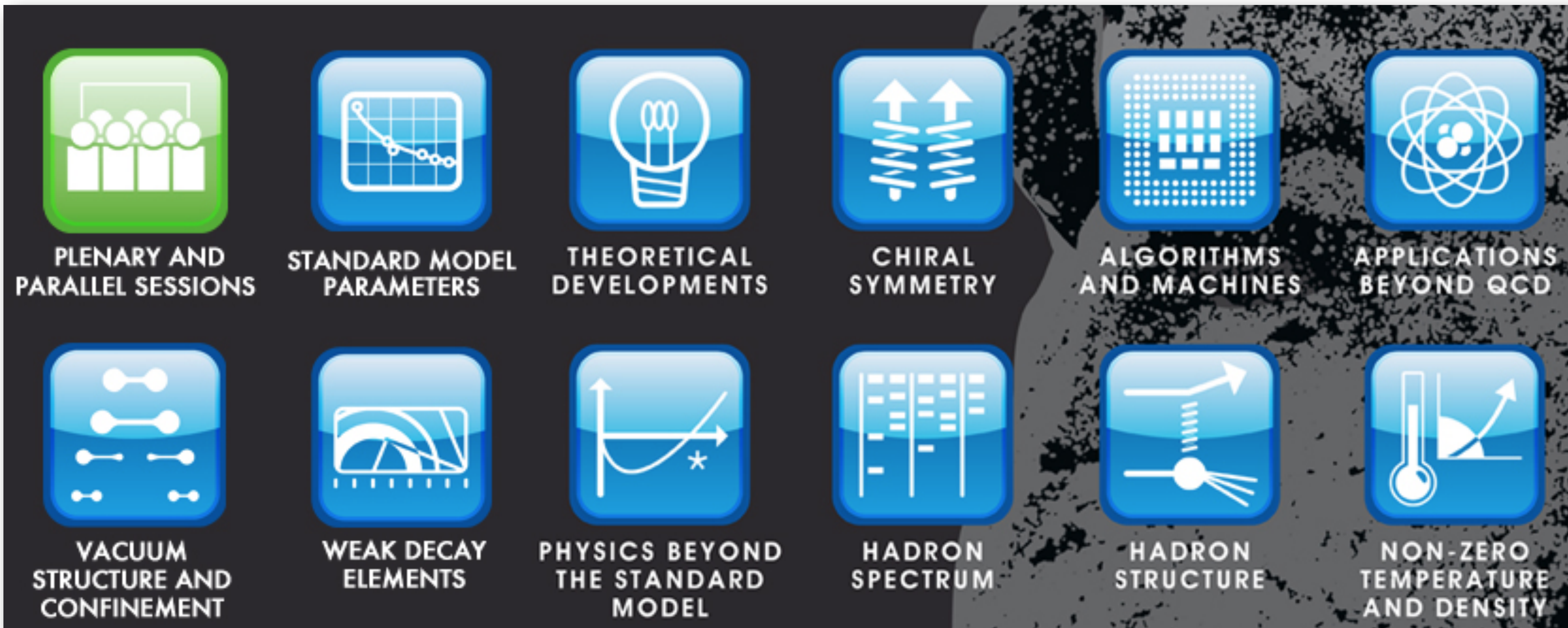
- motivation: strong interaction(s) and non-perturbative physics

- lattice field theory
  - QFT in Euclidean space
  - matter and gauge fields on a lattice
  - interacting gauge theories on a lattice: QCD

- numerical aspects
  - Monte Carlo techniques for non-perturbative QFT
  - reach of QCD computations
  - anatomy of an example

- overview of physics capabilities
  - FLAG
  - selected lattice QCD results
  - beyond the SM

overview of current QCD physics from the lattice



overview of current QCD physics from the lattice

very competitive area, several large collaborations (Europe/Japan/USA)
example collaboration:

- CERN
- DESY
- Dublin
- Humboldt
- Madrid
- Mainz
- Milan
- Münster
- Regensburg
- Rome Sapienza
- Rome ToV
- Valencia
- Wuppertal

CLS based

overview of current QCD physics from the lattice

effort by the lattice community to summarise and qualify results for non-experts

FLAG 2013
Flavour Lattice Averaging Group

advisory board: S. Aoki (J), C. Bernard (US), C. Sachrajda (EU)

editorial board: G. Colangelo, H. Leutwyler, A. Vladikas, U. Wenger

working groups:

quark masses	T. Blum, L. Lellouch, V. Lubicz
V_{ud}, V_{us}	A. Jüttner, T. Kaneko, S. Simula
LECs	S. Dürr, H. Fukaya, S. Necco
B_K	J. Laiho, S. Sharpe, H. Wittig
α_s	T. Onogi, J. Shigemitsu, R. Sommer
f_B, B_B, f_D	Y. Aoki, M. Della Morte, A. El Khadra
$B, D \to H\ell\nu$	E. Lunghi, CP, R. Van de Water

FLAG-2 review partially published online, full version to appear within year end
new published review every 2nd year; regular web updates in between.

overview of current QCD physics from the lattice

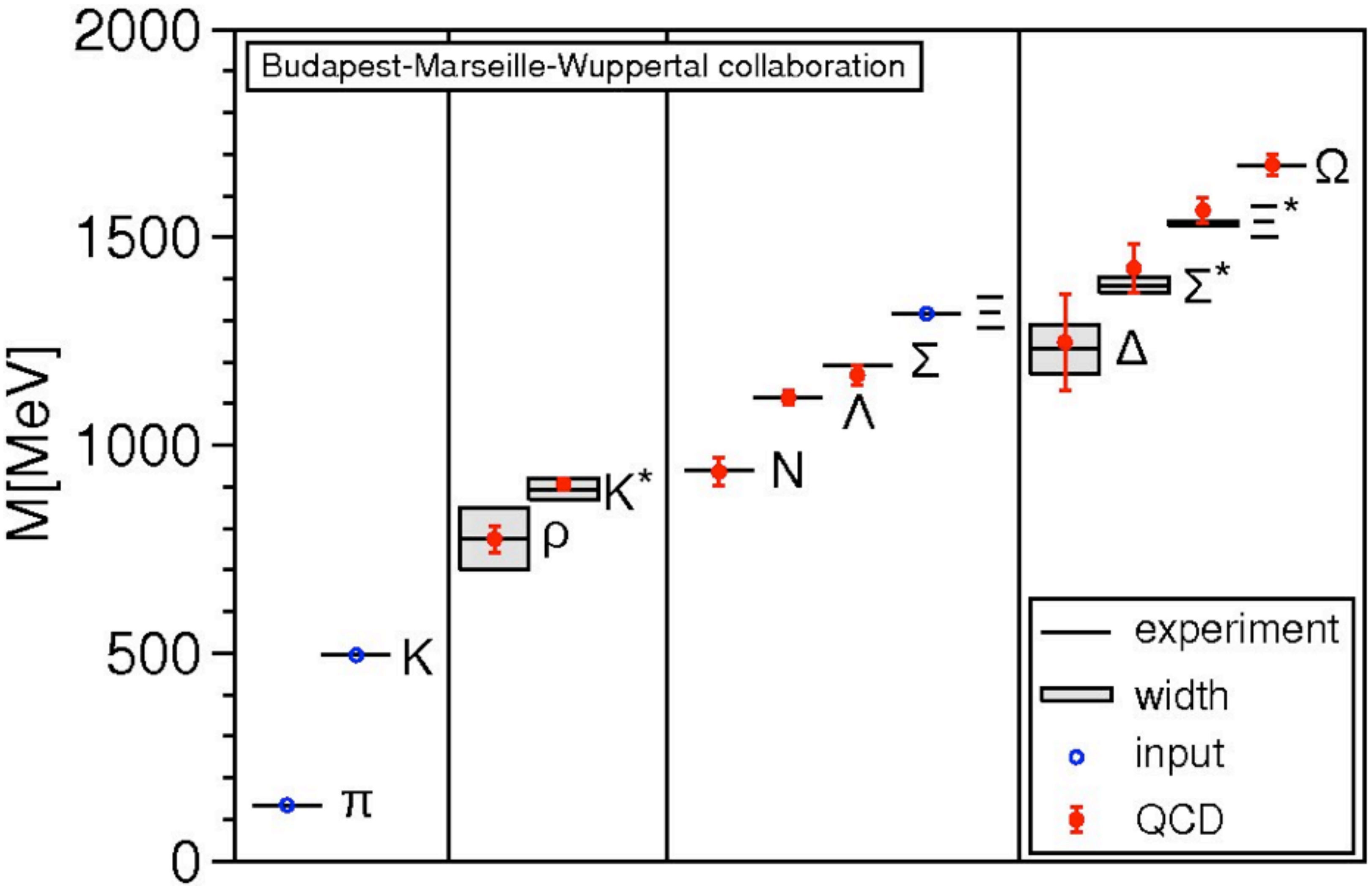
FLAG quantities

- light quark masses
- LECs (light hadron dynamics)
- decay constants
- pion and kaon form factors
- kaon bag parameter
- D meson leptonic and semileptonic decays
- B meson leptonic and semileptonic decays, mxing
- strong coupling constant

For each quantity provide:

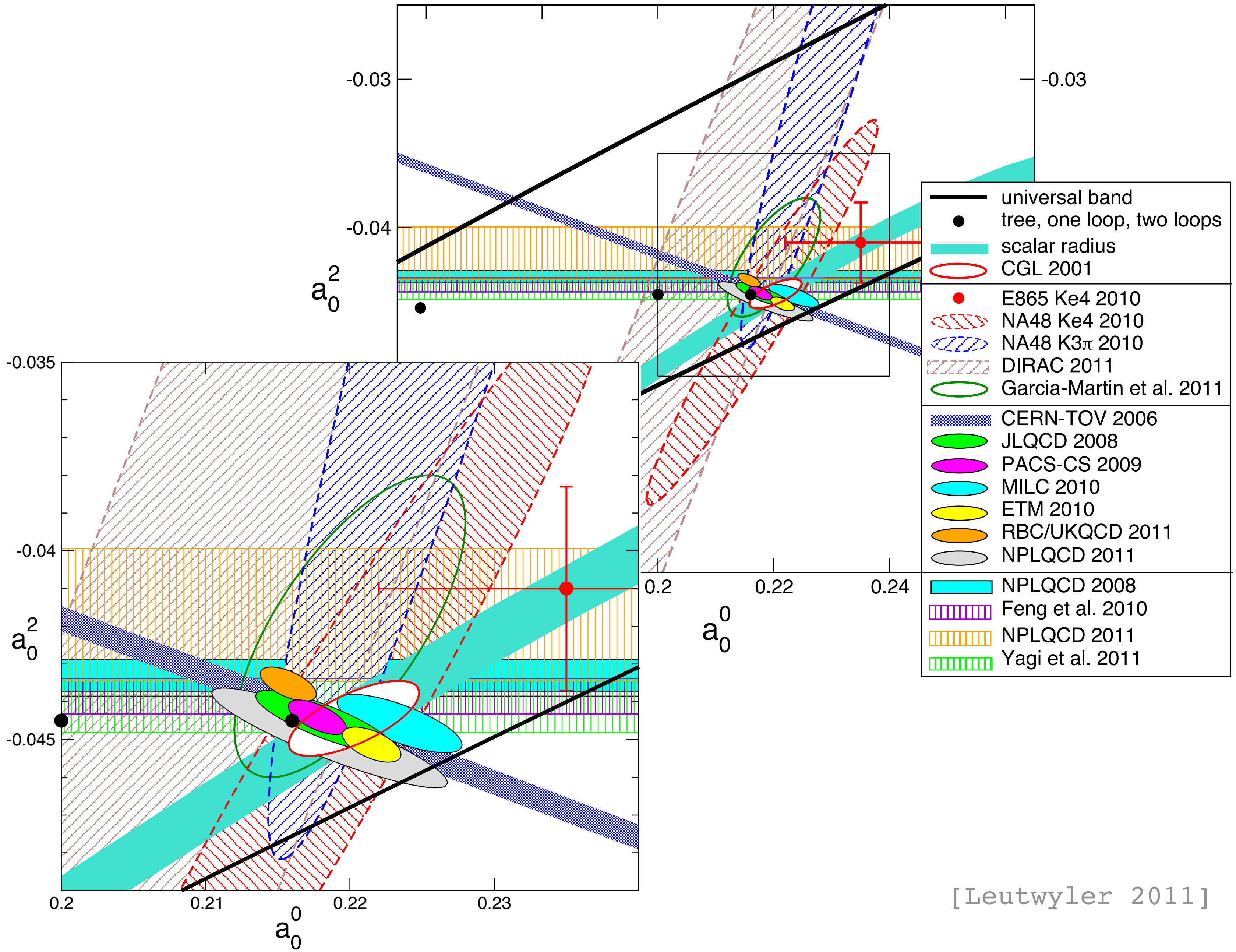
- complete list of references, summary of formulae/notations, ...
- summary of essential aspects of each computation
- averages (if sensible)
- "lattice dictionary" for non-experts

light hadron spectrum



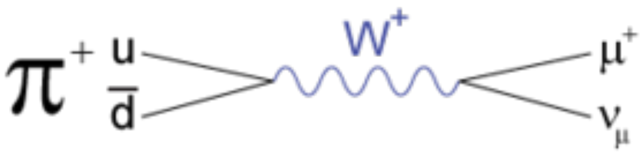
[BMW Collaboration 2008]

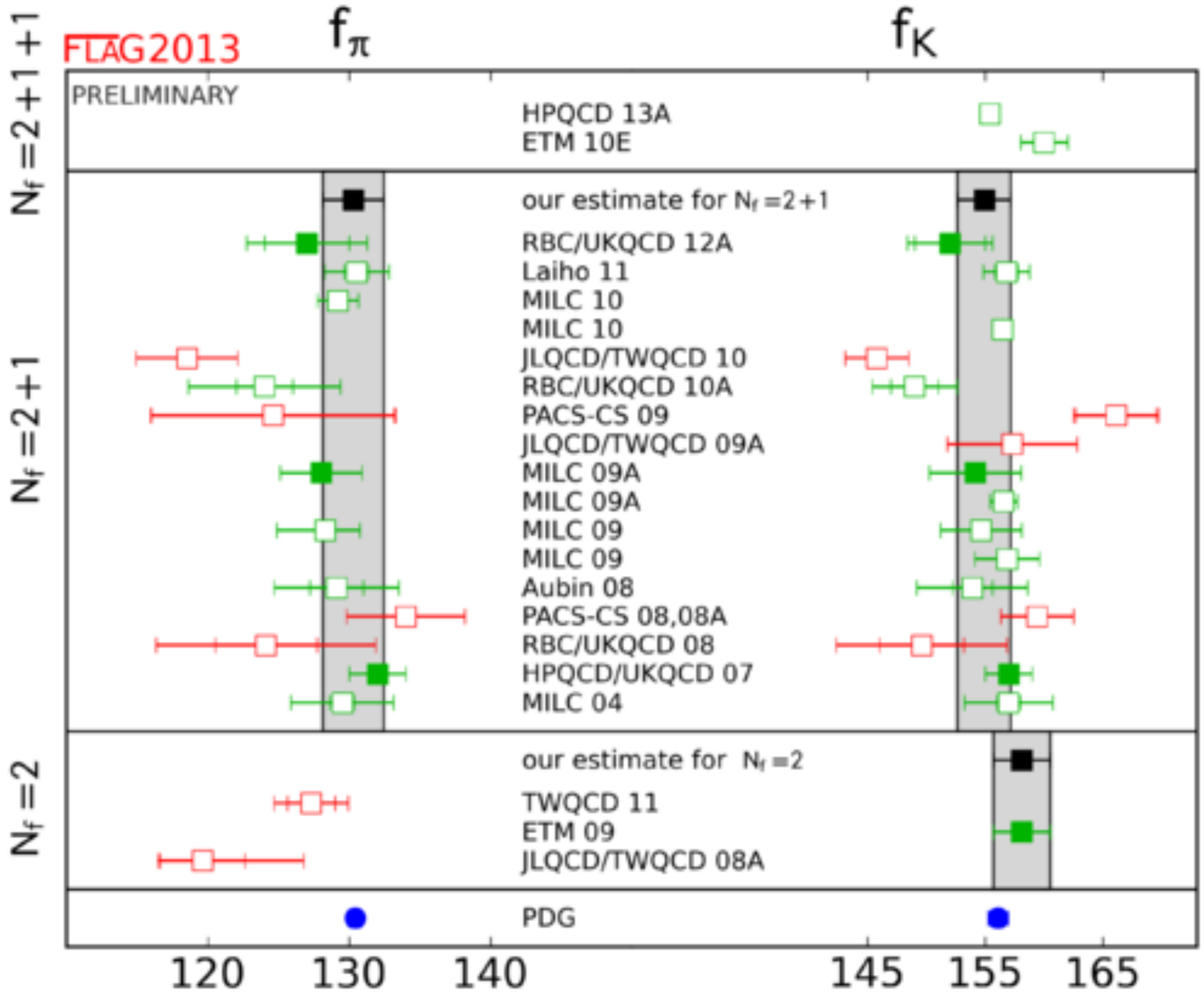
light hadron dynamics: pion scattering



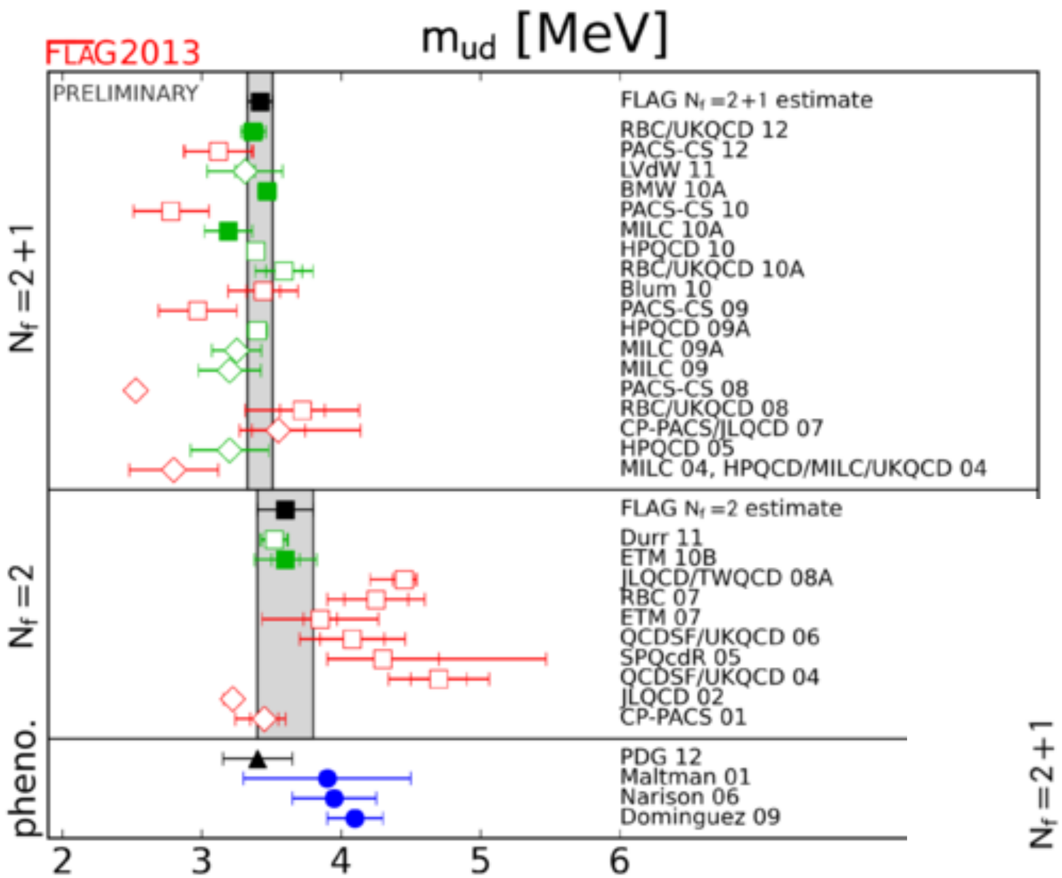
[Leutwyler 2011]

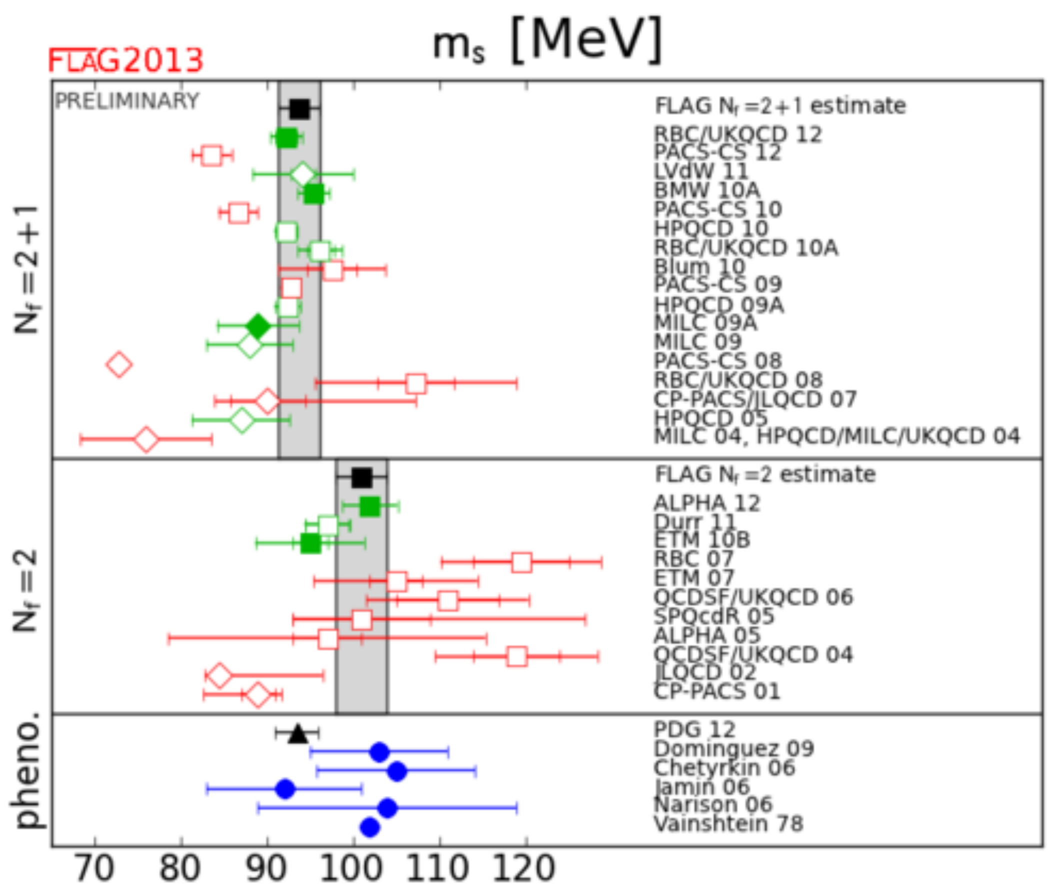
light hadron dynamics: kaon and pion decay



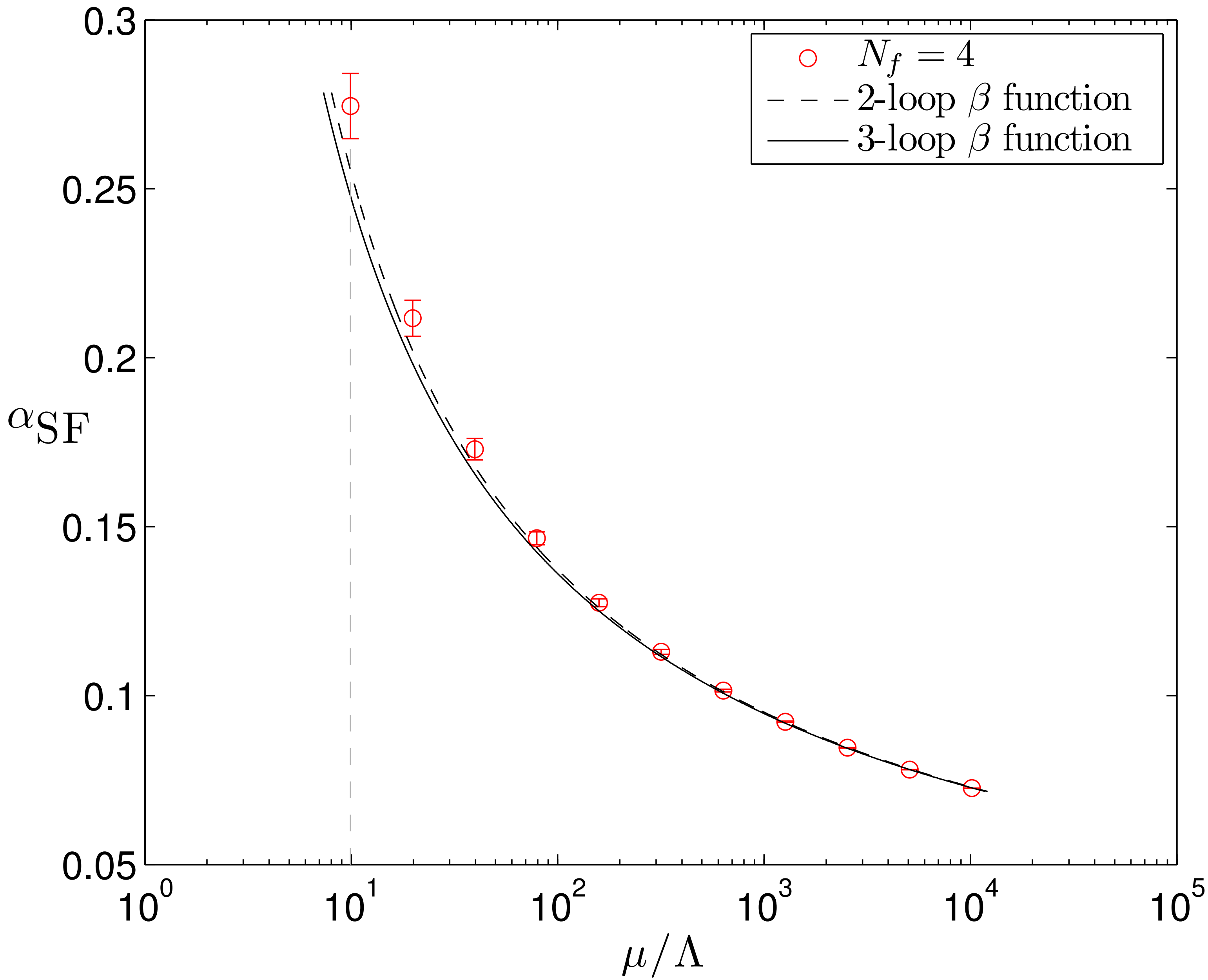


fundamental parameters: quark masses





fundamental parameters: strong coupling constant

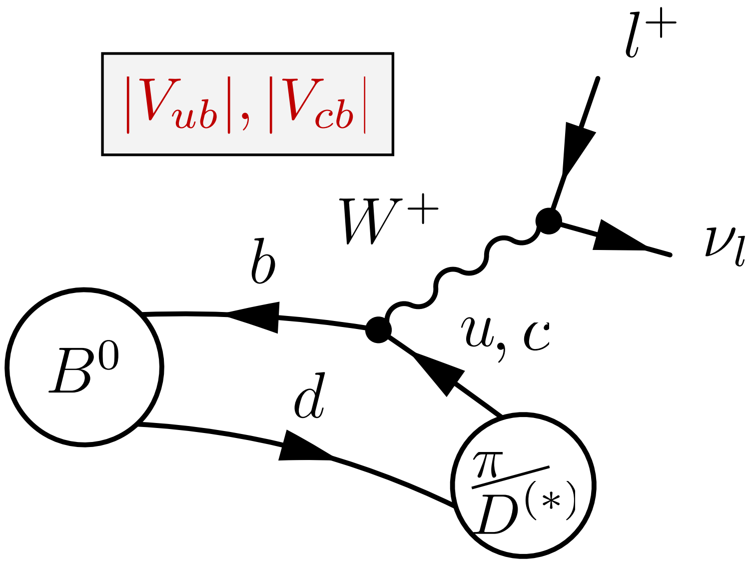


[ALPHA Collaboration, 2010]

flavour physics + fundamental parameters: CKM

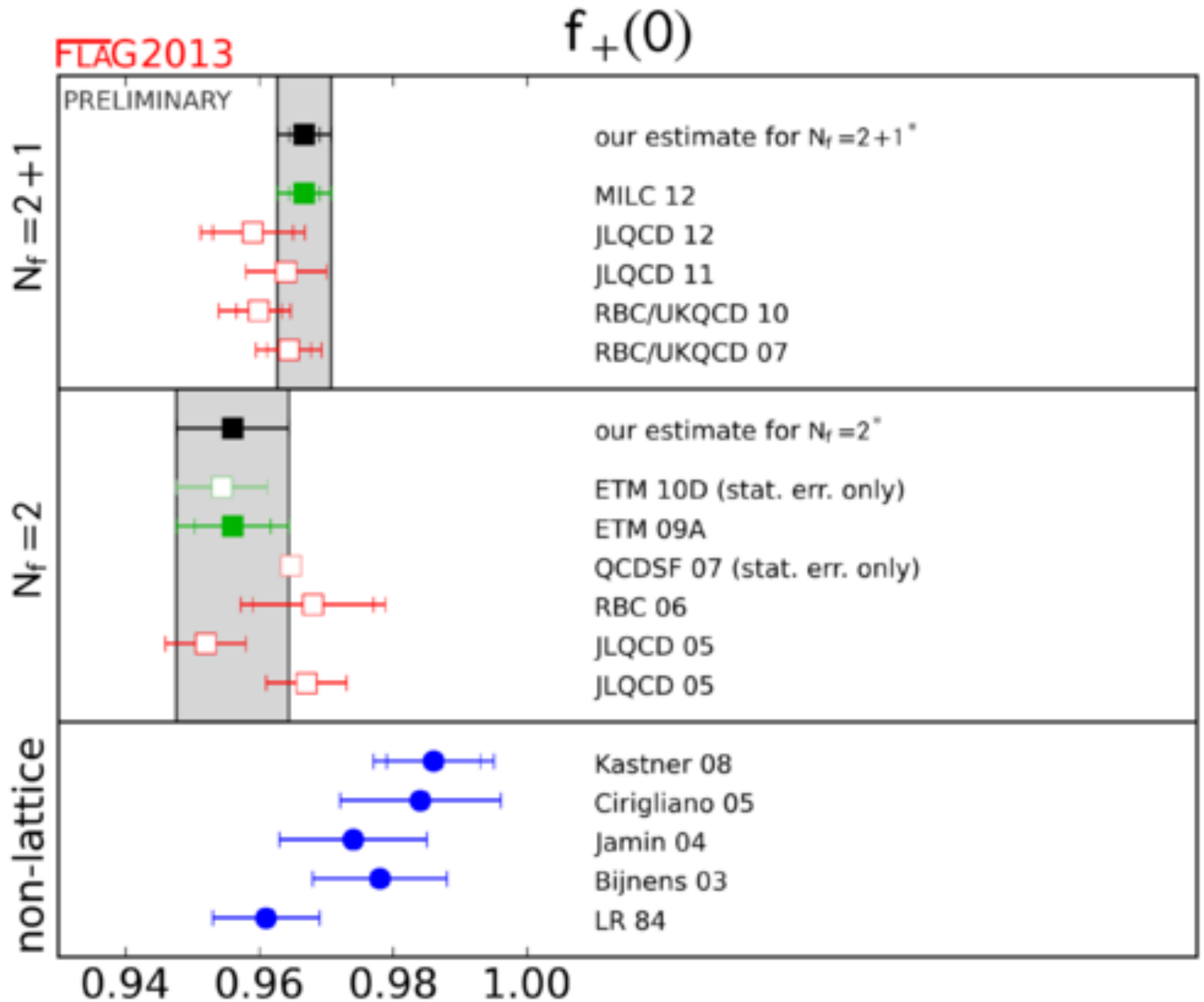
$$\Gamma(P \to P'\ell\nu) = \frac{G_F^2 M_P^5}{192\pi^3} |V_{ij}|^2 \left[c_+(q^2)|f_+(q^2)|^2 + c_0(q^2)|f_0(q^2)|^2\right]$$

$\boxed{|V_{ub}|, |V_{cb}|}$



flavour physics + fundamental parameters: CKM

$$\Gamma(P \to P'\ell\nu) = \frac{G_F^2 M_P^5}{192\pi^3} |V_{ij}|^2 \left[c_+(q^2)|f_+(q^2)|^2 + c_0(q^2)|f_0(q^2)|^2\right]$$



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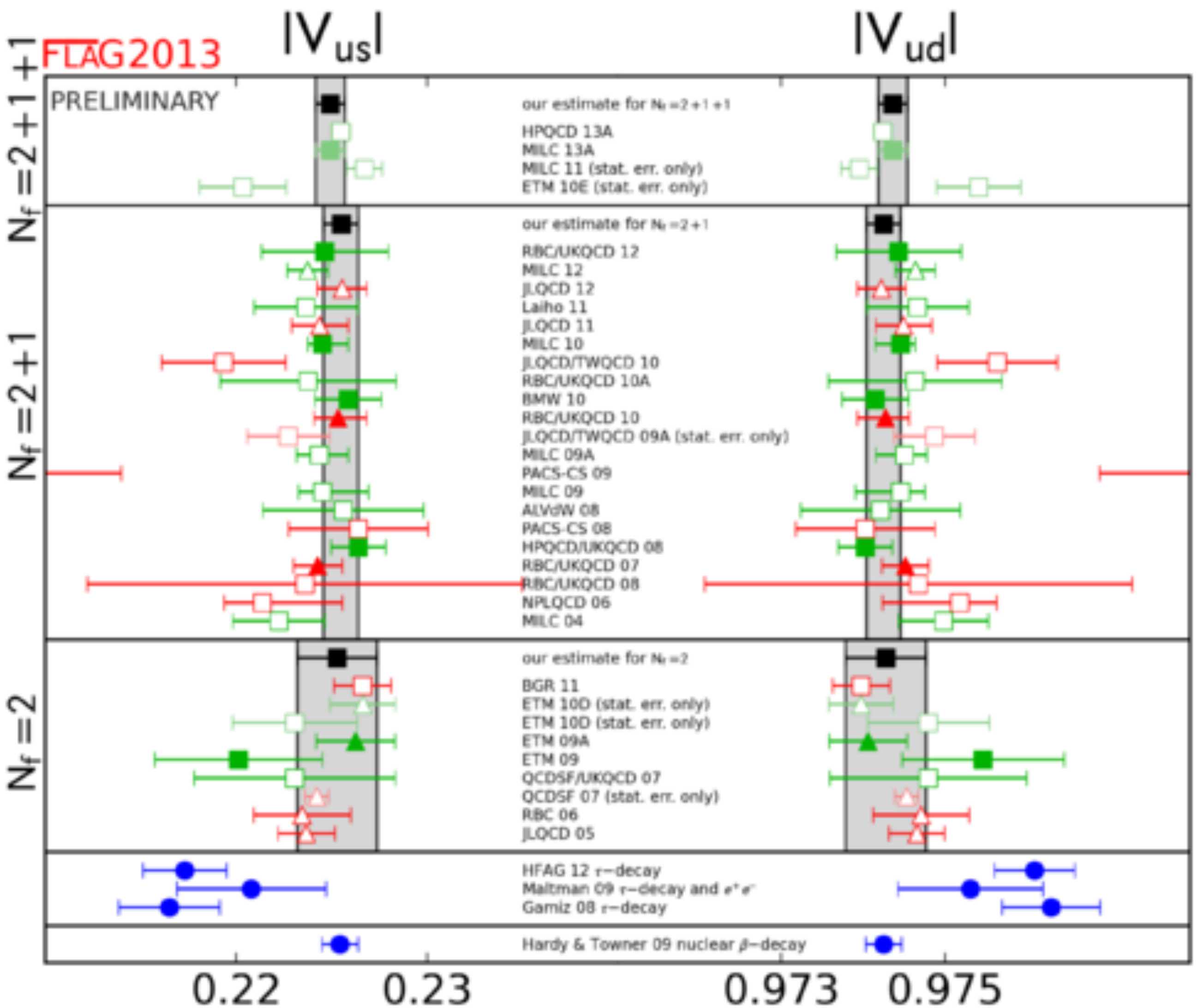

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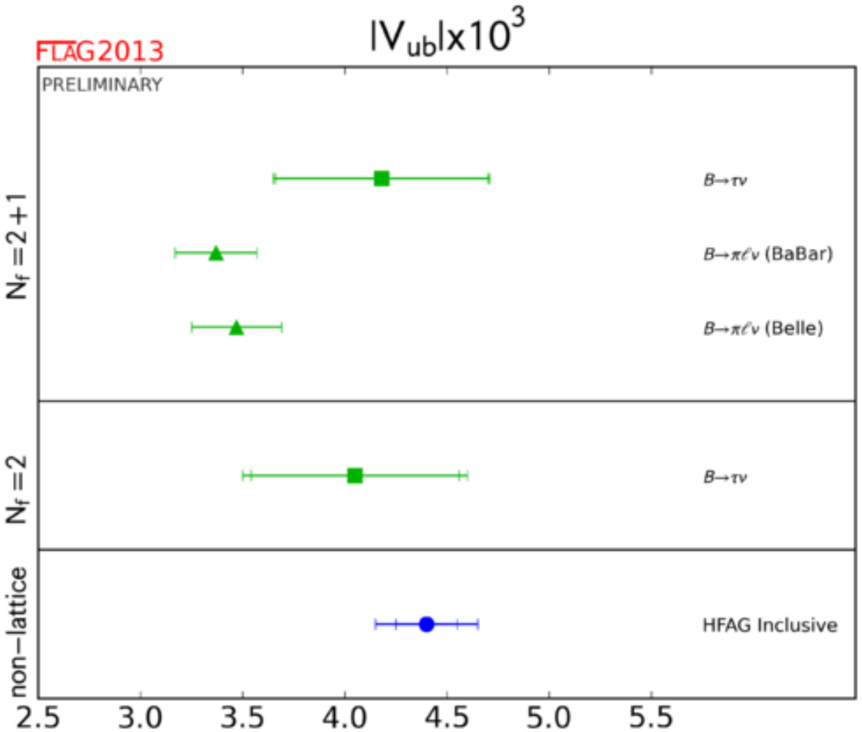
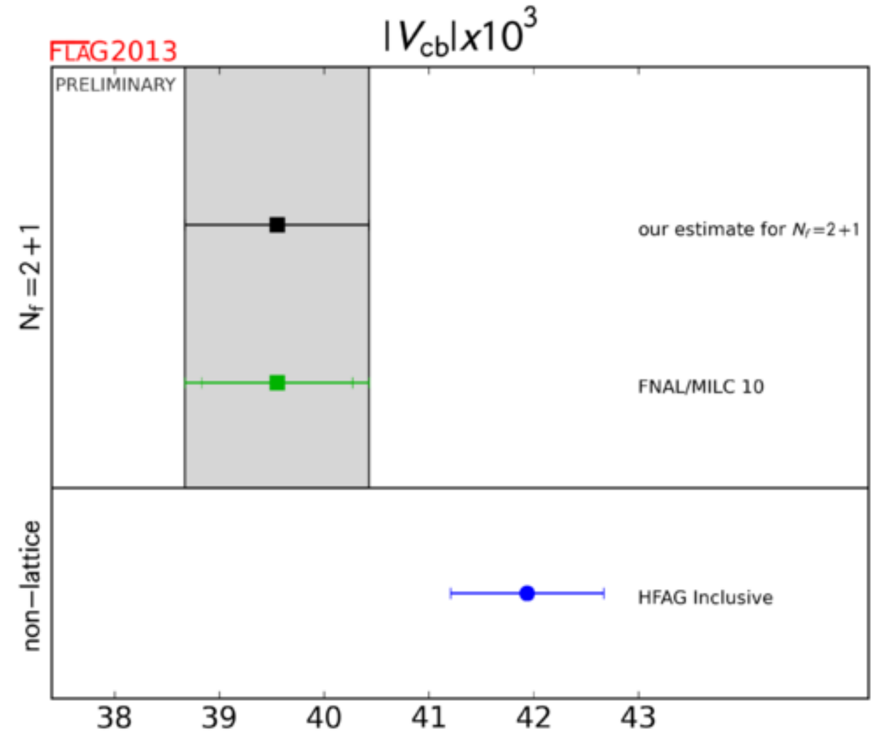
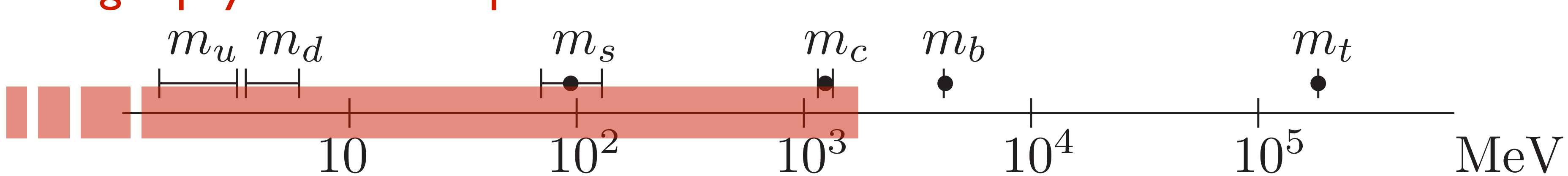
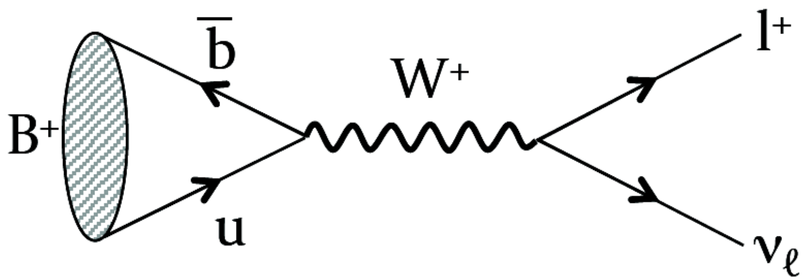


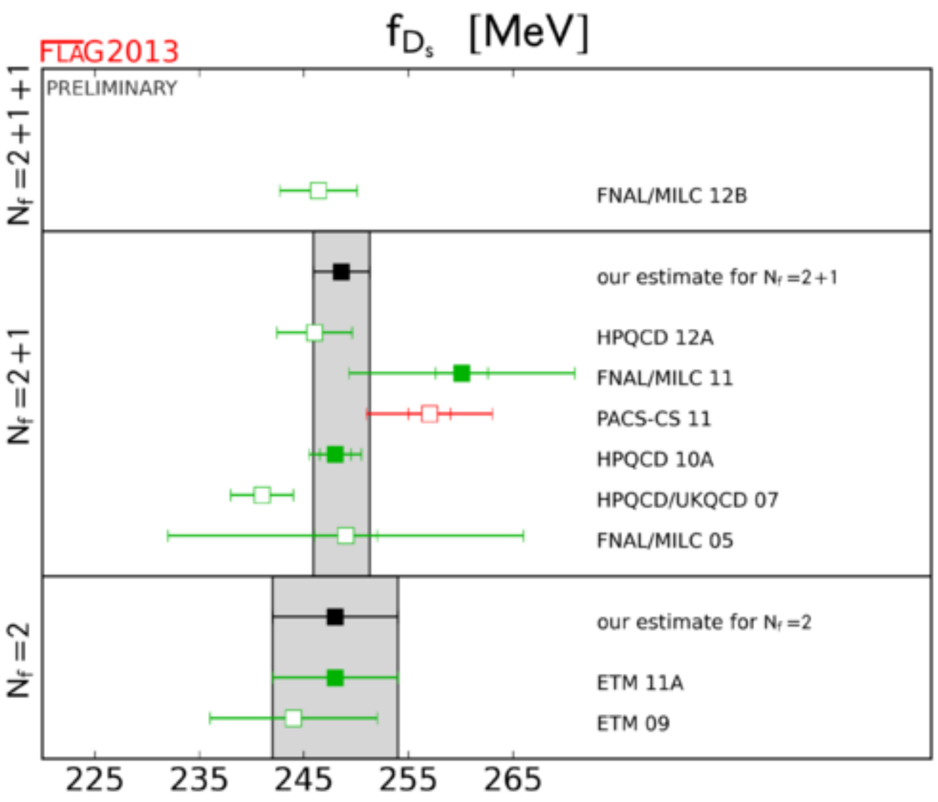


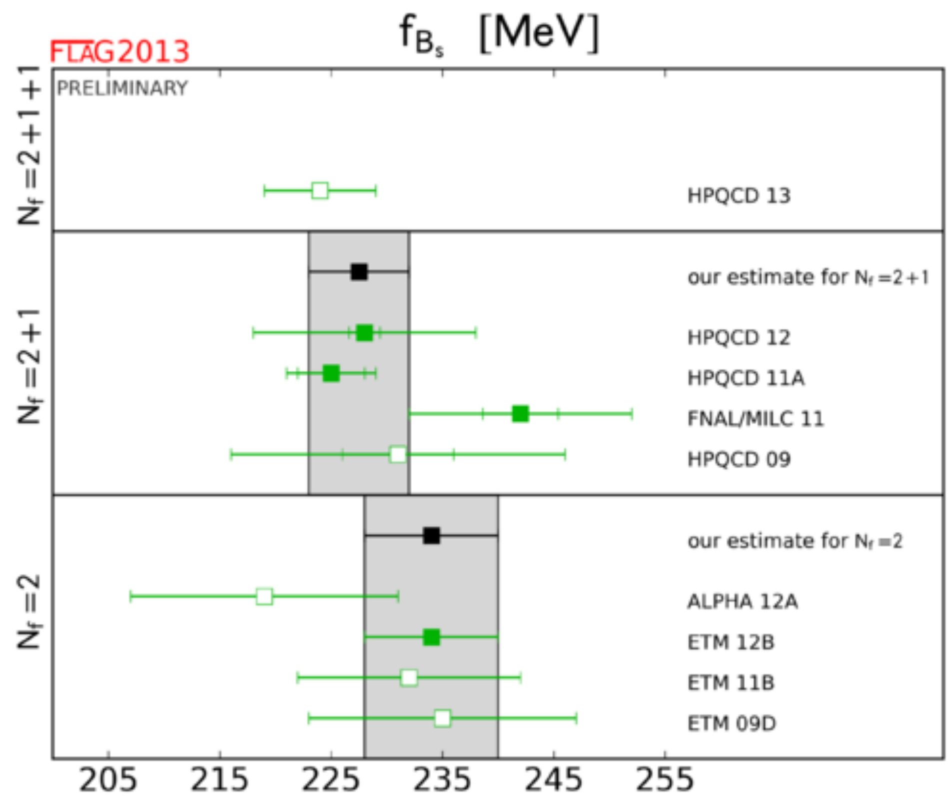
n.b.: computations involving heavy quarks still face several issues, are less developed than light physics counterparts



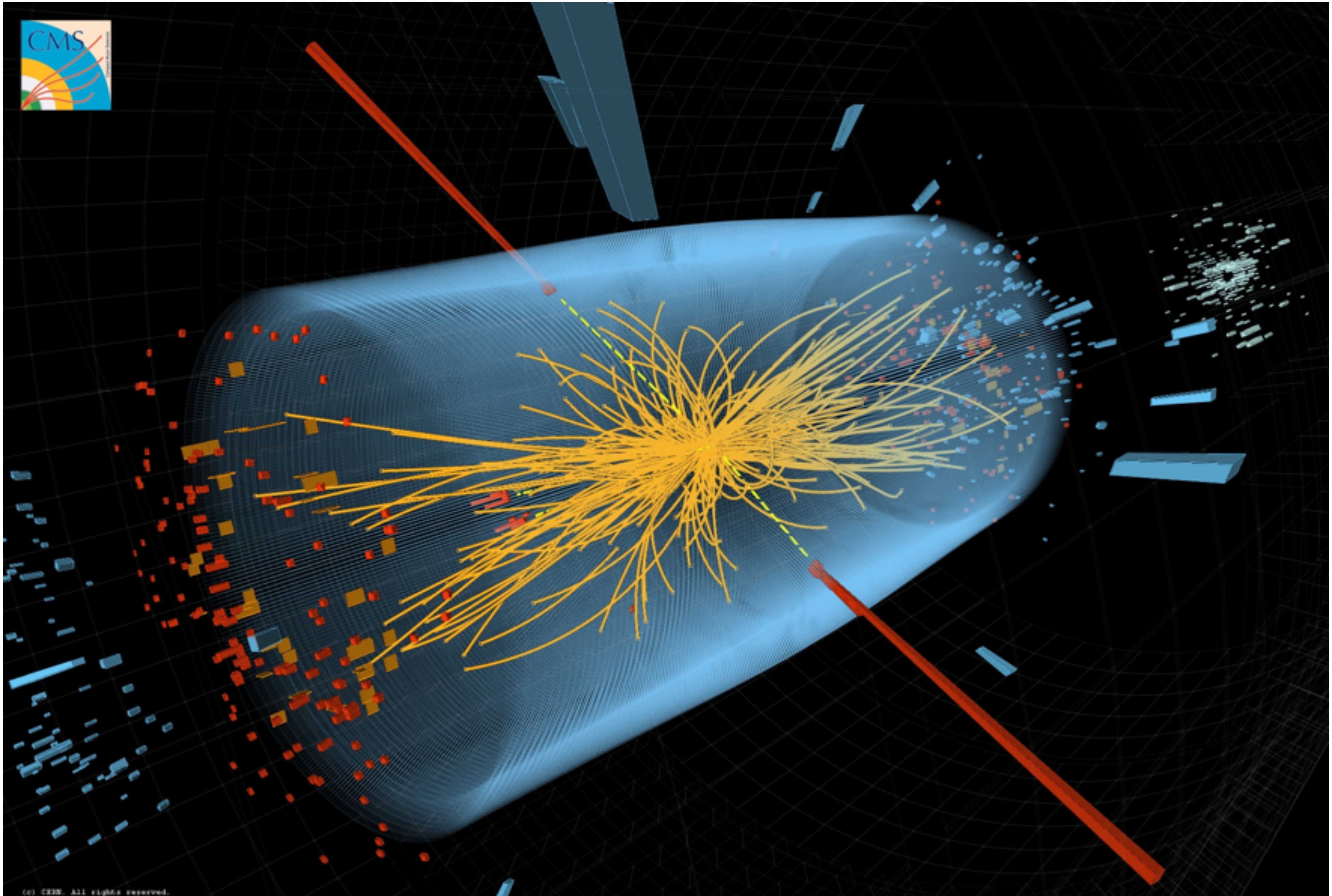
heavy quark dynamics: decay constants



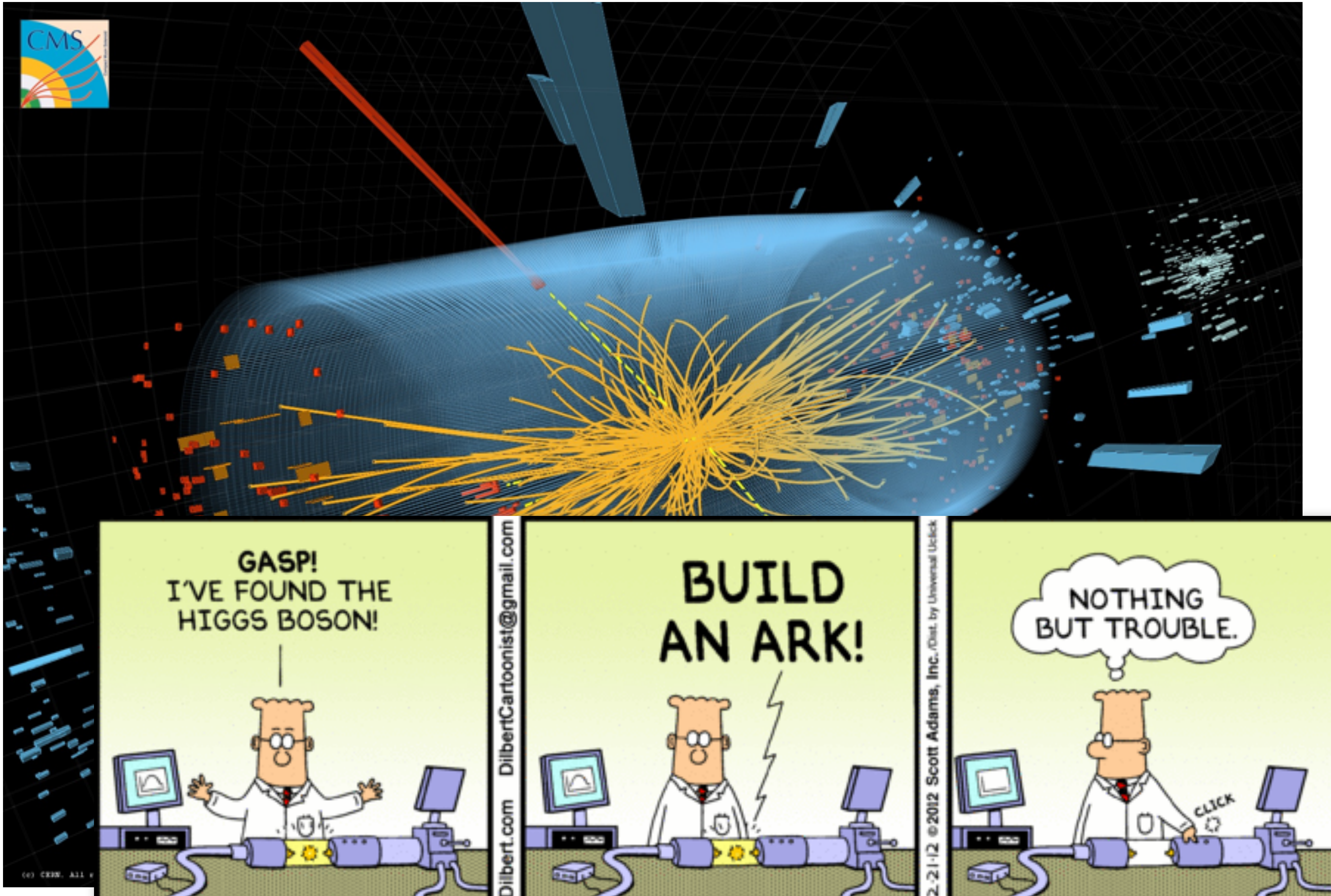




beyond QCD



beyond QCD


CMS
GASP!
I'VE FOUND THE HIGGS BOSON!
Dilbert.com DilbertCartoonist@gmail.com
BUILD AN ARK!
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NOTHING BUT TROUBLE.
CLICK

electroweak symmetry breaking

two classes of models:

- EW symmetry broken by weakly coupled scalar field(s):
  - without SUSY (hierarchy problem);
  - with SUSY (natural, but plethora of soft-breaking parameters);

- EWSB degrees of freedom are actual Nambu-Goldstone bosons:
  - immediately connects EWSB and flavour;
  - dynamics necessarily strongly coupled.

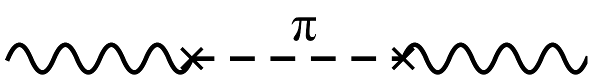
Emphasis biased because of technical difficulties posed by strongly coupled dynamics.

May the success story of lattice QCD make up for that?

strong electroweak symmetry breaking

- technicolour
- composite Higgs

Chiral SSB in QCD provides qualitatively correct mechanism for *W* mass generation:

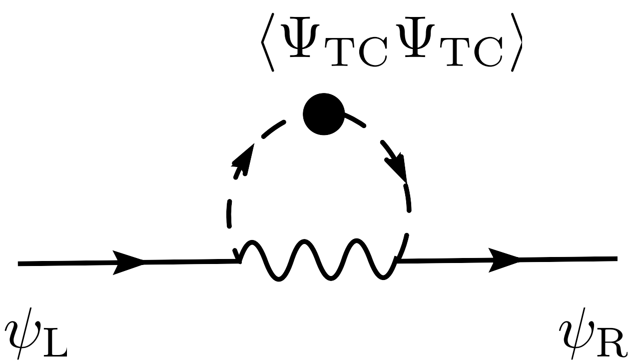


$$M_W \approx \frac{g f_\pi}{2} \simeq 29\ \mathrm{MeV}$$

Postulate super-strong interaction with $f_\pi \sim 250\ \mathrm{GeV}$.

[Weinberg, Susskind 1979]

flavour:



$$m_{\mathrm{q}} \sim \frac{\Sigma_{\mathrm{TC}}}{M_X^2} \sim \frac{\Sigma_{\mathrm{TC}}}{f_{\mathrm{TC}}^2}$$

mixing ($\subset$ FCNC)

[Dimopoulos et al. 1979; Eichten et al. 1980]

strong electroweak symmetry breaking

LATTICE GAUGE THEORIES AT THE ENERGY FRONTIER

Thomas Appelquist, Richard Brower, Simon Catterall, George Fleming,
Joel Giedt, Anna Hasenfratz, Julius Kuti, Ethan Neil, and David Schaich

(USQCD Collaboration)

(Dated: February 11, 2013)

Abstract

This White Paper has been prepared as a planning document for the Division of High Energy Physics of the U. S. Department of Energy. Recent progress in lattice-based studies of physics beyond the standard model is summarized, and major current goals of USQCD research in this area are presented. Challenges and opportunities associated with the recently discovered 126 GeV Higgs-like particle are highlighted. Computational resources needed for reaching important goals are described. The document was finalized on February 11, 2013 with references that are not aimed to be complete, or account for an accurate historical record of the field.

extremely active field, providing crucial input for the understanding of Higgs physics

conclusions

- (a large part of) QCD in the hadronic regime tamed by the lattice

- essential tool in several studies at the frontier of particle physics
  - understand hadron dynamics
  - study flavour physics
  - explore dynamics underlying EWSB and the flavour sector

- challenges for the immediate future
  - does new physics appear in flavour?
  - is dynamical EWSB compatible with LHC findings? does it offer room to understand flavour?