TAE 2013

Advanced Quantum Field Theory

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Exercise Sheet

1. **Anomalies in the Schwinger Model.** In this exercise we work in two dimensions and use light-cone coordinates $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1)$ in which the line element reads $ds^2 = 2dx^+dx^-$.

   The Schwinger model is a theory of a positive chirality fermion ψ_+ in two dimensions

$$\gamma_-\psi_+ = 0, \qquad \gamma^\pm = \frac{1}{\sqrt{2}}(\gamma^0 \pm \gamma^1), \qquad \gamma_5 = \gamma^0\gamma^1. \tag{1}$$

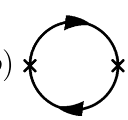
   coupled to an external classical U(1) gauge field

$$S[\psi_+, \overline{\psi}_+, \mathcal{A}_-] = \int d^2x \left(i\overline{\psi}_+ \partial\!\!\!/\, \psi_+ - e\,\mathscr{A}_- J_+\right), \qquad J_+ = \overline{\psi}_+\gamma_+\psi_+. \tag{2}$$

   a) Write the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1}$ and the chirality matrix γ_5 in terms of $\gamma^\pm$. Setting the external field to zero, show that fermions with different chirality propagate in opposite directions.

   b) Using contour integration, compute the correlation function

$$U_{++}(p) = \int d^2x\, \langle 0|T[J_+(x)J_+(0)]|0\rangle e^{ip\cdot x} \equiv J_+(p)$$



$$J_+(-p)$$

$$= -\int \frac{d^2k}{(2\pi)^2} \frac{\mathrm{Tr}[\gamma_+(p\!\!\!/ + k\!\!\!/)\gamma_+ k\!\!\!/]}{[(p+k)^2 + i\epsilon](k^2 + i\epsilon)}. \tag{3}$$

   and derive the gauge anomaly in the Schwinger model

$$p_- U_{++}(p) = \frac{i}{2\pi} p_+ \quad \Longrightarrow \quad \partial_-\langle J_+(x)\rangle_{\mathscr{A}} = -\frac{e}{2\pi}\mathscr{F}_{+-}(x). \tag{4}$$

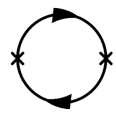
   [Hint: the identity $\mathrm{Tr}\,(\gamma_+\gamma_-\gamma_+\gamma_-) = 4$ is useful in the evaluation of the Feynman integral.]

   c) Draw the two-loop diagrams contributing to to $U_{++}(p)$ and show that they vanish. [Hint: use the algebraic properties of the γ-matrices]

   d) Show that the axial anomaly in the Schwinger model is given by

$$\int d^2x\, e^{ip\cdot x}\partial_\mu\langle J^\mu_{\mathrm{A}}(x)\rangle_{\mathscr{A}} = \mathscr{A}_-(-p)p_-U_{++}(p) - \mathscr{A}_+(-p)p_+U_{--}(p), \tag{5}$$

where $J^\mu_{\rm A}(x) = \overline{\psi}\gamma^\mu\gamma_5\psi$, $U_{++}(p)$ is defined in Eq. (4), and

$$U_{++}(p) \quad = \quad J_{-}(p)$$

$$J_{-}(-p) \tag{6}$$

Compute the axial anomaly in momentum and position space.

e) Using the result of the axial anomaly computed above and the Maxwell equations for the external field

$$\partial_\mu \mathscr{F}^{\mu\nu}(x) = \langle J^\nu(x)\rangle_{\mathscr{A}}, \tag{7}$$

show that the pseudoscalar field $\mathscr{F}^* = \frac{1}{2}\epsilon_{\mu\nu}\mathscr{F}^{\mu\nu}$ satisfies a massive Klein-Gordon equation. Discuss the physical meaning of this result. [Hint: use the identity $\gamma^\mu\gamma_5 = -\epsilon^{\mu\nu}\gamma_\nu$ valid in two dimensions.]

2. **Anomaly cancellation in the standard model (the general case).** The aim of this exercise is to show how the cancellation of anomalies fixes the assignment of hypercharges in the standard model.

For a single standard model family, let us take the representations of SU(3)×SU(2)×U(1)$_Y$ to be (the notation matches the one used in the lectures)

$$\begin{array}{lll} (N_c,2)^L_{q_L} & (1,2)^L_{\ell_L} & \\ (N_c,1)^R_{u_R} & (N_c,1)^R_{d_R} & (1,1)^R_{e_R} \end{array} \tag{8}$$

For the time being we leave the number of colors N_c and the hypercharges q_L, ℓ_L, u_R, d_R, and e_R undetermined.

a) Imposing the cancellation of both gauge and mixed gauge-gravitational anomalies, write four equations to be satisfied by N_c and the hypercharges. Show that these equations do not fix the global normalization of the hypercharges.

b) Fixing this global normalization such that $e_R = -1$, find the solutions for the hypercharges in terms of N_c. Particularize the results to the case $N_c = 3$ and discuss the results.