

TAE Statistics Problem

The purpose of this exercise is to design a statistical test to discover a signal process such as dark matter by counting events in a detector. Suppose the detector can for each event measure a quantity x with $0 \leq x \leq 1$, for which probability density functions (pdfs) are for signal (s) and background (b),

$$f(x|s) = 3(1-x)^2, \quad (1)$$

$$f(x|b) = 3x^2. \quad (2)$$

1(a) Suppose for each event we test the hypothesis that it is background. We reject this hypothesis if the observed value of x is less than a specified cut value x_{cut} . Find the value of x_{cut} such that the probability to reject the background hypothesis (i.e., accept as signal) if it is background is $\alpha = 0.05$. (The value α is the *size* or significance level of the test.)

1(b) For the value of x_{cut} that you find, what is the probability to accept an event with $x < x_{\text{cut}}$ given that it is signal. (This is the *power* of the test with respect to the signal hypothesis or equivalently the signal efficiency.)

1(c) Suppose that the expected number of background events is $b_{\text{tot}} = 100$ and for a given signal model one expects $s_{\text{tot}} = 10$ signal events. Find the expected numbers of events s and b of signal and background events that will satisfy $x < x_{\text{cut}}$ using the value of $x_{\text{cut}} = 0.1$.

1(d) Assuming the numbers from 1(c), the prior probabilities for an event to be signal or background are

$$\pi_s = \frac{s_{\text{tot}}}{s_{\text{tot}} + b_{\text{tot}}} = 0.09, \quad (3)$$

$$\pi_b = \frac{b_{\text{tot}}}{s_{\text{tot}} + b_{\text{tot}}} = 0.91. \quad (4)$$

Based on these values, what is the probability for an event to be signal given that one finds $x < x_{\text{cut}}$. (Recall Bayes' theorem or consult [arXiv:1307.2487](https://arxiv.org/abs/1307.2487).)

1(e) Now suppose we do the experiment and observe n_{obs} events in the search region $x < x_{\text{cut}}$. We now want to test the hypothesis that $s = 0$ (the background-only hypothesis or “ b ”), against the alternative that signal is present with $s \neq 0$ (the “ $s + b$ ” hypothesis).

The actual number of events n found in the experiment with $x < x_{\text{cut}}$ can be modeled as following a Poisson distribution with a mean value of $s + b$. That is, the probability to find n events is

$$P(n|s, b) = \frac{(s + b)^n}{n!} e^{-(s+b)}. \quad (5)$$

Suppose for a certain x_{cut} one has $b = 0.5$ and we find there $n_{\text{obs}} = 3$ events. The p -value of the background-only hypothesis is the probability, assuming $s = 0$, to find $n \geq n_{\text{obs}}$.

$$p = P(n \geq n_{\text{obs}} | s = 0, b) = \sum_{n=n_{\text{obs}}}^{\infty} \frac{b^n}{n!} e^{-b} = 1 - \sum_{n=0}^{n_{\text{obs}}-1} \frac{b^n}{n!} e^{-b}. \quad (6)$$

Find the p -value and from this find the *significance* with which one can reject the $s = 0$ hypothesis, defined as

$$Z = \Phi^{-1}(1 - p), \quad (7)$$

where Φ is the standard cumulative Gaussian distribution and Φ^{-1} is its inverse (the standard Gaussian quantile). For more information see Sec. 10 of [arXiv:1307.2487](https://arxiv.org/abs/1307.2487). You will need the cumulative chi-square distribution and the quantile of the Gaussian distribution, which from ROOT are available as `1 - TMath::Prob` and `TMath::NormQuantile`.

1(f) The expected (median) significance assuming the $s + b$ hypothesis of the test of the $s = 0$ hypothesis is a measure of sensitivity and this is what one tries to maximize when designing an experiment. It can be approximated with a number of different formulas. For $s \ll b$ one can use $\text{med}[Z_b | s + b] = s / \sqrt{b}$. If $s \ll b$ does not hold, a better approximation is

$$\text{med}[Z_b | s + b] = \sqrt{2 \left((s + b) \ln \left(1 + \frac{s}{b} \right) - s \right)}. \quad (8)$$

Using Eq. (8), find me median significance for $x_{\text{cut}} = 0.1$. If you have time, try to write a program to find the value of x_{cut} that maximizes the median significance.

1(g) Now suppose that for each event we do not simply count the events having x in a certain region but we design a test that exploits each measured value in the entire range $0 \leq x \leq 1$. The data thus consist of the number n of events, which follows a Poisson distribution with mean of $s + b$, and the n values x_1, \dots, x_n .

We can define a test statistic to test the background-only hypothesis that is a monotonic function of the likelihood ratio L_{s+b}/L_b ,

$$q = -2 \sum_{i=1}^n \left[1 + \frac{s_{\text{tot}}}{b_{\text{tot}}} \frac{f(x_i | s)}{f(x_i | b)} \right] \quad (9)$$

The motivation for this statistic is described further in Sec. 5.1 of [arXiv:1307.2487](https://arxiv.org/abs/1307.2487).

From <http://www.pp.rhul.ac.uk/~cowan/stat/invisibles/mc/> download the program `invisibleMC.cc` and the makefile. Build and run the program. This will produce histograms of q under the $s + b$ hypothesis, and also a histogram of q (called `h_q_sb`) and it will find the median q , $\text{med}[q | s + b]$.

You should add code in analogy with this that generates data according to the background-only ($s = 0$) hypothesis. Generate 10^7 experiments and count how many have $q < \text{med}[q | s + b]$. The fraction with $q < \text{med}[q | s + b]$ is the median p -value of the background-only hypothesis. Find this and from it find the median significance Z (the sensitivity). Compare to the values you found from Eq. (8).