HIGH DENSITY MATTER

1. Problem 1.

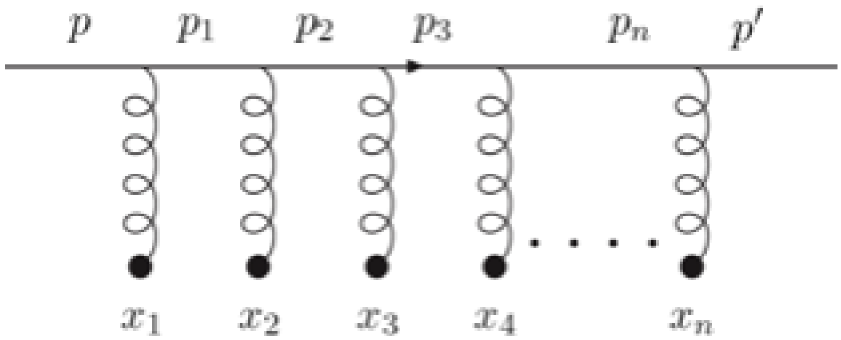

Figure 1. example caption

Consider a massless, fast-moving quark moving propagating through a "dense" medium (it could be a nucleus or a QGP) along the z (3rd) direction. It is convenient to work in light-cone coordinates:

$$a^\mu = (a^+, a^-, a_\perp), \quad \text{with} \quad a^\pm = \frac{a^0 \pm a^3}{\sqrt{2}} \; ; \, a_\perp \equiv (a^1, a^2) \tag{1}$$

In these coordinates the scalar product is written as

$$a \cdot b = a^+ b^- + a^- b^+ + a_\perp \cdot b_\perp. \tag{2}$$

Thus, the momentum of the incoming quark can be written as $p^\mu \simeq (p^+, 0, 0_\perp)$.

Question 1: How does the metric look like in these coordinates? How do you raise and lower indeces?

The quark will interact with the color sources in the medium through multiple gluon exchange. The contribution to the scattering matrix from a single scattering is

$$S_1(p,p') = \int d^4x \, e^{i(p-p')\cdot x} \bar{u}^{s'}(p') \, igA^a_\mu \, t^a \, \gamma^\mu \, u^s(p), \tag{3}$$

where p' is the outgoing quark momentum.

Question 2: Using the *eikonal* approximation, $p \sim p'$ (why?), and the relation $\frac{1}{2}\sum_{spins} \bar{u}^{s'}(p)\gamma^\mu u^s(p) = 2p^\mu$, show that S_1 can be writen as:

$$S_1(p,p') \simeq 2\pi\delta(p'^+ - p^+)2p^+ \int dx_\perp e^{ix_\perp\cdot(p'_\perp - p_\perp)} \left[ig \int dx^+ A^-(x^+, x_\perp)\right], \tag{4}$$

where $A^\mu \equiv A^\mu t^a$ henforth. Relying again in the eikonal approximation, $p^+ >>$, demonstrate that the contribution from two rescatterings is given by

$$S_2(p',p) \simeq 2\pi\delta(p'^+ - p^+)2p^+ \int dx_\perp e^{-ix_\perp\cdot(p'_\perp - p_\perp)} \frac{1}{2}\mathcal{P}\left[ig \int dx^+ A^-(x^-, x^+)\right]^2 \tag{5}$$

Finally, yse the expression above to guess the contribution from n rescatterings, $S_n(p',p)$. Then, demonstrate that the total scattering matrix is given by:

$$S(p',p) = \sum_{n=0}^{\infty} S_n(p',p) \simeq 2\pi\delta(p'^+ - p^+)2p^+ \int dx_\perp e^{-ix_\perp\cdot(p'_\perp - p_\perp)} W(x_\perp), \tag{6}$$

where $W(x_\perp$ is the following Wilson line:

$$W(x_\perp) = \mathcal{P}\exp\left[ig \int dx^+ A^-(x^-, x_\perp)\right]. \tag{7}$$

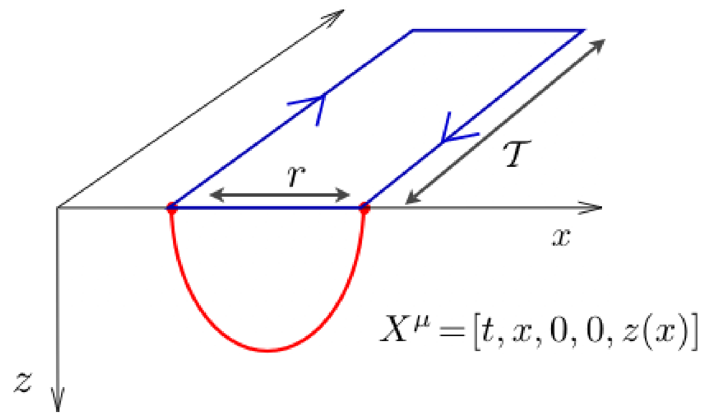

Figure 2. example caption

2. Problem 2

We want to calculate the heavy quark potential in $\mathcal{N} = 4$ Super Yang Mills theory using the AdS/CFT correspondence (Maldacena; hep-th/9803002). In the gauge theory, the heavy quark potential $V_{Q\bar{Q}}(r)$ is given by a static (time-like) Wilson loop connecting the quark and the antiquark (see figure 2):

$$\langle W(r) \rangle_{static} = \lim_{T \to \infty} \exp \left[-V_{Q\bar{Q}}(r) \, \mathcal{T} \right] \tag{8}$$

where r is the separation between the quark and the antiquark and $\mathcal{T}$ the temporal extent of the loop. On the gravity side, the interaction is represented by a string connecting the quark and the antiquark and “holding“ down in the 5th dimension, z. The prescription to calculate the Wilson loop under the correspondence is the following:

$$\langle W(r) \rangle_{static} = \exp \left[-S_{NG} \right], \tag{9}$$

where S_{NG} is the (classical) Nambu-Goto action of the two-dimensional worldsheet spanned by the string. It is defined as

$$S_{NG} = -\frac{1}{2\pi \, \alpha'} \int d\sigma d\tau \det \sqrt{-g_{MN} \partial_a X^M \partial_b X^N}, \tag{10}$$

where $\{M, N\} = 1...5$, $\{a, b\}$ are the two coordinates that parametrize the two-dimensional string worldsheet and α' is the string tension. The metric of the AdS$_5$

space can be written as follows

$$ds^2 = g_{MN}dx^M dx^N = \frac{L^2}{z^2}\left[-dt^2 + d\vec{x}^2 + dz^2\right], \tag{11}$$

where $(t, \vec{x})$ are the usual 4 dimensions, z is the coordinate describing the 5th dimension and L is the curvature of the AdS$_5$ space (constant). It is convenient to parametrize the string coordinates in terms of the $x-t$ coordinates, i.e. $(\sigma, \tau) = (x, t)$. Since the problem at hand is static, we have

$$X^M = \left[X^1 = t, X^2 = x, X^3 = cte, X^4 = cte, X^5 = z(x)\right] \tag{12}$$

Question 1: Using Eqs. (10-12), derive the Nambu-Goto action.

Question 2: Extremize the obtained NG action by deriving and solving the corresponding Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial\, z(x)} - \frac{\partial}{\partial\, x}\frac{\partial \mathcal{L}}{\partial z'(x)} = 0, \tag{13}$$

with the appropriate boundary conditions: $z(x = \pm r/2) = 0$. $\mathcal{L}$ is the Lagrangian density.

Question 3: Evaluate the classical NG action. Notice that it is divergent when $z \to 0$. In order to obtain a finite result subtract the action corresponding to two straight strings hanging from the quark and antiquark positions to the end of the AdS space, i. e, replace

$$S_{NG} \to S_{NG} - 2S_{NG}^{mass}, \quad \text{with } S_{NG}^{mass} = \mathcal{T}\int_0^\infty dz \lim_{z'\to\infty} \mathcal{L}_{NG}(t, z(x)). \tag{14}$$

What is the physical meaning of the subtracted term, $2S_{NG}^{mass}$? Using (8-9), calculate the heavy quark potential. Interpret physically the result.

Question 4: Using exactly the same rationale as above, derive the heavy quark potential at finite temperature. The only difference is that now the background metric is given by the “black brane“ gravity solution:

$$ds^2 = g_{MN}dx^M dx^N = \frac{L^2}{z^2}\left[-\left(1 - \frac{z^4}{z_H^4}\right)dt^2 + d\vec{x}^2 + \frac{dz^2}{1 - \frac{z^4}{z_H^4}}\right], \tag{15}$$

where $z_H = 1/(\pi\, T)$ is the location of the black hole horizon along the 5th dimension and T is the temperature in the gauge theory.

Hints

- Use the following relation between the string tension, α' the curvature, L, and the 't Hooft coupling, $\lambda = g_{YM}^2 N_c$, to simplify the expressions:

$$\frac{1}{\alpha'} = \frac{\sqrt{\lambda}}{L^2} \tag{16}$$

- The Euler-Lagrange equation is a second order differential equation in x. It has, therefore, two constants of integration. The first one can be sorted out from the boundary conditions $z(x = \pm r/2 = 0)$. The second one can be obtained from the condition that the string maximum along the 5th coordinate, z_m, is reached at $x = 0$, i.e $z(x = 0) = z_m$, $z'(x = 0) = 0$.
- Some formulae

$$\int_0^{z_m} dz \frac{z^2}{\sqrt{z_m^4 - z^4}} = \frac{z_m}{2c_0}\,, \quad \text{with } c_0 = \frac{\Gamma\left(\frac{1}{4}\right)^2}{(2\pi)^{3/2}} \tag{17}$$

$$\int_\epsilon^{z_m} \frac{dz}{z^2} \frac{z_m^2}{\sqrt{z_m^4 - z^4}} = \frac{1}{\epsilon} - \frac{1}{2\,c_0\,z_m} + \mathcal{O}(\epsilon) \tag{18}$$