Exercise 1: Covariant derivative

Prove that the term $\overline{\Psi}\not{D}\Psi$ where the covariant derivative is given by:

$$D_\mu = \partial_\mu - \mathrm{i}g\widetilde{W}_\mu \;, \quad \widetilde{W}_\mu = T_a W^a_\mu$$

is invariant under gauge transformations:

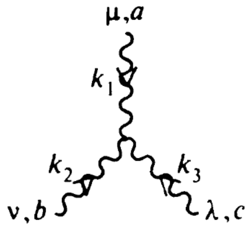
$$\Psi \mapsto \Psi' = U\Psi \;, \quad U = \exp\{-\mathrm{i}T_a\theta^a(x)\}$$
$$\widetilde{W}_\mu \mapsto \widetilde{W}'_\mu = U\widetilde{W}_\mu U^\dagger - \frac{\mathrm{i}}{g}(\partial_\mu U)U^\dagger$$

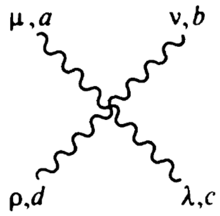
$$\begin{aligned}
\Psi &\mapsto U\Psi \\
\widetilde{W}_\mu &\mapsto U\widetilde{W}_\mu U^\dagger - \frac{\mathrm{i}}{g}(\partial_\mu U)U^\dagger \\
D_\mu\Psi = (\partial_\mu - \mathrm{i}g\widetilde{W}_\mu)\Psi &\mapsto (\partial_\mu - \mathrm{i}gU\widetilde{W}_\mu U^\dagger - (\partial_\mu U)U^\dagger)U\Psi \\
&= (\partial_\mu U + U\partial_\mu - \mathrm{i}gU\widetilde{W}_\mu - \partial_\mu U)\Psi \\
&= U(\partial_\mu - \mathrm{i}g\widetilde{W}_\mu)\Psi \\
&= UD_\mu\Psi
\end{aligned}$$

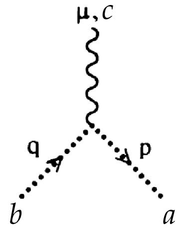
$$\Rightarrow \overline{\Psi}\not{D}\Psi \mapsto \overline{\Psi}U^\dagger U\not{D}\Psi = \overline{\Psi}\not{D}\Psi$$

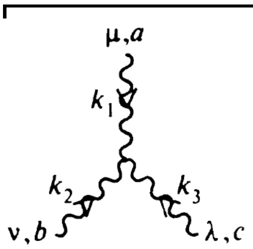
Exercise 2: Feynman rules of general non-Abelian gauge theories

Obtain the Feynman rules for cubic and quartic self-interactions among gauge fields in a general non-Abelian gauge theory, as well as those for the interactions of Faddeev-Popov ghosts with gauge fields:









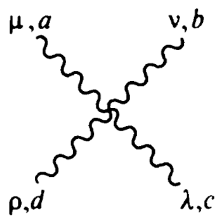
$$\mathcal{L}_{\text{cubic}} = -\frac{1}{2}g f_{abc}\,(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu)W^{b,\mu}W^{c,\nu}$$

$$\Rightarrow \quad g f_{abc}\,[g_{\mu\nu}(k_1 - k_2)_\lambda + g_{\nu\lambda}(k_2 - k_3)_\mu + g_{\lambda\mu}(k_3 - k_1)_\nu]$$

Proof:
Note that there is a summation over repeated indices abc and assume totally antisymmetric structure constants f_{abc}. Then, fixing abc the summation is over all possible permutations of three indices 123 labeling the momenta and polarization vectors. Write $-\mathrm{i}k$ for every incoming momentum and a polarization vector $\epsilon(k)$ for every incoming vector boson. The polarization vectors factor out of the vertex definition:

$$\begin{aligned}
\Gamma_{\mu\nu\lambda}[V^a(k_1), V^b(k_2), V^c(k_3)]\epsilon_1^\mu\epsilon_2^\nu\epsilon_3^\lambda &= -\frac{\mathrm{i}}{2} g f_{abc}\ (-\mathrm{i}k_{a\mu}\epsilon_{a\nu} + \mathrm{i}k_{a\nu}\epsilon_{a\mu})\epsilon_b^\mu\epsilon_c^\nu \\
&\qquad + (ab) + (ac) + (bc) + (abc) + (acb) \\
&= -\frac{1}{2} g f_{abc}\ [(k_1\epsilon_2)(\epsilon_1\epsilon_3) - (k_1\epsilon_3)(\epsilon_1\epsilon_2) \\
&\qquad - (k_2\epsilon_1)(\epsilon_2\epsilon_3) + (k_2\epsilon_3)(\epsilon_2\epsilon_1) \\
&\qquad - (k_3\epsilon_2)(\epsilon_3\epsilon_1) + (k_3\epsilon_1)(\epsilon_3\epsilon_2) \\
&\qquad - (k_1\epsilon_3)(\epsilon_1\epsilon_2) + (k_1\epsilon_2)(\epsilon_1\epsilon_3) \\
&\qquad + (k_2\epsilon_3)(\epsilon_2\epsilon_1) - (k_2\epsilon_1)(\epsilon_2\epsilon_3) \\
&\qquad + (k_3\epsilon_1)(\epsilon_3\epsilon_2) - (k_3\epsilon_2)(\epsilon_3\epsilon_1)] \\
&= g f_{abc}\ \{[(k_1\epsilon_3) - (k_2\epsilon_3)](\epsilon_1\epsilon_2) \\
&\qquad + (k_2\epsilon_1) - (k_3\epsilon_1)](\epsilon_2\epsilon_3) \\
&\qquad + (k_3\epsilon_2) - (k_3\epsilon_2)](\epsilon_3\epsilon_1)\}
\end{aligned}$$

$$\Rightarrow \Gamma_{\mu\nu\lambda}[V^a(k_1), V^b(k_2), V^c(k_3)] = g f_{abc}\ [g_{\mu\nu}(k_1 - k_2)_\lambda + g_{\nu\lambda}(k_2 - k_3)_\mu + g_{\lambda\mu}(k_3 - k_1)_\nu]$$



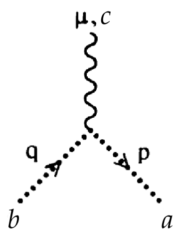
$$\begin{aligned}
\mathcal{L}_{\text{quartic}} &= -\frac{1}{4} g^2 f_{abe} f_{cde}\ W^a_\mu W^b_\nu W^{c,\mu} W^{d,\nu} \\
\Rightarrow &\quad -\mathrm{i}g^2\ [\ f_{abe} f_{cde}\ (g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}) \\
&\qquad + f_{ace} f_{dbe}\ (g_{\mu\rho}g_{\nu\lambda} - g_{\mu\nu}g_{\lambda\rho}) \\
&\qquad + f_{ade} f_{bce}\ (g_{\mu\nu}g_{\lambda\rho} - g_{\mu\lambda}g_{\nu\rho})]
\end{aligned}$$

Proof:
Fixing $abcd$ the summation over repeated indices leads to the following set of permutations:

$$\begin{aligned}
&\Gamma_{\mu\nu\lambda\rho}[V^a(k_1), V^b(k_2), V^c(k_3), V^d(k_4)]\epsilon_1^\mu\epsilon_2^\nu\epsilon_3^\lambda\epsilon_4^\rho \\
&\quad = -\frac{\mathrm{i}}{4} g^2 \{ f_{abe} f_{cde}\ [(\epsilon_a\epsilon_c)(\epsilon_b\epsilon_d) + (ac) + (bd) + (ac)(bd)] \\
&\qquad + f_{cbe} f_{ade}\ [(\epsilon_c\epsilon_a)(\epsilon_b\epsilon_d) + (ac) + (bd) + (ac)(bd)] && \Leftarrow a \leftrightarrow c \\
&\qquad + f_{dbe} f_{cae}\ [(\epsilon_d\epsilon_c)(\epsilon_b\epsilon_a) + (ab) + (cd) + (ab)(cd)] && \Leftarrow a \leftrightarrow d \\
&\qquad + f_{ace} f_{bde}\ [(\epsilon_a\epsilon_b)(\epsilon_c\epsilon_d) + (ab) + (cd) + (ab)(cd)] && \Leftarrow b \leftrightarrow c \\
&\qquad + f_{ade} f_{cbe}\ [(\epsilon_a\epsilon_c)(\epsilon_d\epsilon_b) + (ac) + (bd) + (ac)(bd)] && \Leftarrow b \leftrightarrow d \\
&\qquad + f_{dce} f_{bae}\ [(\epsilon_d\epsilon_b)(\epsilon_c\epsilon_a) + (ac) + (bd) + (ac)(bd)]\} && \Leftarrow a \leftrightarrow d\ \&\ b \leftrightarrow c
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\mathrm{i}}{4}g^2\{f_{abe}f_{cde}\ [(\epsilon_1\epsilon_3)(\epsilon_2\epsilon_4)-(\epsilon_2\epsilon_3)(\epsilon_1\epsilon_4)-(\epsilon_1\epsilon_4)(\epsilon_2\epsilon_3)+(\epsilon_2\epsilon_4)(\epsilon_1\epsilon_3)]\\
&\qquad +f_{cbe}f_{ade}\ [(\epsilon_3\epsilon_1)(\epsilon_2\epsilon_4)-(\epsilon_2\epsilon_1)(\epsilon_3\epsilon_4)-(\epsilon_3\epsilon_4)(\epsilon_2\epsilon_1)+(\epsilon_2\epsilon_4)(\epsilon_3\epsilon_1)]\\
&\qquad +f_{dbe}f_{cae}\ [(\epsilon_4\epsilon_3)(\epsilon_2\epsilon_1)-(\epsilon_2\epsilon_3)(\epsilon_4\epsilon_1)-(\epsilon_4\epsilon_1)(\epsilon_2\epsilon_3)+(\epsilon_2\epsilon_1)(\epsilon_4\epsilon_3)]\\
&\qquad +f_{ace}f_{bde}\ [(\epsilon_1\epsilon_2)(\epsilon_3\epsilon_4)-(\epsilon_3\epsilon_2)(\epsilon_1\epsilon_4)-(\epsilon_1\epsilon_4)(\epsilon_3\epsilon_2)+(\epsilon_3\epsilon_4)(\epsilon_1\epsilon_2)]\\
&\qquad +f_{ade}f_{cbe}\ [(\epsilon_1\epsilon_3)(\epsilon_4\epsilon_2)-(\epsilon_4\epsilon_3)(\epsilon_1\epsilon_2)-(\epsilon_1\epsilon_2)(\epsilon_4\epsilon_3)+(\epsilon_4\epsilon_2)(\epsilon_1\epsilon_3)]\\
&\qquad +f_{dce}f_{bae}\ [(\epsilon_4\epsilon_2)(\epsilon_3\epsilon_1)-(\epsilon_3\epsilon_2)(\epsilon_4\epsilon_1)-(\epsilon_4\epsilon_1)(\epsilon_3\epsilon_2)+(\epsilon_3\epsilon_1)(\epsilon_4\epsilon_2)]\}\\
&= -\mathrm{i}g^2\{f_{abe}f_{cde}\ [(\epsilon_1\epsilon_3)(\epsilon_2\epsilon_4)-(\epsilon_1\epsilon_4)(\epsilon_2\epsilon_3)]\\
&\qquad +f_{ace}f_{dbe}\ [(\epsilon_1\epsilon_4)(\epsilon_2\epsilon_3)-(\epsilon_1\epsilon_2)(\epsilon_3\epsilon_4)]\\
&\qquad +f_{ade}f_{bce}\ [(\epsilon_1\epsilon_2)(\epsilon_3\epsilon_4)-(\epsilon_1\epsilon_3)(\epsilon_2\epsilon_4)]\}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \Gamma_{\mu\nu\lambda\rho} = -\mathrm{i}g^2\ [\ &f_{abe}f_{cde}\ (g_{\mu\lambda}g_{\nu\rho}-g_{\mu\rho}g_{\nu\lambda})\\
+&f_{ace}f_{dbe}\ (g_{\mu\rho}g_{\nu\lambda}-g_{\mu\nu}g_{\lambda\rho})\\
+&f_{ade}f_{bce}\ (g_{\mu\nu}g_{\lambda\rho}-g_{\mu\lambda}g_{\nu\rho})]
\end{aligned}
$$



$$\mathcal{L}_{\mathrm{FP}} \supset -gf_{abc}(\partial^\mu \bar{c}_a)c_b W^c_\mu$$

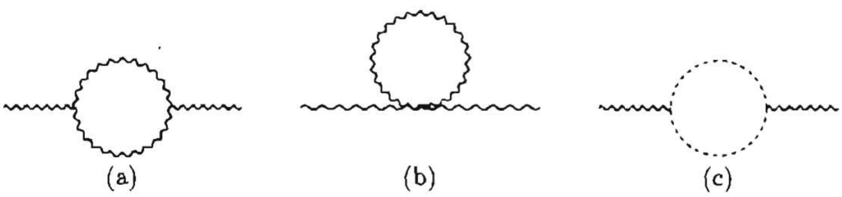
$$\Rightarrow \quad g f_{abc} p_\mu$$

Proof:
Momentum p is outgoing. Then:

$$\Gamma_\mu = -\mathrm{i}g f_{abc}\mathrm{i}p_\mu = g f_{abc} p_\mu$$

Exercise 3: Faddeev-Popov ghosts and gauge invariance

Consider the 1-loop self-energy diagrams for non-Abelian gauge theories in the figure. Calculate the diagrams in the 't Hooft-Feynman gauge and show that the sum does not have the tensor structure $g_{\mu\nu}k^2 - k_\mu k_\nu$ required by the gauge invariance of the theory unless diagram (c) involving ghost fields is included.



Hint: Take Feynman rules from previous excercise and use dimensional regularization. It is convenient to use the Passarino-Veltman tensor decomposition of loop integrals:

$$\frac{\mathrm{i}}{16\pi^2}\{B_0,\, B_\mu,\, B_{\mu\nu}\} = \mu^\epsilon \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{\{1,\, q_\mu,\, q_\mu q_\nu\}}{q^2(q+k)^2}$$

$$\text{where} \quad B_0 = \Delta_\epsilon + \text{finite}$$

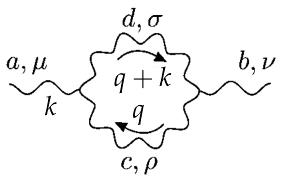
$$B_\mu = k_\mu B_1\,, \quad B_1 = -\frac{\Delta_\epsilon}{2} + \text{finite}$$

$$B_{\mu\nu} = g_{\mu\nu} B_{00} + k_\mu k_\nu B_{11}\,, \quad B_{00} = -\frac{k^2}{12}\Delta_\epsilon + \text{finite}\,, \quad B_{11} = \frac{\Delta_\epsilon}{3} + \text{finite}$$

with $\Delta_\epsilon = 2/\epsilon - \gamma + \ln 4\pi$ and $D = 4 - \epsilon$. You may check that the ultraviolet divergent part has the expected structure or find the final result in terms of scalar integrals, that for this configuration of masses and momenta read:

$$B_1 = -\frac{1}{2}B_0\,, \quad B_{00} = -\frac{k^2}{4(D-1)}B_0\,, \quad B_{11} = \frac{D}{4(D-1)}B_0\,.$$

Do not forget a symmetry factor (1/2) in front of (a) and (b), and a factor (-1) in (c).



$$= \frac{1}{2}\mu^\epsilon \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{-\mathrm{i}}{q^2} \frac{-\mathrm{i}}{(q+k)^2} g^2 f_{acd} f_{bcd} N_{\mu\nu}$$

$$= -\frac{g^2 C_2(G)\delta_{ab}}{2}\mu^\epsilon \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{N_{\mu\nu}}{q^2(q+k)^2}$$

where $f_{acd} f_{bcd} \equiv C_2(G)\delta_{ab}$

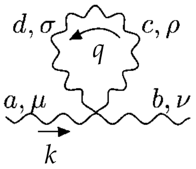
and $N_{\mu\nu} = -g_{\mu\nu}(2q^2 + 5k^2 + 2qk) - 10 q_\mu q_\nu - 5(q_\mu k_\nu + k_\nu q_\mu) + 2k_\mu k_\nu$

$$= -\frac{\mathrm{i}g^2 C_2(G)\delta_{ab}}{32\pi^2}\left\{-g_{\mu\nu}(2g_{\rho\sigma}B^{\rho\sigma} + 5k^2 B_0 + 2k_\rho B^\rho) - 10 B_{\mu\nu} - 5(B_\mu k_\nu + B_\nu K_\mu) + 2k_\mu k_\nu B_0\right\}$$

$$= -\frac{\mathrm{i}g^2 C_2(G)\delta_{ab}}{32\pi^2}\left\{-g_{\mu\nu}[(2D+10)B_{00} + 2k^2 B_{11} + 2k^2 B_1 + 5k^2 B_0] - k_\mu k_\nu(10 B_{11} + 10 B_1 - 2B_0)\right\}$$

$$= -\frac{\mathrm{i}g^2 C_2(G)\delta_{ab}}{32\pi^2}\frac{B_0}{2(D-1)}\left\{-(8D-13)g_{\mu\nu}k^2 - (9D-14)k_\mu k_\nu\right\}$$

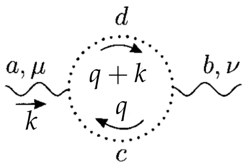
$$= -\frac{\mathrm{i}g^2 C_2(G)\delta_{ab}}{32\pi^2}\Delta_\epsilon\left\{-\frac{19}{6}g_{\mu\nu}k^2 + \frac{11}{3}k_\mu k_\nu\right\} + \text{finite}$$



$$= \frac{1}{2}\mu^\epsilon \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{-\mathrm{i}g^{\rho\sigma}}{q^2}\delta_{cd}(-\mathrm{i}g^2)[\; f_{abe} f_{cde}\,(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma})$$
$$+ f_{ace} f_{dbe}\,(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\nu}g_{\sigma\rho})$$
$$+ f_{ade} f_{bce}\,(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\sigma}g_{\nu\rho})]$$

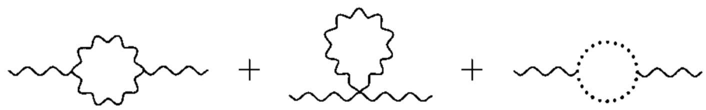
$$= -g^2 C_2(G)\delta_{ab}\mu^\epsilon \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{(D-1)g_{\mu\nu}}{q^2}$$

$$
\begin{aligned}
&= -g^2 C_2(G)\delta_{ab}\mu^\epsilon \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{(D-1)g_{\mu\nu}}{q^2} \frac{q^2+2qk+k^2}{(q+k)^2} \\
&= -\frac{\mathrm{i}g^2 C_2(G)\delta_{ab}}{16\pi^2}(D-1)g_{\mu\nu}(g_{\rho\sigma}B^{\rho\sigma} + 2k_\rho B^\rho + k^2 B_0) \\
&= -\frac{\mathrm{i}g^2 C_2(G)\delta_{ab}}{16\pi^2}(D-1)g_{\mu\nu}(DB_{00} + k^2 B_{11} + 2k^2 B_1 + k^2 B_0) \\
&= 0
\end{aligned}
$$



$$
\begin{aligned}
&= (-1)\mu^\epsilon \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{\mathrm{i}}{q^2} \frac{\mathrm{i}}{(q+k)^2} g^2 f_{dca}\,(q_\mu + k_\mu)\, f_{cdb}\, q_\nu \\
&= -g^2 C_2(G)\delta_{ab}\mu^\epsilon \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{(q_\mu + k_\mu)q_\nu}{q^2(q+k)^2} \\
&= -\frac{\mathrm{i}g^2 C_2(G)\delta_{ab}}{16\pi^2}(B_{\mu\nu} + k_\mu B_\nu) \\
&= -\frac{\mathrm{i}g^2 C_2(G)\delta_{ab}}{16\pi^2}(g_{\mu\nu}B_{00} + k_\mu k_\nu B_{11} + k_\mu k_\nu B_1) \\
&= -\frac{\mathrm{i}g^2 C_2(G)\delta_{ab}}{32\pi^2}\frac{B_0}{2(D-1)}\left\{g_{\mu\nu}k^2 - (2-D)k_\mu k_\nu\right\} \\
&= -\frac{\mathrm{i}g^2 C_2(G)\delta_{ab}}{32\pi^2}\Delta_\epsilon \left\{-\frac{1}{6}g_{\mu\nu}k^2 - \frac{1}{3}k_\mu k_\nu\right\} + \text{finite}
\end{aligned}
$$

Summing all three diagrams (actually diagram (b) does not contribute):



$$
= -\frac{\mathrm{i}g^2 C_2(G)\delta_{ab}}{16\pi^2}\frac{B_0}{D-1}(2D-3)\left\{g_{\mu\nu}k^2 - k_\mu k_\nu\right\} = -\frac{\mathrm{i}g^2 C_2(G)\delta_{ab}}{16\pi^2}\Delta_\epsilon \frac{5}{3}\left\{g_{\mu\nu}k^2 - k_\mu k_\nu\right\} + \text{finite}
$$

Exercise 4: Propagator of a massive vector boson field

Consider the Proca Lagrangian of a massive vector boson field

$$
\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2 A_\mu A^\mu\,, \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,.
$$

Show that the propagator of A_μ is

$$
\widetilde{D}_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 - M^2 + \mathrm{i}\epsilon}\left[-g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2}\right]
$$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2 A_\mu A^\mu = -\frac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu) + \frac{1}{2}M^2 A_\mu A^\mu$$

Euler-Lagrange: $\dfrac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = 0 \quad \Rightarrow \quad M^2 A^\nu + \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = 0$

$$\Rightarrow \quad [g^{\mu\nu}(\Box + M^2) - \partial^\mu \partial^\nu] A_\mu = 0$$

The propagator is i times the inverse of the operator in square brackets. In momentum space:

$$\widetilde{D}_{\mu\nu}(k) = \mathrm{i}[-g^{\mu\nu}(k^2 - M^2) + k^\mu k^\nu]^{-1} = \frac{\mathrm{i}}{k^2 - M^2 + \mathrm{i}\epsilon}\left[-g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2}\right]$$

where the Feynman's prescription has been included. To show that this is actually the inverse, check:

$$\widetilde{D}_{\mu\nu}(k)[-g^{\nu\rho}(k^2 - M^2) + k^\nu k^\rho] = \mathrm{i}\delta_\mu^\rho$$

Exercise 5: Propagator of a massive gauge field

Consider the U(1) gauge invariant Lagrangian $\mathcal{L}$ with gauge fixing $\mathcal{L}_{\mathrm{GF}}$:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^\dagger(D^\mu\phi) - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2$$

$$\mathcal{L}_{\mathrm{GF}} = -\frac{1}{2\xi}(\partial_\mu A^\mu - \xi M_A \chi)^2\,, \quad \text{with} \quad D_\mu = \partial_\mu + \mathrm{i}eA_\mu\,, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

where $M_A = ev$ after spontanteous symmetry breaking ($\mu^2 < 0$, $\lambda > 0$) when the complex scalar field ϕ acquires a VEV and is parameterized by

$$\phi(x) = \frac{1}{\sqrt{2}}[v + \varphi(x) + \mathrm{i}\chi(x)]\,, \quad \mu^2 = -\lambda v^2.$$

Show that the propagators of φ, χ and the gauge field A_μ are respectively

$$\widetilde{D}^\varphi(k) = \frac{\mathrm{i}}{k^2 - M_\varphi^2 + \mathrm{i}\epsilon} \quad \text{with } M_\varphi^2 = -2\mu^2 = 2\lambda v^2$$

$$\widetilde{D}^\chi(k) = \frac{\mathrm{i}}{k^2 - \xi M_A^2 + \mathrm{i}\epsilon}\,, \quad \widetilde{D}_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 - M_A^2 + \mathrm{i}\epsilon}\left[-g_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2 - \xi M_A^2}\right]$$

Writing ϕ in terms of φ and χ:

$$\begin{aligned}\mathcal{L} + \mathcal{L}_{\mathrm{GF}} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M_A^2 A_\mu A^\mu - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \\ &+ \frac{1}{2}(\partial_\mu\chi)(\partial^\mu\chi) - \frac{1}{2}\xi M_A^2\chi^2 \\ &+ \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - \lambda v^2\varphi^2 + \ldots\end{aligned}$$

– Propagator of A_μ:

$$\text{Euler-Lagrange:}\quad \frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = 0 \quad \Rightarrow \quad \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) + \frac{1}{\xi}\partial^\nu\partial^\mu A_\mu = 0$$

$$\Rightarrow \quad \left[g^{\mu\nu}\Box - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu\right] A_\mu = 0$$

The propagator is i times the inverse of the operator in square brackets. In momentum space:

$$\widetilde{D}_{\mu\nu}(k) = \mathrm{i}\left[-g^{\mu\nu}k^2 + \left(1-\frac{1}{\xi}\right)k^\mu k^\nu\right]^{-1} = \frac{\mathrm{i}}{k^2 - M^2 + \mathrm{i}\epsilon}\left[-g_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]$$

where the Feynman's prescription has been included. In fact,

$$\widetilde{D}_{\mu\nu}(k)\left[-g^{\nu\rho}k^2 + \left(1-\frac{1}{\xi}\right)k^\nu k^\rho\right] = \mathrm{i}\delta_\mu^\rho$$

– Propagator of χ:

$$\text{Euler-Lagrange:}\quad \frac{\partial \mathcal{L}}{\partial \chi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \chi)} = 0 \quad \Rightarrow \quad [\Box\chi - \xi M_A^2]\chi = 0$$

The propagator is $-\mathrm{i}$ times the inverse of the operator in square brackets. In momentum space:

$$\widetilde{D}(k) = -\mathrm{i}[-k^2 - \xi M_A^2]^{-1} = \frac{\mathrm{i}}{k^2 - \xi M_A^2 + \mathrm{i}\epsilon}$$

– Propagator of φ. Similarly to previous case:

$$\widetilde{D}(k) = \frac{\mathrm{i}}{k^2 - M_\varphi^2 + \mathrm{i}\epsilon}\,, \quad M_\varphi^2 = 2\lambda v^2$$

Exercise 6: The conjugate Higgs doublet

Show that $\Phi^c \equiv i\sigma_2\Phi^* = \begin{pmatrix}\phi^{0*}\\ -\phi^-\end{pmatrix}$ transforms under SU(2) like $\Phi = \begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix}$, with $\phi^- = (\phi^+)^*$. What are the weak isospins, hypercharges and electric charges of ϕ^0, ϕ^{0*}, ϕ^+, ϕ^-?
Hint: Use the property of Pauli matrices: $\sigma_i^* = -\sigma_2\sigma_i\sigma_2$.

Consider an infinitesimal SU(2) transformation:

$$\Phi = \begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix} \mapsto \left(1 - \mathrm{i}\frac{\sigma_i}{2}\delta\theta^i\right)\Phi$$

$$\Rightarrow \Phi^c = \begin{pmatrix}-\phi^{0*}\\ \phi^-\end{pmatrix} = \mathrm{i}\sigma_2\Phi^* \mapsto \mathrm{i}\sigma_2\left(1 - \mathrm{i}\frac{\sigma_i}{2}\delta\theta^i\right)^*\Phi^* = \left(1 + \mathrm{i}\sigma_2\frac{\sigma_i^*}{2}\sigma_2\delta\theta^i\right)\mathrm{i}\sigma_2\Phi^*$$

$$= \left(1 - \mathrm{i}\frac{\sigma_i}{2}\delta\theta^i\right)\Phi^c$$

Under a U(1) transformation:

$$\Phi \mapsto \mathrm{e}^{-\mathrm{i}\alpha Y}\Phi \quad \Rightarrow \quad \Phi^c \mapsto \mathrm{e}^{\mathrm{i}\alpha Y}\Phi^c$$

Then, using $Q = T_3 + Y$ and taking Φ with hypercharge $y = \frac{1}{2}$ we have

	ϕ^0	ϕ^+	ϕ^{0*}	ϕ^-
T_3	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
Y	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
Q	0	1	0	-1

Exercise 7: Lagrangian and Feynman rules of the Standard Model

Try to reproduce the Lagrangian and the corresponding Feynman rules of as many Standard Model interactions as you can. Of particular interest/difficulty are [VVV] and [VVVV].

Check your results in http://www.ugr.es/local/jillana/SM/FeynmanRulesSM.pdf (taken from FeynArts)

Exercise 8: Z pole observables at tree level

Show that

(a) $\Gamma(f\bar{f}) \equiv \Gamma(Z \to f\bar{f}) = N_c^f \frac{\alpha M_Z}{3}(v_f^2 + a_f^2)\ , \quad N_c^f = 1\ (3)$ for $f =$ lepton (quark)

(b) $\sigma_{\text{had}} = 12\pi \frac{\Gamma(\text{e}^+\text{e}^-)\Gamma(\text{had})}{M_Z^2\Gamma_Z^2}$

(c) $A_{FB} = \frac{3}{4}A_f\ , \quad \text{with } A_f = \frac{2v_f a_f}{v_f^2 + a_f^2}$

(a) Amplitude for $Z \to \overline{f}(p_1)f(p_2)$ in the SM at tree level:

$$\text{i}\mathcal{M} = \overline{v}(p_1)\text{i}e\gamma^\mu(v_f - a_f\gamma_5)u(p_2)\epsilon_\mu(\lambda)$$

Averaging over the 3 initial polarizations and summing over final polarizations:

$$\begin{aligned}\widetilde{\sum}|\mathcal{M}|^2 &= \frac{e^2}{3}\sum_{\lambda=\pm,0}\epsilon_\mu(\lambda)\epsilon_\nu^*(\lambda)\ v(p_1)\gamma^\mu(v_f - a_f\gamma_5)u(p_2)\overline{u}(p_2)(v_f + a_f\gamma_5)\gamma^\nu v(p_1)\\ &= -\frac{e^2}{3}\text{Tr}\{\not{p}_1\gamma^\mu(v_f - a_f\gamma_5)\not{p}_2(v_f + a_f\gamma_5)\gamma_\mu\}\\ &= \frac{e^2}{3}8p_1p_2(v_f^2 + a_f^2) = \frac{e^2}{3}4M_Z^2(v_f^2 + a_f^2)\end{aligned}$$

where fermion masses have been neglected and we have inserted:

$$\sum_{\lambda=\pm,0}\epsilon_\mu(\lambda)\epsilon_\nu^*(\lambda) = -g_{\mu\nu} + \frac{k_\mu k_\nu}{M_Z^2}$$

(the second term does not contribute in the limit of massless fermions) and substituted:

$$2p_1p_2 = (p_1 + p_2)^2 = M_Z^2$$

The differential width is then:

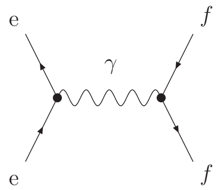
$$\frac{\mathrm{d}\Gamma_Z}{\mathrm{d}\Omega} = \frac{1}{32\pi^2}\frac{|\vec{p}|}{M_Z^2}|\mathcal{M}|^2 = \frac{1}{64\pi^2 M_Z}|\mathcal{M}|^2 = N_c^f \frac{\alpha}{12\pi} M_Z (v_f^2 + a_f^2)$$

(isotropic) where a factor has been included to account for a sum over 3 colors in the case of the fermion being a quark, and the total width is:

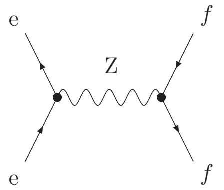
$$\Gamma_Z = 4\pi \frac{\mathrm{d}\Gamma_Z}{\mathrm{d}\Omega} = N_c^f \frac{\alpha M_Z}{3}(v_f^2 + a_f^2)$$

(b) Amplitude for $\mathrm{e}^+(p_1)\mathrm{e}^-(p_2) \to \overline{f}(p_3) f(p_4)$ with $f \neq \mathrm{e}$ in the SM at tree level (unitary gauge):

$$\mathcal{M} = \mathcal{M}_\gamma + \mathcal{M}_Z$$



$$\mathrm{i}\mathcal{M}_\gamma = \overline{u}(p_4)\ (-\mathrm{i}eQ_f)\gamma^\mu\ v(p_3) \frac{-\mathrm{i}g_{\mu\nu}}{s}\ \overline{v}(p_1)\ (-\mathrm{i}eQ_e)\gamma^\nu\ u(p_2)$$



$$\mathrm{i}\mathcal{M}_Z = \overline{u}(p_4)\ \mathrm{i}e\gamma^\mu(v_e - a_e\gamma_5)\ v(p_3) \frac{\mathrm{i}(-g_{\mu\nu} + k_\mu k_\nu / M_Z^2)}{s - M_Z^2 + \mathrm{i}M_Z\Gamma_Z} \times \overline{v}(p_1)\ \mathrm{i}e\gamma^\nu(v_f - a_f\gamma_5)\ u(p_2)$$

where the term proportional to $k_\mu k_\nu$ is irrelevant in the limit of $m_e = 0$. The cross-section in the CM (unpolarized case and $m_e = 0$) is, after some Diracology:

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{\beta_f}{64\pi^2}\frac{N_c^f}{4}\widetilde{\sum}|\mathcal{M}|^2 \\ &= N_c^f \frac{\alpha^2}{4s}\beta_f \left\{ \left[1 + \cos^2\theta + (1-\beta_f^2)\sin^2\theta\right] G_1(s) + 2(\beta_f^2 - 1)G_2(s) + 2\beta_f \cos\theta G_3(s) \right\}\end{aligned}$$

with
$$\begin{aligned} G_1(s) &= Q_e^2 Q_f^2 + 2Q_e Q_f v_e v_f \mathrm{Re}\chi_Z(s) + (v_e^2 + a_e^2)(v_f^2 + a_f^2)|\chi_Z(s)|^2 \\ G_2(s) &= (v_e^2 + a_e^2) a_f^2 |\chi_Z(s)|^2 \\ G_3(s) &= 2Q_e Q_f a_e a_f \mathrm{Re}\chi_Z(s) + 4 v_e v_f a_e a_f |\chi_Z(s)|^2 \end{aligned}$$

$$\Rightarrow \quad \sigma(s) = N_c^f \frac{2\pi\alpha^2}{3s}\beta_f \left[(3 - \beta_f^2)G_1(s) - 3(1-\beta_f^2)G_2(s)\right]$$

where $\chi_Z(s) \equiv \dfrac{s}{s - M_Z^2 + \mathrm{i}M_Z\Gamma_Z}$ and $N_c^f = 1$ (3) for $f =$ lepton (quark). It is easy to guess which terms come from the exchange of γ, Z and the interference.

At the Z pole ($s = M_Z^2$) the interference terms vanish. Neglecting all fermion masses and the γ exchange, the total cross-section is

$$\sigma(M_Z^2) = N_c^f \frac{4\pi\alpha^2}{3M_Z^2} G_1(M_Z^2) = N_c^f \frac{4\pi\alpha^2}{3\Gamma_Z^2}(v_e^2 + a_e^2)(v_f^2 + a_f^2)$$

We see that in fact

$$\sigma_{\text{had}} = 12\pi \frac{\Gamma(\mathrm{e}^+\mathrm{e}^-)\Gamma(\text{had})}{M_Z^2\Gamma_Z^2} = 3\frac{4\pi\alpha^2}{3\Gamma_Z^2}(v_e^2 + a_e^2)(v_f^2 + a_f^2)$$

(c) The forward-backward asymmetry at the Z pole, neglecting fermion masses and γ exchange, is:

$$A_{FB} = \frac{\sigma_F - \sigma_B}{\sigma_F + \sigma_B} = \frac{3}{4}\frac{G_3(M_Z^2)}{G_1(M_Z^2)} = \frac{3}{4}A_f A_e \quad \text{with } A_f = \frac{2v_f a_f}{v_f^2 + a_f^2}$$