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The Electromagnetic Vacuum of Random Media

M.D., Phys. Rev. A 83, 022502 (2011)
M.D., SPIE Conference Proceedings AOP2011
M.D., arXiv:1103.1154

QFEXT2011-Benasque

Motivation - Questions

-Relation between the permittivity of a dielectric and its vacuum/binding energy.

-Relation between the shifts in the resonant frequency of the dielectric constant and the level shifts in the dielectric.

-Is Schwinger's approach to the vacuum energy of an effective medium sufficient to study the Lamb shift?

Outline

-Renormalized polarizability and polarized EM vacuum

Fixed configuration vs. random dielectric medium

-The Lamb shift and the vacuum energy density

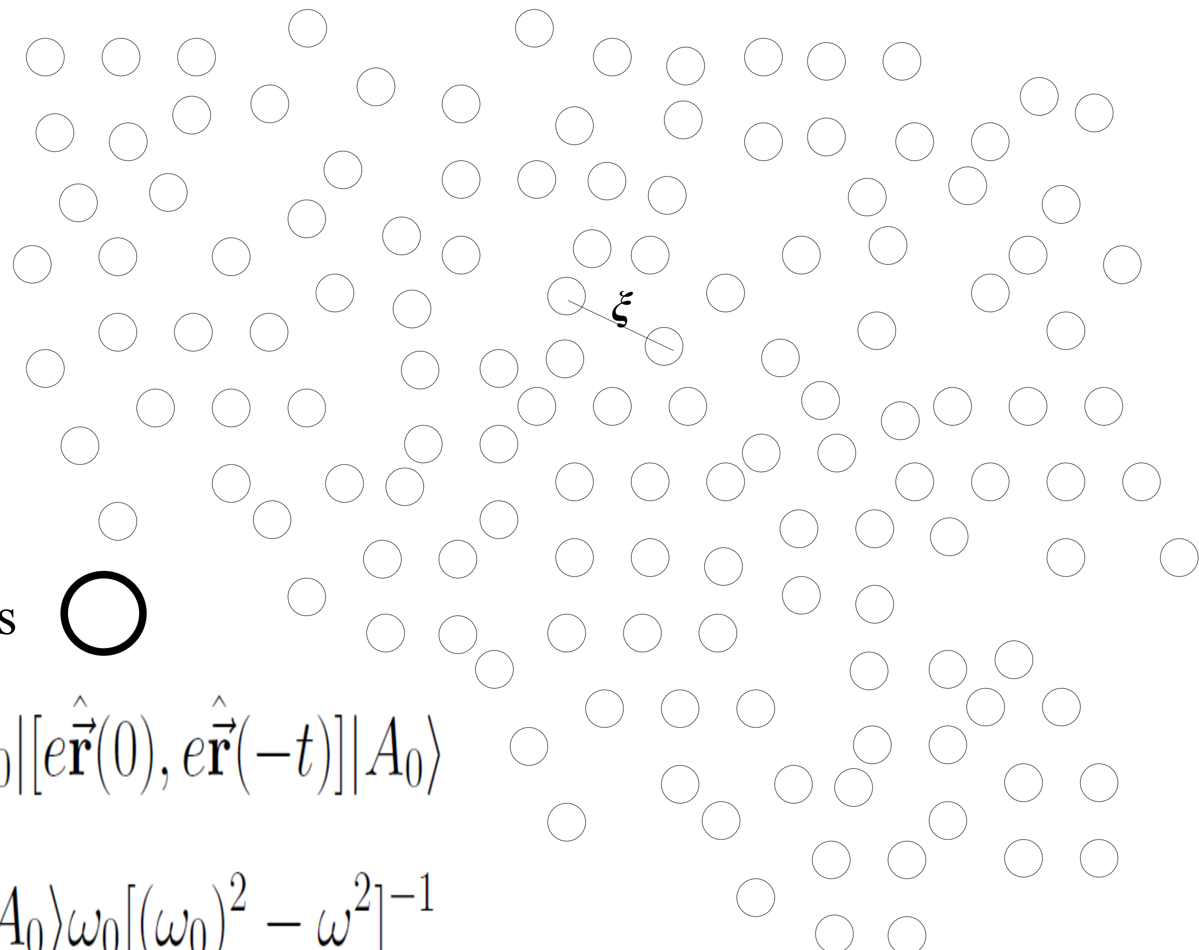
-The vacuum energy of an effective medium

Electrostatic energy of an MG dielectric

Radiative energy from Schwinger's approach

-Conclusions

Molecular dielectric



Zero temperature limit

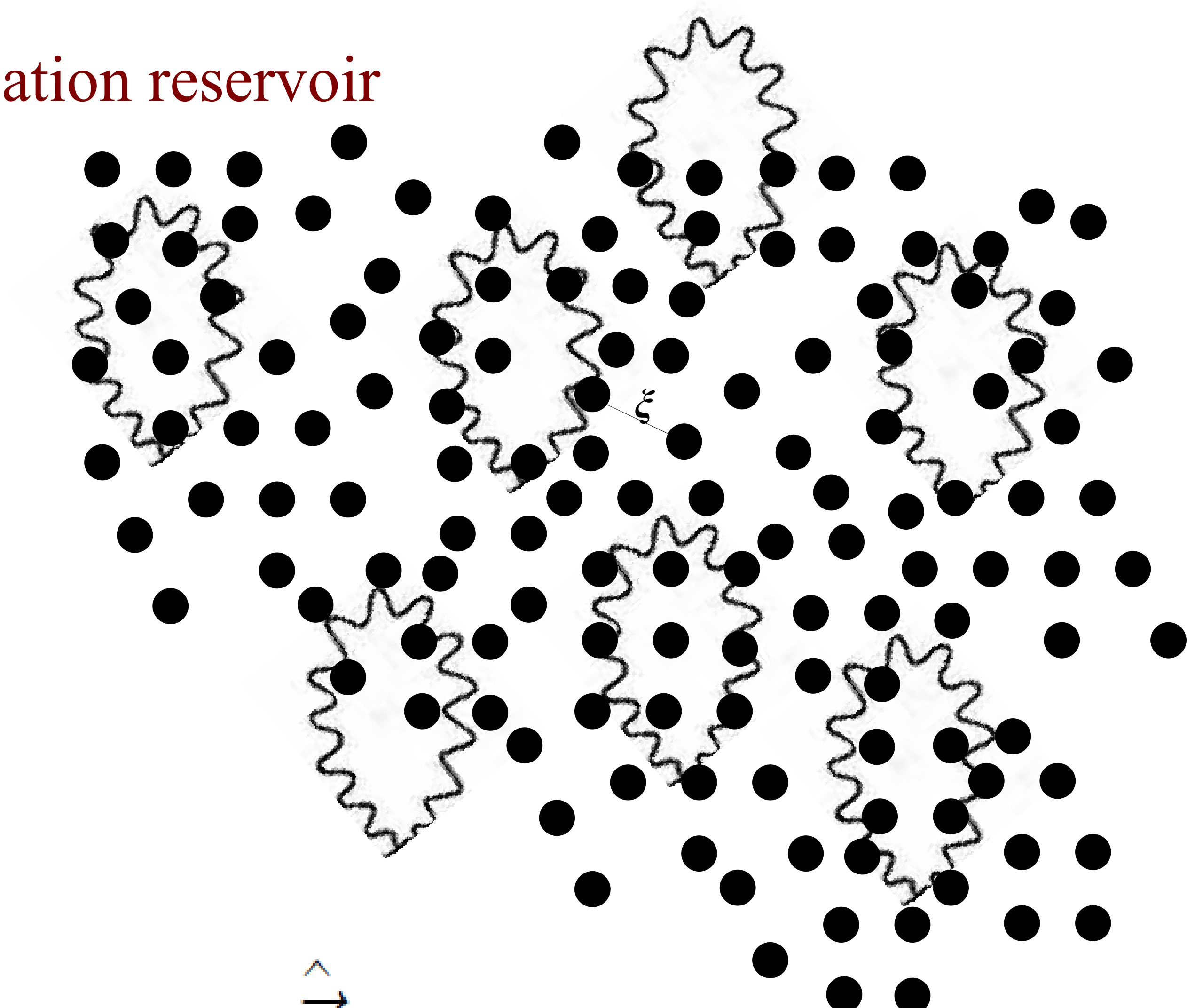
$$k_B T \ll \hbar\omega_0, mv^2, U_{max}$$

bare polarizability of 2-level (AB) atoms

$$\begin{aligned}\bar{\alpha}_0(\omega) &\equiv i(\epsilon_0\hbar)^{-1}\int \mathrm{d}t\,\exp[i\omega t]\Theta(t)\langle A_0|[e\hat{\vec{\mathbf{r}}}(0), e\hat{\vec{\mathbf{r}}}(-t)]|A_0\rangle \\ &= 2(\epsilon_0\hbar)^{-1}\langle A_0|e\hat{\vec{\mathbf{r}}}^S|B_0\rangle\langle B_0|e\hat{\vec{\mathbf{r}}}^S|A_0\rangle\omega_0[(\omega_0)^2-\omega^2]^{-1} \\ &= \frac{2}{3}(\epsilon_0\hbar)^{-1}\omega_0[\omega_0^2-\omega^2]^{-1}\mu^2\bar{\mathbb{I}}\end{aligned}$$

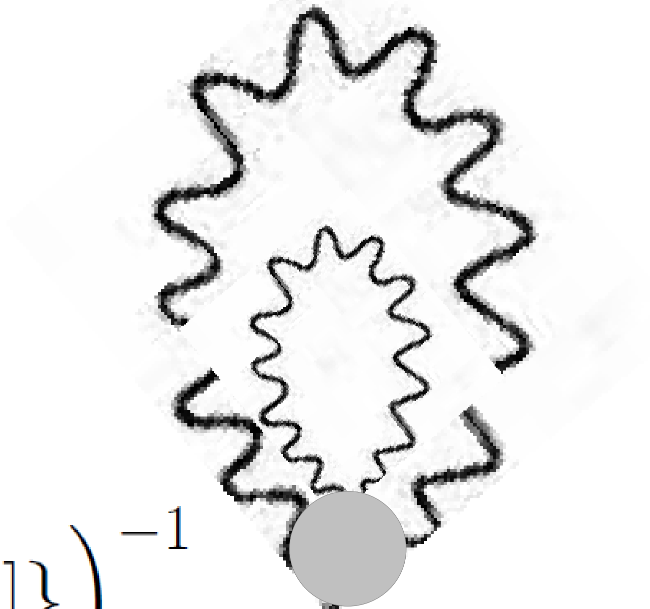
$$\vec{p}^{\omega} = \epsilon_0\bar{\alpha}_0(\omega)\cdot\vec{E}^{\omega}_{tot}$$

Molecular dielectric x Radiation reservoir



$$\hat{H}_{int}(t) = -e\hat{\vec{\mathbf{r}}}(\vec{r},t)\cdot\hat{\vec{\mathbf{E}}}(\vec{r},t)$$

An only dipole in free space



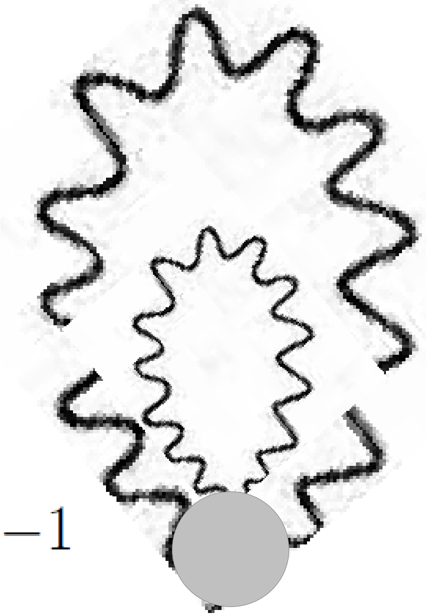
Dressed polarizability in free space

$$\bar{\alpha}(\omega) \equiv \bar{\alpha}_0\Big(1 + ik^2\mathrm{Tr}\{\bar{\alpha}_0 \cdot \Im[\bar{G}^{(0)}(\vec{r},\vec{r};\omega)]\}\Big)^{-1}$$

Radiation-reaction field propagator in free space

$$\vec{p}^{\omega}(\vec{R}) = \epsilon_0 \bar{\alpha}(\omega) \cdot \vec{E}^{\omega}_{ext}(\vec{R})$$

An only dipole in free space



Dressed polarizability in free space

$$\bar{\alpha}(\omega) \equiv \bar{\alpha}_0\left(1 + ik^2\mathrm{Tr}\{\bar{\alpha}_0 \cdot \Im[\underline{\bar{G}^{(0)}(\vec{r},\vec{r};\omega)}]\}\right)^{-1}$$

Radiation-reaction field propagator in free space

$$\vec{p}^{\omega}(\vec{R}) = \epsilon_0\bar{\alpha}(\omega)\cdot\vec{E}^{\omega}_{ext}(\vec{R})$$

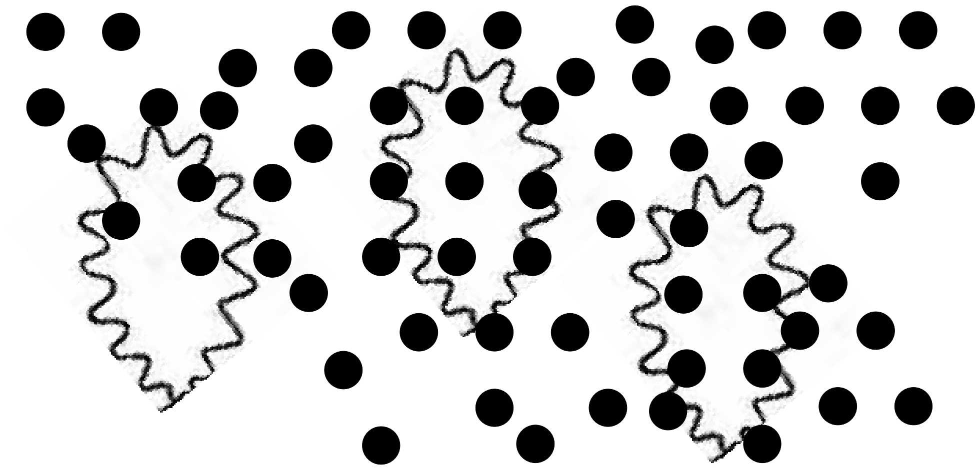
Incident field propagator in free space

$$\bar{G}^{(0)}(\vec{r}-\vec{r}';\omega) = i\epsilon_0\hbar^{-1}\int \mathrm{d}t\,\exp\,[i\omega t]\Theta(t)\Theta(t)\langle\Omega_0|[\hat{\vec{\mathbf{E}}}(\vec{r},0),\hat{\vec{\mathbf{E}}}(\vec{r}',-t)]|\Omega_0\rangle$$

$$\left[\frac{\omega^2}{c^2}\bar{\mathbb{I}} - \vec{\nabla}\times\vec{\nabla}\times\right]\bar{G}^{(0)}(\vec{r}-\vec{r}';\omega) = \delta^{(3)}(\vec{r}-\vec{r}')\bar{\mathbb{I}}$$

A specific dipole configuration

$\{\vec{R}^i\},\ i=1,..,N$



Polarization propagator

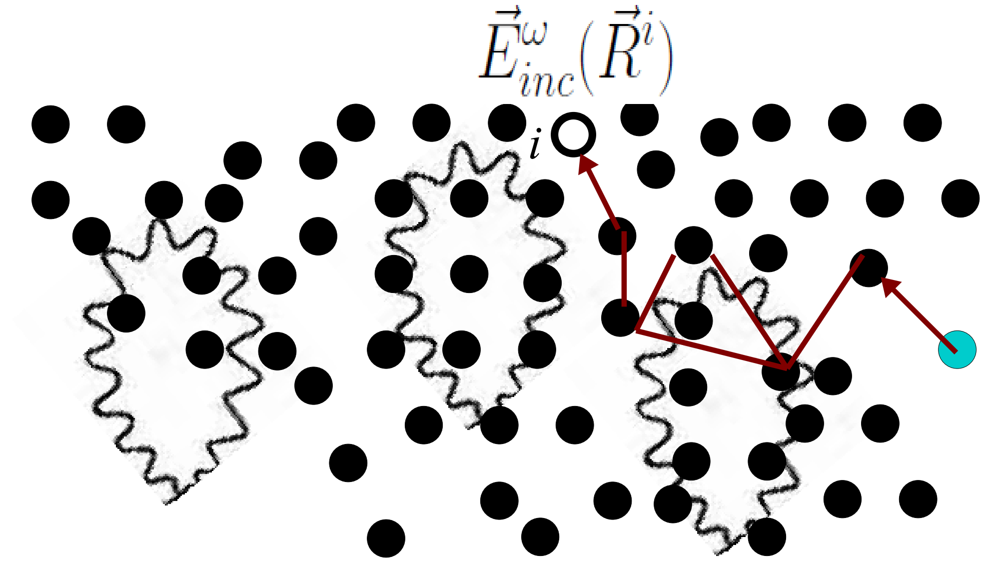
$$\bar{\pi}^{\omega}(\vec{R}^i,\vec{R}^j) \;=\; \int \mathrm{d}t e^{i\omega t} i(\epsilon_0\hbar)^{-1}\Theta(t)\langle A^i|[e\hat{\vec{\mathbf{r}}}_i(0),e\hat{\vec{\mathbf{r}}}_j(-t)]|A^j\rangle$$

$$\;=\; [\alpha_0^{-1}\bar{\bar{\mathbb{I}}}\delta_{ij}+k^2\bar{G}^{(0)}(\vec{R}^i,\vec{R}^j)]^{-1}$$

$$\vec{p}^{\omega}(\vec{R}^i)=\epsilon_0\sum_j^N \bar{\pi}^{\omega}(\vec{R}^i,\vec{R}^j)\cdot\vec{E}^{\omega}_{ext}(\vec{R}^j)$$

A specific dipole configuration

$$\{\vec{R}^i\},\ i = 1, .., N$$



Incident 'polarized' field propagator

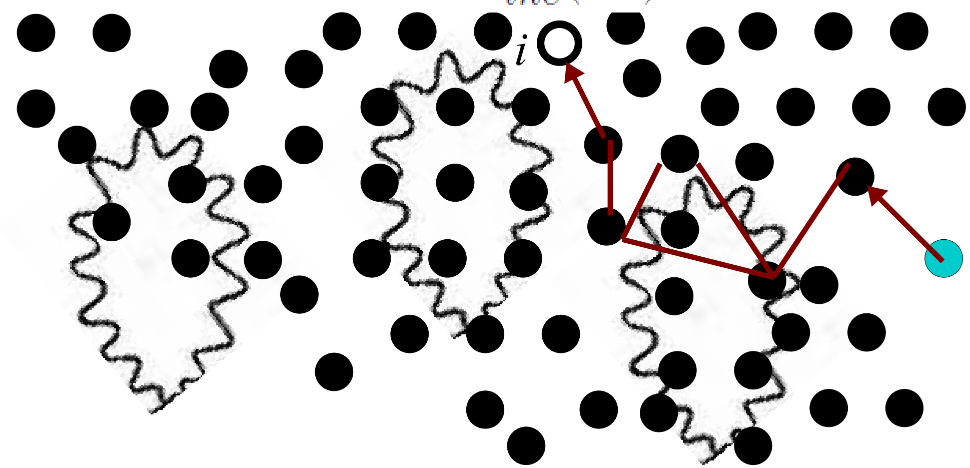
$$\left[\frac{\omega^2}{c^2}e_i^{\omega}(\vec{r})\bar{\bar{\mathbb{I}}} - \vec{\nabla}\times\vec{\nabla}\times\right]\bar{\mathfrak{g}}(\vec{r},\vec{R}^i;\omega) = \delta^{(3)}(\vec{r}-\vec{R}^i)\bar{\bar{\mathbb{I}}}$$

$$\tilde{e}^{\omega}(\vec{r}) = 1 + \alpha_0\sum_{j=1,N}\delta^{(3)}(\vec{r}-\vec{R}^j) \qquad e_i^{\omega}(\vec{r}) = \tilde{e}^{\omega}(\vec{r}) - \alpha_0\delta^{(3)}(\vec{r}-\vec{R}^i)$$

$$\vec{E}^{\omega}_{inc}(\vec{R}^i) = k^2\epsilon_0^{-1}\bar{\mathfrak{g}}(\vec{R}^i,\vec{r}';\omega)\cdot\vec{\mu}^{\omega}_{ext}$$

A specific dipole configuration

$\{\vec{R}^i\},\ i=1,..,N$



Renormalized ith polarizability

$$\bar{\breve{\alpha}}^i = \bar{\alpha}_0[1 + k^2\mathrm{Tr}\{\bar{\mathfrak{g}}(\vec{R}^i,\vec{R}^i;\omega)\cdot\bar{\alpha}_0\}]^{-1}$$

Radiation-reaction 'polarized' field propagator

$$\bar{\mathfrak{g}}(\vec{R}^i,\vec{R}^i;\omega) = \sum_{j=0}^{N}\bar{G}^{(0)}(\vec{R}^i-\vec{R}^j)\cdot\bar{\pi}^{\omega}(\vec{R}^j,\vec{R}^i)[\bar{\breve{\alpha}}^i]^{-1}$$

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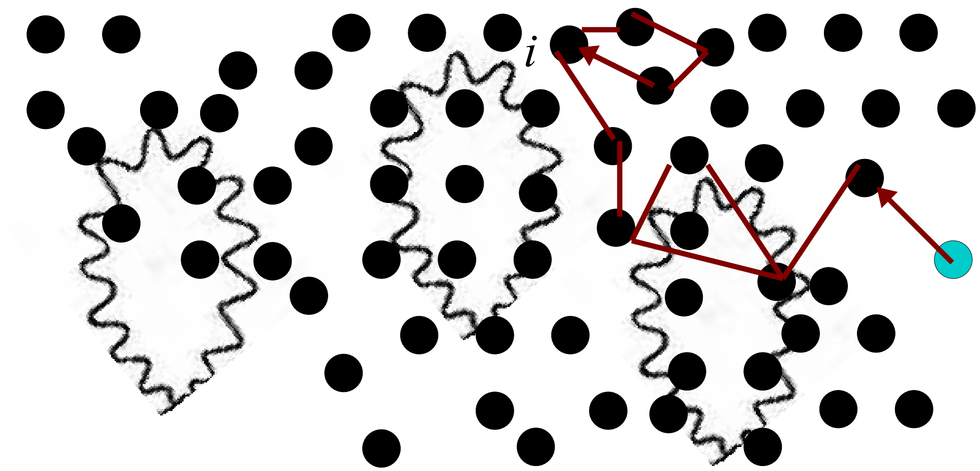
A specific dipole configuration

$\{\vec{R}^i\}, i = 1, .., N$



$$\vec{E}^{\omega}_{tot}(\vec{R}^i) = \vec{E}^{\omega}_{inc}(\vec{R}^i) + \vec{E}^{\omega}_{rr}(\vec{R}^i) = k^2\epsilon_0^{-1}\bar{\alpha}_0^{-1}\cdot\bar{\breve{\alpha}}^i\cdot\bar{\mathfrak{g}}(\vec{R}^i,\vec{r}';\omega)\cdot\vec{\mu}^{\omega}_{ext}$$

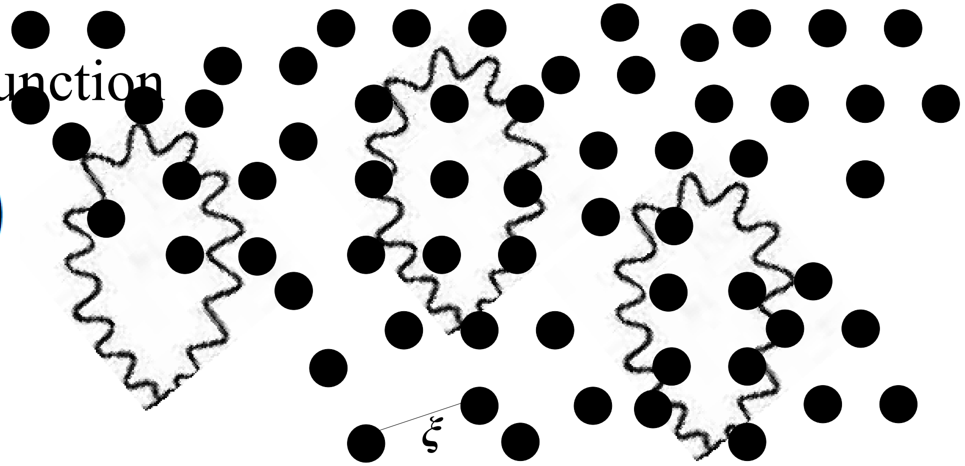
Source field propagator

$$\bar{\tilde{\mathfrak{g}}}(\vec{R}^i,\vec{r}';\omega) \equiv \left[\bar{\mathbb{I}} + k^2\bar{\mathfrak{g}}(\vec{R}^i,\vec{R}^i)\cdot\bar{\alpha}_0\right]^{-1}\cdot\bar{\mathfrak{g}}(\vec{R}^i,\vec{r}')$$

Random medium

Characterized by n-point spatial correlation function

$$h_{\xi}^{(n)}(r_1,..,r_{n-1})$$



Random medium

Characterized by n-point spatial correlation function

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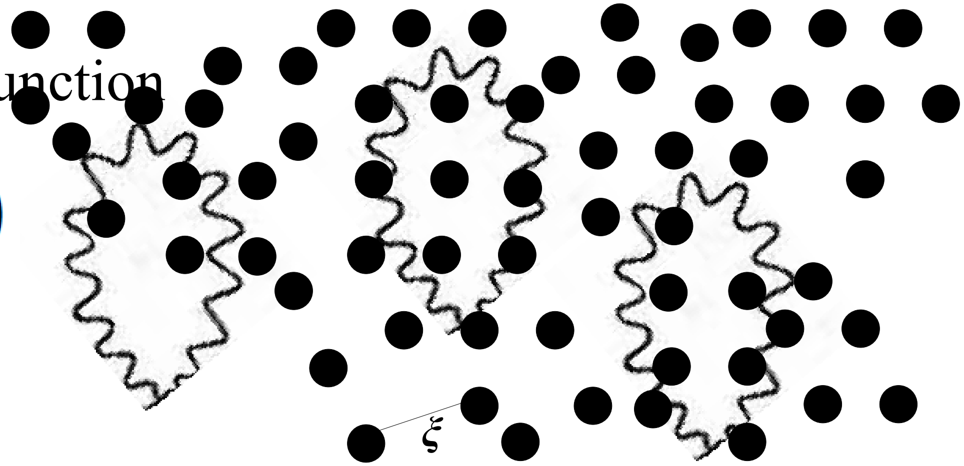
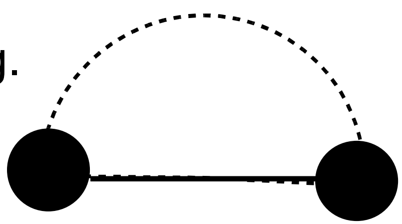


Optical response (electrical susceptibility)
cluster expansion

$$\chi_{\perp,\parallel}(q;\omega)=\sum_{n=1}\chi^{(n)}_{\perp,\parallel}(q;\omega)=\sum_{n=1,m=0}^{\infty}X^{(n,m)}_{\perp,\parallel}(q;\omega)\rho^n\tilde{\alpha}^{n+m}$$

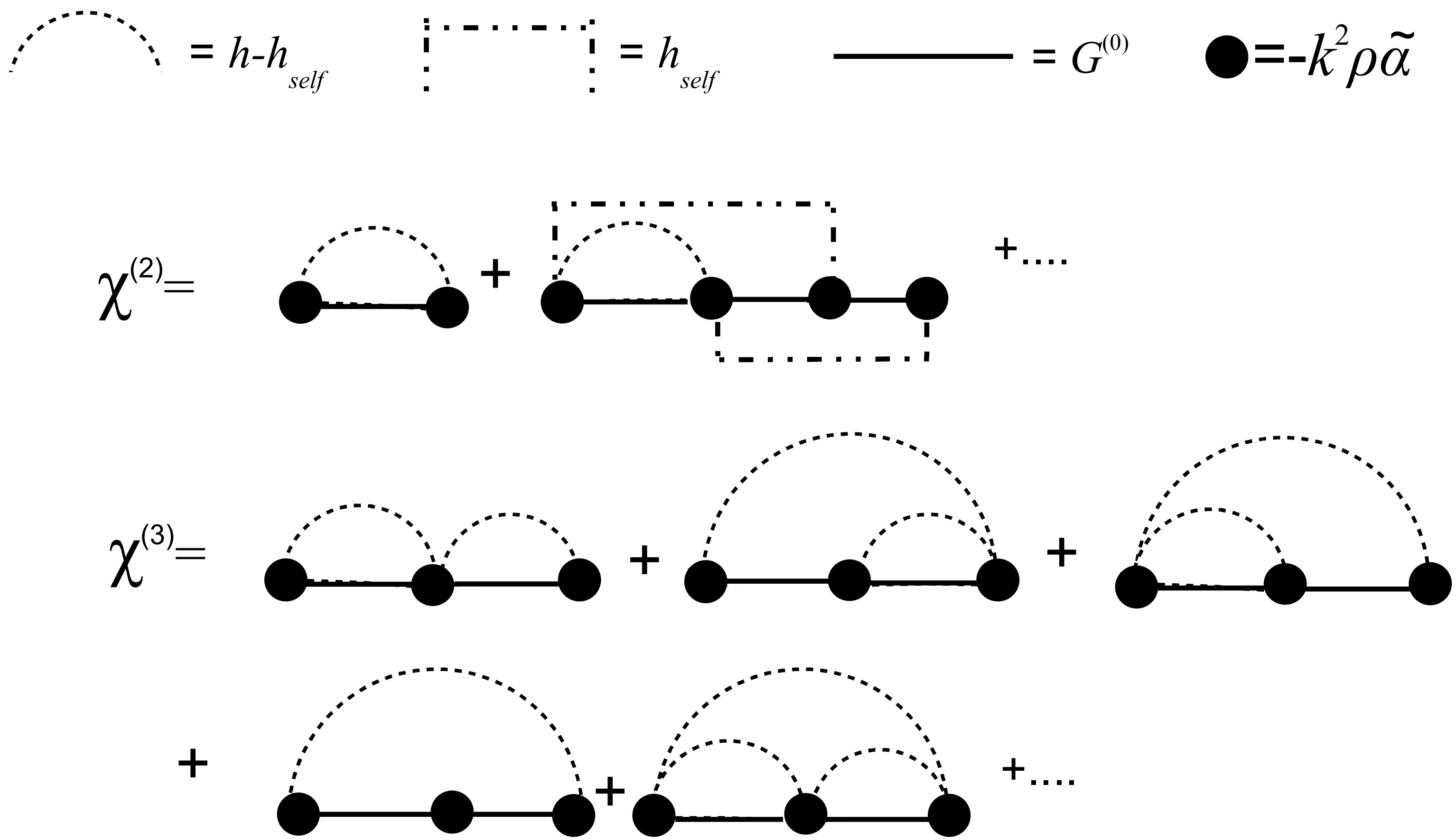
eg.



$$\chi_\perp^{(2,0)}(q)=\frac{(\tilde{\alpha}\rho)^2k^2}{2}\int \mathrm{d}^3r e^{i\vec{q}\cdot\vec{r}}h(r-\xi)\mathrm{Tr}\{\bar{G}^{(0)}(r)\cdot[\mathbb{I}-\hat{q}\otimes\hat{q}]\}$$

Random medium

Electrical susceptibility=Sum of 1PI diagrams

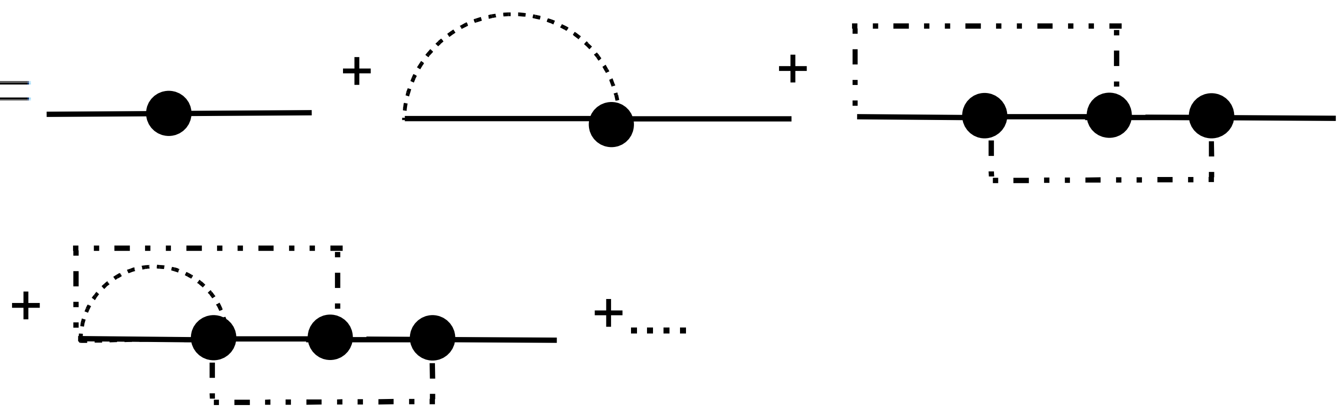


(*c*)

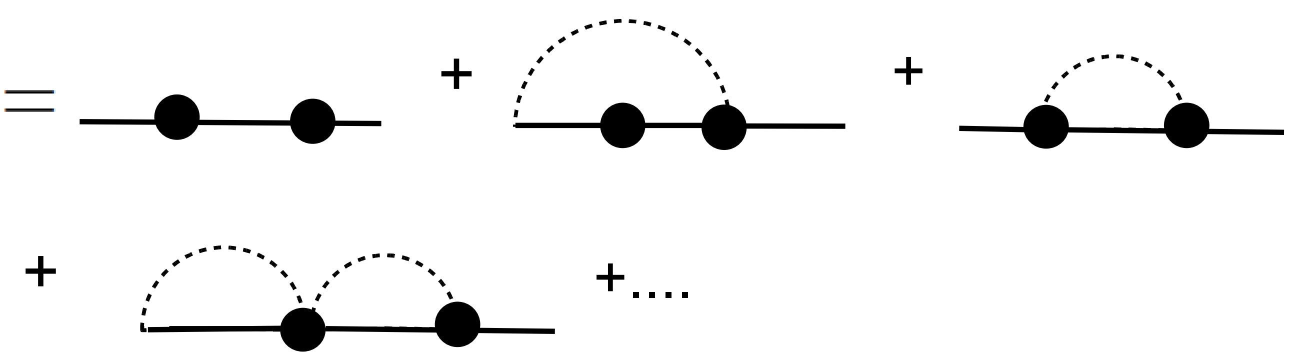
Random medium

Average incident EM field propagator

$$\mathcal{G}^{(1)} =$$

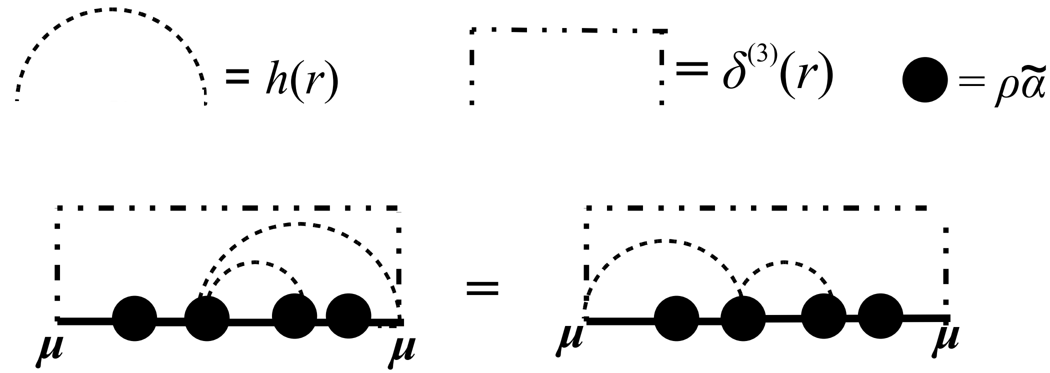


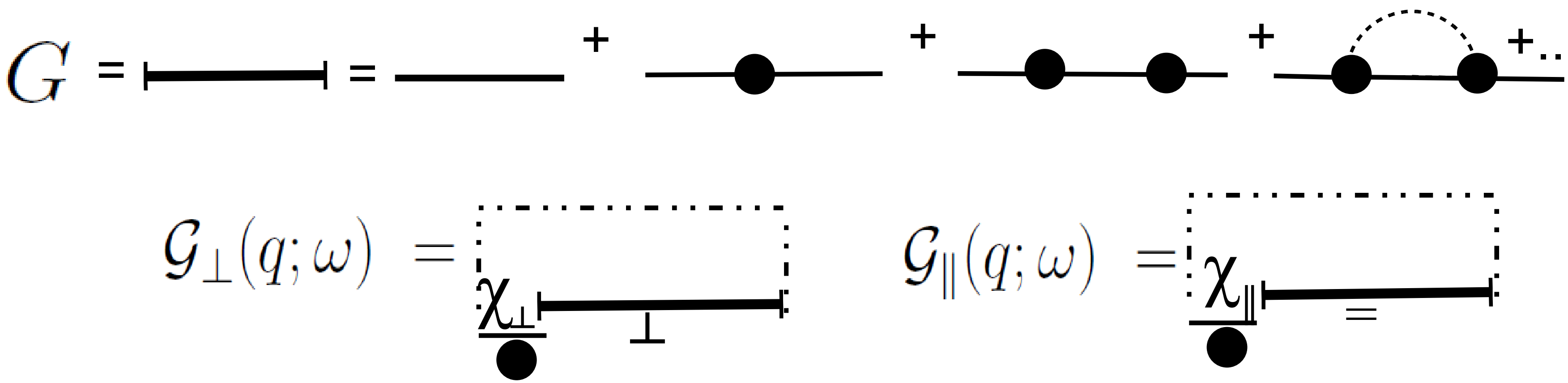
$$\mathcal{G}^{(2)} =$$



Random medium

Radiation-reaction field





$$\mathcal{G}_{\perp}(q;\omega) = \frac{\chi_{\perp}(q;\omega)}{\rho\tilde{\alpha}} G_{\perp}(q;\omega) = \frac{\chi_{\perp}(q;\omega)/(\rho\tilde{\alpha})}{k^2[1+\chi_{\perp}(q;\omega)] - q^2}$$

$$\mathcal{G}_{\parallel}(q;\omega) = \frac{\chi_{\parallel}(q;\omega)}{\rho\tilde{\alpha}} G_{\parallel}(q;\omega) = \frac{1}{\rho\tilde{\alpha}} \frac{\chi_{\parallel}(q;\omega)}{k^2[1+\chi_{\parallel}(q;\omega)]}$$

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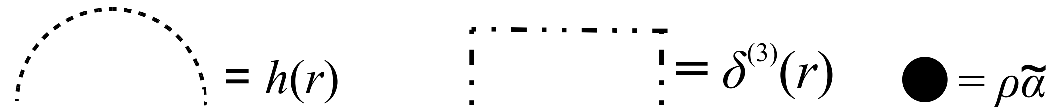
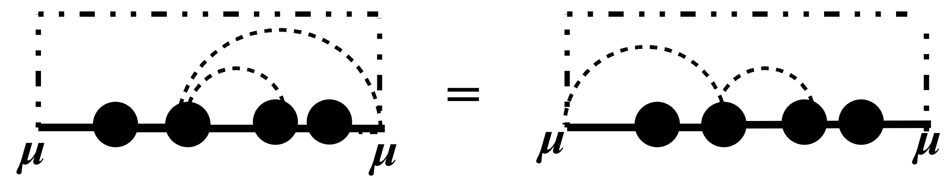
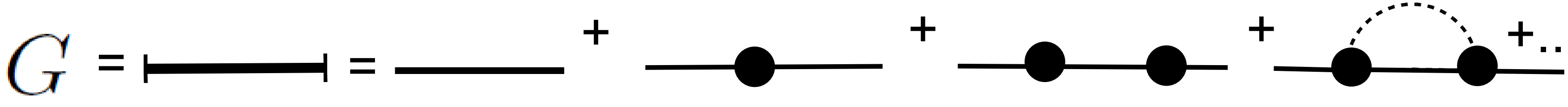
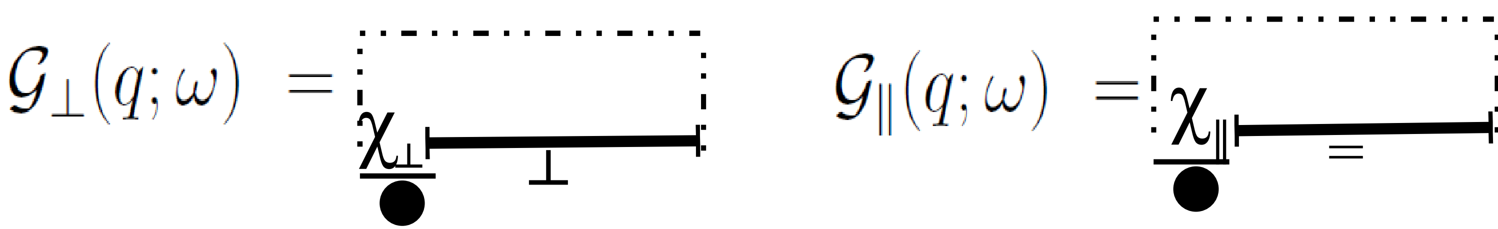
Random medium



Radiation-reaction field







$$\mathcal{G}_\perp(q;\omega) = \boxed{\frac{\chi_\perp(q;\omega)}{\rho\tilde{\alpha}}} G_\perp(q;\omega) = \frac{\chi_\perp(q;\omega)/(\rho\tilde{\alpha})}{k^2[1+\chi_\perp(q;\omega)]-q^2}$$

$$\mathcal{G}_\parallel(q;\omega) = \boxed{\frac{\chi_\parallel(q;\omega)}{\rho\tilde{\alpha}}} G_\parallel(q;\omega) = \frac{1}{\rho\tilde{\alpha}}\,\frac{\chi_\parallel(q;\omega)}{k^2[1+\chi_\parallel(q;\omega)]}$$

Local field factors

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Random medium

Average renormalized polarizability

$$\tilde{\alpha} = \alpha_0 (1 + \alpha_0 \mathrm{Tr}\{\bar{\mathcal{G}}(\vec{r},\vec{r};\omega)\})^{-1}$$

Average source field propagator

$$\bar{\tilde{\mathcal{G}}}(\vec{r},\vec{r}\,';\omega) \equiv \bar{\mathcal{G}}(\vec{r},\vec{r}\,';\omega)(1 + \alpha_0 \mathrm{Tr}\{\bar{\mathcal{G}}(\vec{r},\vec{r};\omega)\})^{-1}$$

Average field on a dipole at ***r***

$$\langle \vec{E}^{\omega}_{tot}(\vec{r}) \rangle_{avg} = k^2 \epsilon_0^{-1} \bar{\tilde{\mathcal{G}}}(\vec{r},\vec{r}\,';\omega) \cdot \vec{\mu}^{\omega}_{ext}$$

Lamb shift

Second order perturbation theory

$$\mathcal{E}^{LSh}_{\kappa,i} = \hbar^{-1}\mathrm{Tr}\Big\{\sum_{\gamma}|\langle\gamma, B^i_\kappa|\hat{\mathbf{H}}^S(\vec{R}^i)|A^i_\kappa, \Omega_\kappa\rangle|^2\Big\}\mathrm{PV}\Big[\frac{1}{\omega_\gamma + \omega_0^{\kappa,i}}\Big]$$

$$= -(4\hbar)^{-1}\mathrm{Tr}\Big\{\int_{-\infty}^{\infty}\mathrm{d}\omega\,\Re\Big[\langle\Omega_\kappa|\hat{\vec{\mathbf{E}}}(\vec{R}^i;\omega)\otimes\hat{\vec{\mathbf{E}}}^\dagger(\vec{R}^i;\omega)|\Omega_\kappa\rangle$$

$$\cdot\int \mathrm{d}t\,\exp[i\omega t]i\Theta(t)\langle A^i_\kappa|[e\hat{\vec{\mathbf{r}}}_i(0), e\hat{\vec{\mathbf{r}}}_i(-t)]|A^i_\kappa\rangle\Big]\Big\}$$

$$- (4\hbar)^{-1}\mathrm{Tr}\Big\{\int_{-\infty}^{\infty}\mathrm{d}\omega\,\Re\Big[\langle A^i_\kappa|e\hat{\vec{\mathbf{r}}}_i(\omega)\otimes e\hat{\vec{\mathbf{r}}}^\dagger_i(\omega)|A^i_\kappa\rangle$$

$$\cdot\int \mathrm{d}t\,\exp[i\omega t]i\Theta(t)\langle\Omega_\kappa|[\hat{\vec{\mathbf{E}}}(\vec{R}^i,0), \hat{\vec{\mathbf{E}}}(\vec{R}^i,-t)]|\Omega_\kappa\rangle\Big]\Big\}$$

J. Dalibard *et al.* J. Physique (1984)

Lamb shift...in application of the fluctuation-dissipation theorem

Free-space Lamb shift

$$\mathcal{E}_0^{LSh} = \frac{\hbar}{2\pi}\int_0^\infty \mathrm{d}\omega \Im\{\alpha_0(1+\alpha_0 \mathrm{Tr}\{\Im[\bar{G}^{(0)}(\vec{r},\vec{r})]\})^{-1}\mathrm{Tr}\{\Im[\bar{G}^{(0)}(\vec{r},\vec{r})]\}\}$$

Lamb shift of the ith atom of a specific configuration

$$\mathcal{E}^{LSh,i} = \frac{\hbar}{2\pi}\int_0^\infty \mathrm{d}\omega \sum_{j=1}^{N} \Im\{\mathrm{Tr}[\bar{G}^{(0)}(\vec{R}^i,\vec{R}^j)\cdot\bar{\pi}^\omega(\vec{R}^j,\vec{R}^i)]\}$$

Average Lamb shift

$$\mathcal{E}_{avg}^{LSh} = \frac{\hbar}{2\pi\rho}\int_0^\infty \mathrm{d}\omega k^2 \Im\Big\{\int \frac{\mathrm{d}^3q}{(2\pi)^3}\Big[2\chi_\perp G_\perp^{(0)}[1+k^2 G_\perp^{(0)}\chi_\perp]^{-1} + \chi_\parallel G_\parallel^{(0)}[1+k^2 G_\parallel^{(0)}\chi_\parallel]^{-1}\Big]\Big\}$$

M.D. arXiv:1103.1154 (2011)

which is a function of the optical response function only!!

Vacuum energy.... from the Lamb shift

From the Lamb shift of the ith atom of a specific configuration, applying the Feynman-Pauli theorem, D. Pienes & P. Nozieres (1966)

$$\mathcal{F}_f^V = \mathcal{V}^{-1}\sum_{i=1}^{N}\int_0^{e^2}\frac{\mathcal{E}^{LSh,i}}{e'^2}\delta e'^2$$

the vacuum energy density of a specific configuration is

$$\mathcal{F}_f^V = -\frac{\hbar}{2\pi\mathcal{V}}\int_0^\infty \mathrm{d}\omega\Im\Big\{\mathrm{Tr}\Big[\ln\big[\bar{\pi}_m^\omega(\vec{R}_m^j,\vec{R}_m^i)/\alpha_0\big]\Big]\Big\}$$

G. Agarwal Phys.Rev. A (1975)

$$= -\frac{\hbar}{2\pi\mathcal{V}}\int_0^\infty \mathrm{d}\omega\Im\Big\{\mathrm{Tr}\Big[\ln\big[\bar{\tilde{\mathfrak{g}}}_m(\vec{R}_m^i,\vec{R}_m^j)\cdot[\bar{G}^{(0)}]^{-1}(\vec{R}_m^i,\vec{R}_m^j)\big]\Big]\Big\}$$

Free-space vacuum energy

$$\mathcal{F}_0^V = -\frac{\rho\hbar}{2\pi}\int_0^\infty \mathrm{d}\omega\,\Im\{\ln[\alpha/\alpha_0]\}$$

Vacuum energy

Average vacuum energy density of a random medium

$$\left\langle \mathrm{Tr}\left[\ln\left[\bar{\tilde{\mathfrak{g}}}\cdot[\bar{G}^{(0)}]^{-1}\right]\right]\right\rangle_{avg} \neq \mathrm{Tr}\left[\ln\left[\left\langle\bar{\tilde{\mathfrak{g}}}\cdot[\bar{G}^{(0)}]^{-1}\right\rangle_{avg}\right]\right]$$

No closed form is possible.....

Vacuum energy

Average vacuum energy density of a random medium

$$\left\langle \mathrm{Tr}\left[\ln\left[\bar{\tilde{\mathfrak{g}}}\cdot[\bar{G}^{(0)}]^{-1}\right]\right]\right\rangle_{avg} \neq \mathrm{Tr}\left[\ln\left[\left\langle\bar{\tilde{\mathfrak{g}}}\cdot[\bar{G}^{(0)}]^{-1}\right\rangle_{avg}\right]\right]$$

No closed form is possible

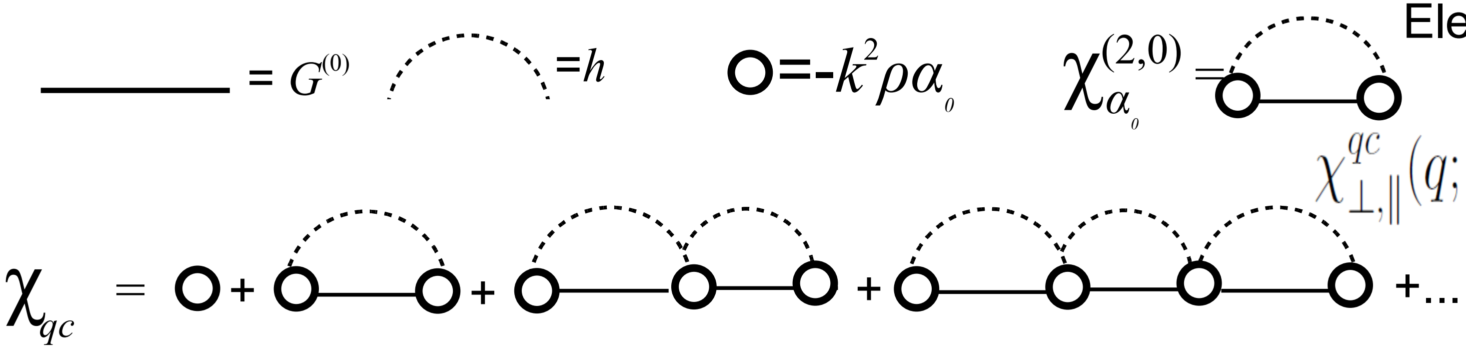
Statistical fluctuations can be treated in the same footing as quantum ones iff time correlation of density fluctuations << 1/ ω_0

Vacuum energy

Average vacuum energy density of a random medium

Quasicrystalline approximation

Electrostatic energy=hbarLLshi



$$\chi^{qc}_{\perp,\parallel}(q;\omega)=\frac{\rho\alpha_0}{1-\chi^{(2,0)}_{\perp,\parallel,\alpha_0}(q;\omega)/\rho\alpha_0}$$

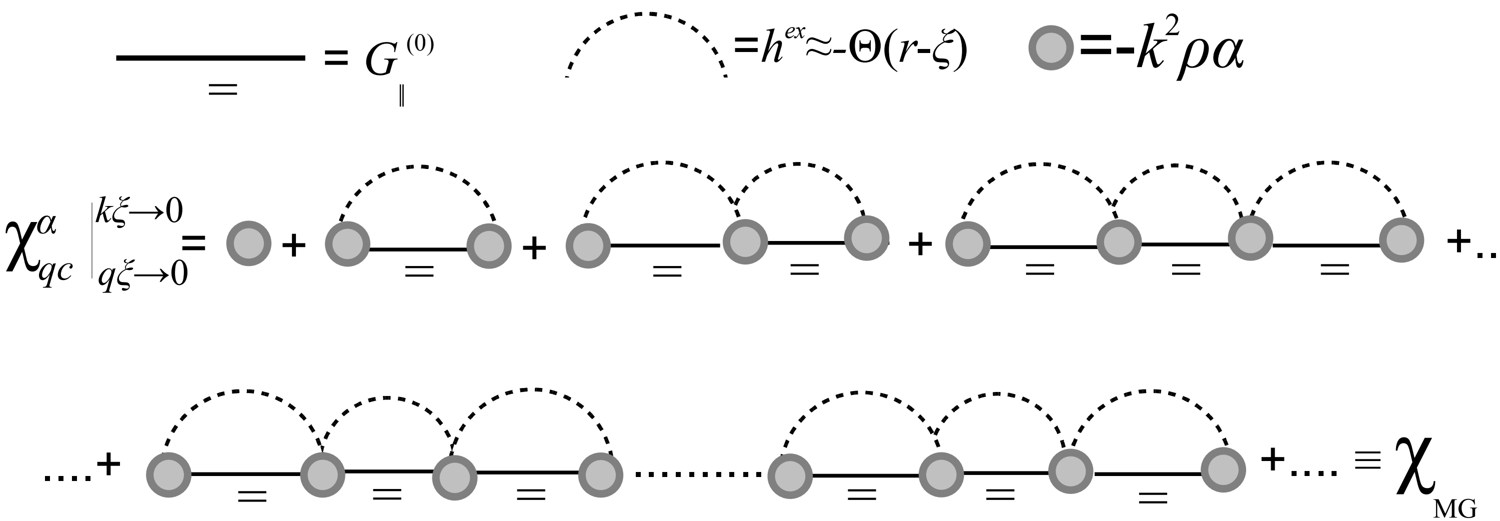
$$\begin{aligned}\mathcal{F}^{V}_{qc} &= -\hbar\Im\Big\{\int_0^\infty\frac{\mathrm{d}\omega}{2\pi}\int\frac{\mathrm{d}^3q}{(2\pi)^3}2\ln\,[\chi^{qc}_{\perp}G^{qc}_{\perp}]+\ln\,[\chi^{qc}_{\parallel}G^{qc}_{\parallel}]\Big\}\\ &+\hbar\Im\Big\{\int_0^\infty\frac{\mathrm{d}\omega}{2\pi}\int\frac{\mathrm{d}^3q}{(2\pi)^3}2\ln\Big[[G^{(0)}_{\perp}]\Big]+\ln\Big[[G^{(0)}_{\parallel}]\Big]\Big\}\\ &+3\hbar\Im\Big\{\int_0^\infty\frac{\mathrm{d}\omega}{2\pi}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\ln\,[\alpha_0]\Big\}\end{aligned}$$

R.K. Bullough & A.S. Obada Chem.Phys.Lett (1969) (in the continuous limit)

M.D. arXiv:1103.1154 (2011)

The vacuum energy of an effective medium

Maxwell Garnett formulation exact in qc approximation



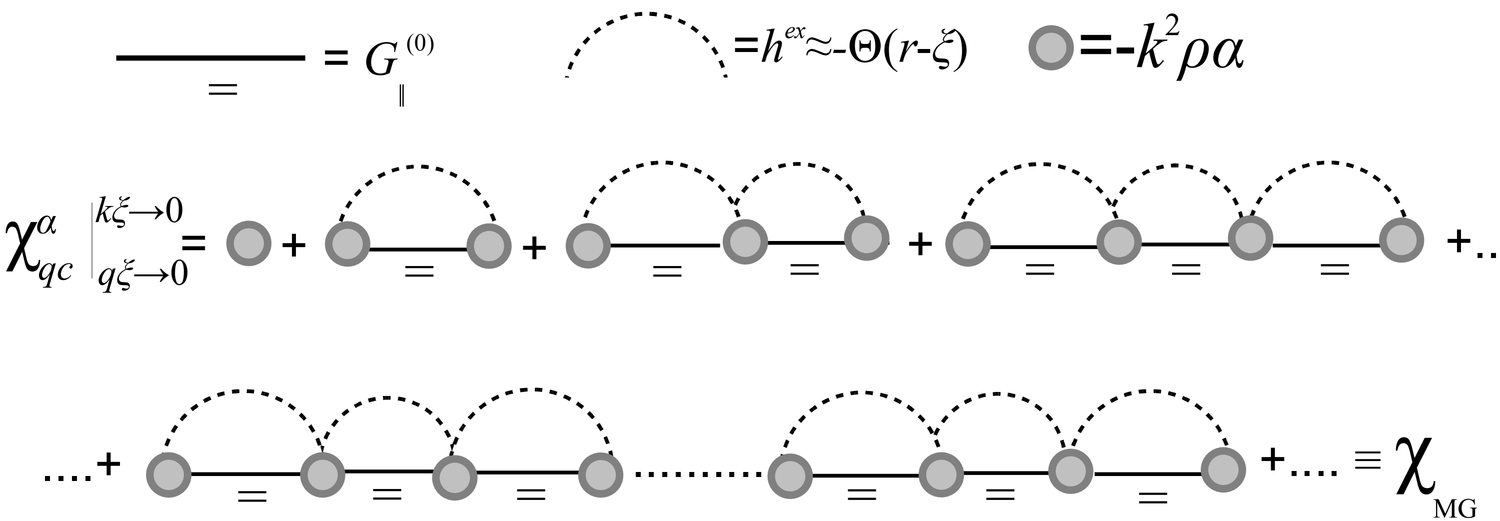
$$\chi_{MG} = \frac{\rho\alpha}{1-\rho\alpha/3}$$

Electrostatic binding energy can be computed ...

$$\mathcal{F}^{V}_{eff}\Big|_{stat} = -\hbar\int_0^{\infty}\frac{\mathrm{d}\omega}{2\pi}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\Im\left\{\ln\left[\frac{\chi_{MG}[\alpha_0]}{\epsilon_{MG}[\alpha_0]\rho\alpha_0}\right]\right\}$$

The vacuum energy of an effective medium

Maxwell Garnett formulation exact in qc approximation



$$\chi_{MG} = \frac{\rho\alpha}{1-\rho\alpha/3}$$

Electrostatic binding energy can be computed ...

$$\mathcal{F}^{V}_{eff}|_{stat} = -\hbar\int_0^\infty \frac{\mathrm{d}\omega}{2\pi}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\Im\left\{\ln\left[\frac{\chi_{MG}[\alpha_0]}{\epsilon_{MG}[\alpha_0]\rho\alpha_0}\right]\right\}$$

...which coincides with the Lorentz-Lorenz resonance shift ....

M.D. arXiv:1103.1154 (2011)

$$\chi_{MG}[\alpha_0] = \frac{\rho\alpha_0\omega_0^2}{\omega_0^2-\omega^2-\boxed{\rho\alpha_0\omega_0^2/3}}$$

$\rightarrow$ LL-shift= $\mathcal{F}^{V}_{eff}|_{stat}/\hbar$

...but it does not yield a good estimation of actual values as near field terms are missing.

The vacuum energy of an effective medium

Schwinger-deRaad-Milton approach Schwinger,deRaad,Milton, Ann.Phys. (1978)

$$H_{int}^{eff} = -\int \mathrm{d}^3r\,\vec{P}_{eff}(\vec{r},t)\cdot\vec{E}^{\perp}_{eff}(\vec{r},t)$$

$$\vec{E}^{\perp}_{eff}(\vec{r};\omega) = k^2\epsilon_0^{-1}\int \mathrm{d}^3r'\,\bar{G}^{eff}_{\perp}(\vec{r},\vec{r}';\omega)\cdot\vec{P}^{\omega}_{eff}(\vec{r}')$$

$$G^{eff}_{\perp}(q;\omega) = (\epsilon_{eff}k^2 - q^2)^{-1} = \text{———} + \text{—}\chi_{eff}\text{—} + \text{—}\chi_{eff}\text{—}\chi_{eff}\text{—} + \text{—}\chi_{eff}\text{—}\chi_{eff}\text{—}\chi_{eff}\text{—} + \ldots$$

$$\mathcal{E}_{Sch.} = -\Re\Big\{\int \mathrm{d}^3r\mathrm{d}\,\langle\Omega_{Sch.}|\hat{\vec{\mathbf{P}}}_{eff}(\vec{r},t)\cdot\hat{\vec{\mathbf{E}}}^{\perp}_{eff}(\vec{r},t)|\Omega_{Sch.}\rangle\Big\}$$

The vacuum energy of an effective medium

Schwinger-deRaad-Milton approach Schwinger,deRaad,Milton, Ann.Phys. (1978)

$$\mathcal{E}_{Sch.} = -\Re\Big\{ \int \mathrm{d}^3 r \mathrm{d}\, \langle \Omega_{Sch.} | \hat{\vec{\mathbf{P}}}_{eff}(\vec{r},t) \cdot \hat{\vec{\mathbf{E}}}^{\perp}_{eff}(\vec{r},t) | \Omega_{Sch.} \rangle \Big\}$$

$$\int \mathrm{d}t \exp[i\omega t] i(\epsilon_0 \hbar)^{-1} \Theta(t) \langle \Omega_{Sch.} | \hat{\vec{\mathbf{P}}}_{eff}(\vec{r},0) \otimes \hat{\vec{\mathbf{P}}}_{eff}(\vec{r}',t) | \Omega_{Sch.} \rangle = \delta\bar{\chi}^{\omega}_{eff} \delta^{(3)}(\vec{r}-$$

$$\delta_{\chi'} \mathcal{E}_{Sch.} = \frac{\hbar}{2\pi\rho} \int_0^\infty \mathrm{d}\omega k^2 \Im \Big\{ \int \frac{\mathrm{d}^3 q}{(2\pi)^3} 2\delta\chi'^{\omega}_{eff} G^{eff}_{\perp} \Big\}$$

$$= \frac{\hbar}{2\pi\rho} \int_0^\infty \mathrm{d}\omega k^2 \Im \Big\{ \int \frac{\mathrm{d}^3 q}{(2\pi)^3} 2\delta\chi'^{\omega}_{eff} G^{(0)}_{\perp} [1 + k^2 G^{(0)}_{\perp} \chi'^{\omega}_{eff}]^{-1} \Big\}$$

The vacuum energy of an effective medium

Schwinger-deRaad-Milton approach

$$\mathcal{F}^{V}_{Sch.} = \frac{-\hbar}{2\pi}\int_0^\infty \mathrm{d}\omega \Im\{\int \frac{\mathrm{d}^3q}{(2\pi)^3} \ln\left[[G^{eff}_{\perp}]^2[G^{(0)}_{\perp}]^{-2}\right]\}$$

Schwinger,deRaad,Milton, Ann.Phys. (1978)
Schwinger, Natl. Acad. Sci. (1992)

No longitudinal modes
No spatial dispersion
No LFFs
Necessary ad-hoc UV cut-off

More importantly: it does not account for internal energies!!
Effective coupling $\propto \sqrt{\chi_{eff}}$

The vacuum energy of an effective medium

Is it possible to compute the Lamb shift of a dipole out of Schwinger's formula?

$$\mathcal{F}^{V}_{Sch.} \simeq \frac{\hbar}{6\pi^2 c^3}\Re\Big\{\int_0^\infty \mathrm{d}\omega\, \omega^3[1-n^3]\Big\}$$

Varying w.r.t. the refractive index....

M. Schaden, L.Spruch, F. Zhou, Phys.Rev.A (1998)

$$\rho^{-1}\Delta_n\mathcal{F}^{V}_{Sch.} = \frac{-\hbar}{4\pi^2 c^3\rho}\Re\{\int_0^\infty \mathrm{d}\omega\, \omega^3 n\Delta\chi_{eff}\},$$

The vacuum energy of an effective medium

Is it possible to compute the Lamb shift of a dipole out of Schwinger's formula?

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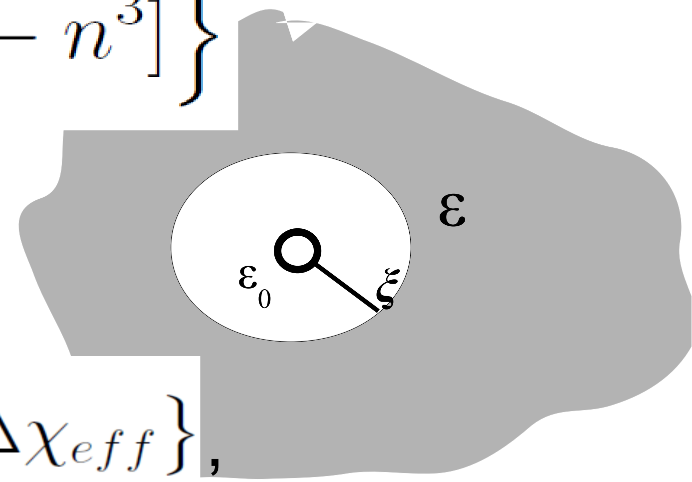

Varying w.r.t. the refractive index....

$$\rho^{-1}\Delta_n\mathcal{F}^{V}_{Sch.} = \frac{-\hbar}{4\pi^2 c^3\rho}\Re\{\int_0^\infty \mathrm{d}\omega\, \omega^3 n\Delta\chi_{eff}\},$$

$$\mathcal{E}^{LSh}_{MSS} = \frac{-\hbar}{4\pi^2 c^3\rho}\Re\{\int_0^\infty \mathrm{d}\omega\, \omega^3 n\rho(\alpha_0^{II}-\alpha_0^{I})\}$$

P.W. Milonni,M. Schaden, L.Spruch, Phys.Rev.A (1999)

...however, in an Onsager cavity,

$$\mathcal{E}^{LSh}_{\mathcal{O}(n-1)}|^{I-II}_{Ons.} = \underbrace{\frac{7}{3}\mathcal{E}^{LSh}_{MSS}|^{I-II}_{BGM}} - \frac{\hbar}{2\pi^2}\Im\overbrace{\{\int \mathrm{d}\omega(n-1)(\alpha_0^{II}-\alpha_0^{I})[\frac{1}{\xi^3}+\frac{\omega^2}{c^2\xi}]\}}^{\text{from near field factors}}$$

from near field factors

1 direct radiation (from Dyson propagator) + 4/3 induced radiation (from LFFs)

M.S. Tomas, Phys. Rev.A (2001)
Trung Dung,Buhmann & Welsch, Phys.Rev. A 74, 023803 (2006)
M.D. arXiv:1103.1154 (2011)

The vacuum energy of an effective medium

For a statistically homogenous medium,

$$\mathcal{E}^{LSh}_{\mathcal{O}(\rho)} = \frac{-\rho\mu^2}{12\epsilon_0}\frac{\Gamma_0}{\omega_0}\left[(\zeta_0^{-3}-\zeta_0^{-1})+\frac{14}{3\pi}(5/6-\gamma_{\mathcal{E}}-\ln[2\zeta_0])\right]$$

M.D. arXiv:1103.1154 (2011)

$$\zeta_0 = \omega_0\xi/c$$

$2\pi c/\xi$ is a natural UV cut-off, both in frequency and momentum space.

Summary

1-. While the Lamb shift is expressible in terms of the optical response function, the vacuum energy is generally not except in some approximations (eg. quasicrystalline approx.)

2-. In the continuum approximation:

2.1. the electrostatic energy of an effective medium cannot be estimated out of the MG formula, although LL-shift= $\mathcal{F}^{V}_{eff}|_{stat}/\hbar$ holds

2.2. Schwinger's formulation does not even account for the energy of retarded radiative modes

3-. Schwinger-deRaad-Milton approach is suitable for the study of Casimir forces between macroscopic dielectrics. However, it is not appropriate to study phenomena which involve variations of internal energies –eg. the Lamb shift