A New Perspective on Path Integral Quantum Mechanics in Curved Space-Time

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Contents

Reference: *D. Singh and N. Mobed, arXiv:1008.1451 [gr-qc]* (under review).

1 Introduction and Motivation

- In the search for a self-consistent quantum theory of gravity over the past ***seventy years*** or more, there now exist many **distinct avenues** available to pursue.
- A partial list of various ***inequivalent*** **approaches**:
  - **String Theory** (Witten, Polchinski, Horowitz, etc.),
  - **Loop Quantum Gravity** (Ashtekar, Smolin, Rovelli, etc.),
  - **Twistor Theory** (Penrose),
  - **Causal Set Theory** (Sorkin),
  - **Regge Calculus** (Williams),
  - **Causal Dynamical Triangulations** (Loll),
  - **Noncommutative Geometry** (Connes),
  - etc.

- Despite their **differences in conceptual and computational details**, all these approaches claim that quantum gravity is only relevant at the Planck length scale of 10^{-33} cm, some *twenty orders of magnitude* **smaller** than the effective **radius of a proton**.
- Given this dilemma, it is worthwhile to ask if there are more **indirect and modest means to seek out quantum gravity**, but strongly driven by a simple desire to acquire some **readily identifiable predictions** involving established theories of gravitation and quantum mechanics when **put under extreme conditions**.

- It is an undeniable fact that the **Feynman path integral approach to quantum mechanics (QM)** has made significant contributions towards the present-day understanding of theoretical physics:
  - subatomic particles,
  - condensed matter,
  - statistical mechanics, etc.
- Arguably, this is **not the simplest approach** for solving practical problems in QM.
- However, it provides a **naturally intuitive and physically insightful overview** about the relationship between **classical and quantum phenomena** within a **unified mathematical framework**.

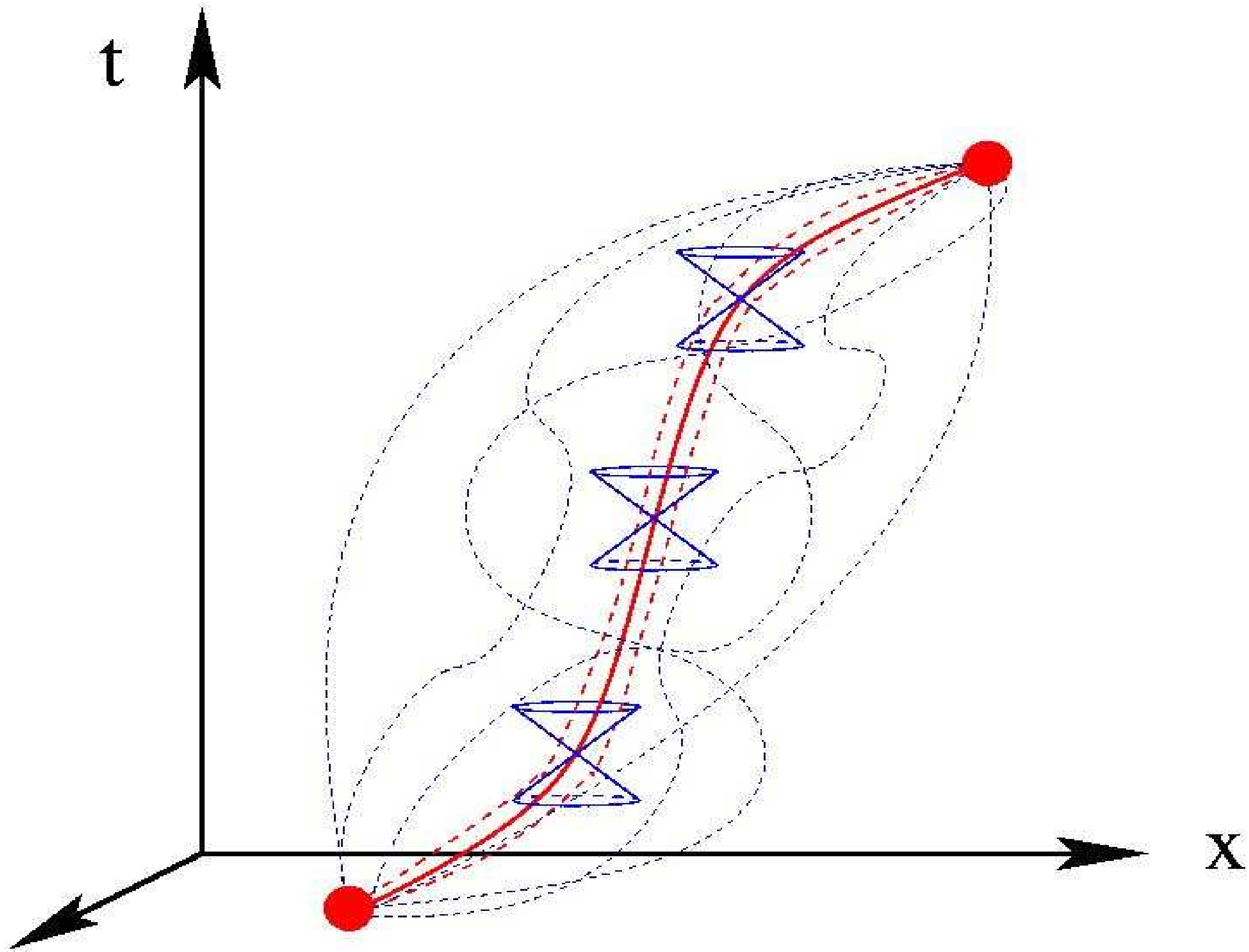

Figure 1: A "sum-over-histories" view about QM, in which all possible classical paths combine to yield the extremized path of classical mechanics plus small quantum fluctuations.

- Because of the **conceptual overlap between classical and quantum mechanics** via the path integral, this approach has obvious implications for **better understanding QM in curved space-time**.
- While the **complete lack of physical data** as a motivating force to provide direction makes ***any approach*** to quantum gravity **problematic**, many useful directions have been explored—including path integral methods—to develop the **mathematical machinery** to make the effort worthwhile.

Relevant Questions About Path Integrals in Quantum Gravity:

To what extent does the **curved space-time manifold** have a **mathematically smooth structure** when applying the path integral approach?

Is it possible to effectively perform the sum-over-histories when the **intermediate classical paths deviate significantly from a classical geodesic**?

Is the presence of **space-time torsion** ***necessarily required*** to correctly define the path integral in curved space-time?

Relevant Questions About Path Integrals in Quantum Gravity:

Are the intermediate classical paths required to **preserve local causality** or should they be **free to trace out causality-violating worldlines**, including ones that imply **propagation into the local past**?

To what extent can a **coarse-grained skeletonized form** of the path integral in curved space-time be identified with a truly **continuum form** in the limit as the **finitely chosen time step becomes** ***infinitesimally small***?

A New Perspective on Path Integral QM in Curved Space-Time:

- Instead of following the **"standard approaches"** on path integral QM established by DeWitt et al., as found in the literature, the approach taken here is ***fundamentally different***, while **simultaneously adhering to Feynman's original approach** as much as possible.
- It employs **Fermi or Riemann normal co-ordinates**, but does so within the context of how **position states are identified within a locally curved background setting**.
- A **scalar particle propagator** is derived that, while satisfying the **expected form in flat space-time**, reveals **interesting new physical predictions** that are not readily evident in the more standard approaches found within the literature.

2 A Representation of Position State Vectors in Normal Co-ordinate Frames

- **Fermi or Riemann Normal Co-ordinates**: $x^\mu = (\tau, \boldsymbol{x}(\tau))$
- **Orthonormal Tetrad**: $\bar{e}^{\hat{\mu}}{}_\nu = \delta^\mu{}_\nu + \tilde{R}^\mu{}_\nu, \qquad \tilde{R}^\mu{}_\nu \ll \delta^\mu{}_\nu.$
- For **Fermi Normal Co-ordinates**:

$$ {}^F\tilde{R}^\mu{}_\nu = \left[\frac{1}{2}\,{}^F R^\mu{}_{lm0}(\tau)\,\delta^0{}_\nu + \frac{1}{6}\,{}^F R^\mu{}_{lmk}(\tau)\,\delta^k{}_\nu\right]\delta x^l\,\delta x^m $$

- For **Riemann Normal Co-ordinates**:

$$ {}^R\tilde{R}^\mu{}_\nu = \frac{1}{6}\,{}^R R^\mu{}_{\alpha\beta\nu}(\tau)\,\delta x^\alpha\,\delta x^\beta $$

- The δx^μ denote **space-time quantum fluctuations**: $|\delta x^\mu| \ll |x^\mu|$.

- **Metric Tensor**: $g_{\mu\nu} = \eta_{\hat{\alpha}\hat{\beta}}\, \bar{e}^{\hat{\alpha}}{}_{\mu}\, \bar{e}^{\hat{\beta}}{}_{\nu} \approx \eta_{\mu\nu} + 2\tilde{R}_{(\mu\nu)}$.
- **Position Ket Vector**: $|x^\mu\rangle = |(\tau, \boldsymbol{x})\rangle$.
- Expression for the position ket vector defined in a **local Lorentz frame**:

$$\left|\boldsymbol{X}^{\hat{\mu}}_{(\mathrm{G})}(\tau, \boldsymbol{x})\right\rangle = \left|\bar{e}^{\hat{\mu}}{}_{\nu}\, x^\nu\right\rangle = U_{\mathrm{Proj.}}(\tau, \boldsymbol{x})\, |x^\mu\rangle$$

$$U_{\mathrm{Proj.}}(\tau, \boldsymbol{x}) = 1 + \frac{i}{\hbar}\, \tilde{R}_{\beta\alpha}\, \boldsymbol{x}^\alpha\, \boldsymbol{p}^\beta\ ,$$

an operator to **project local space-time curvature** onto a local tangent space, in terms of position and canonical momentum operators $\boldsymbol{x}^\alpha$ and $\boldsymbol{p}^\alpha$.

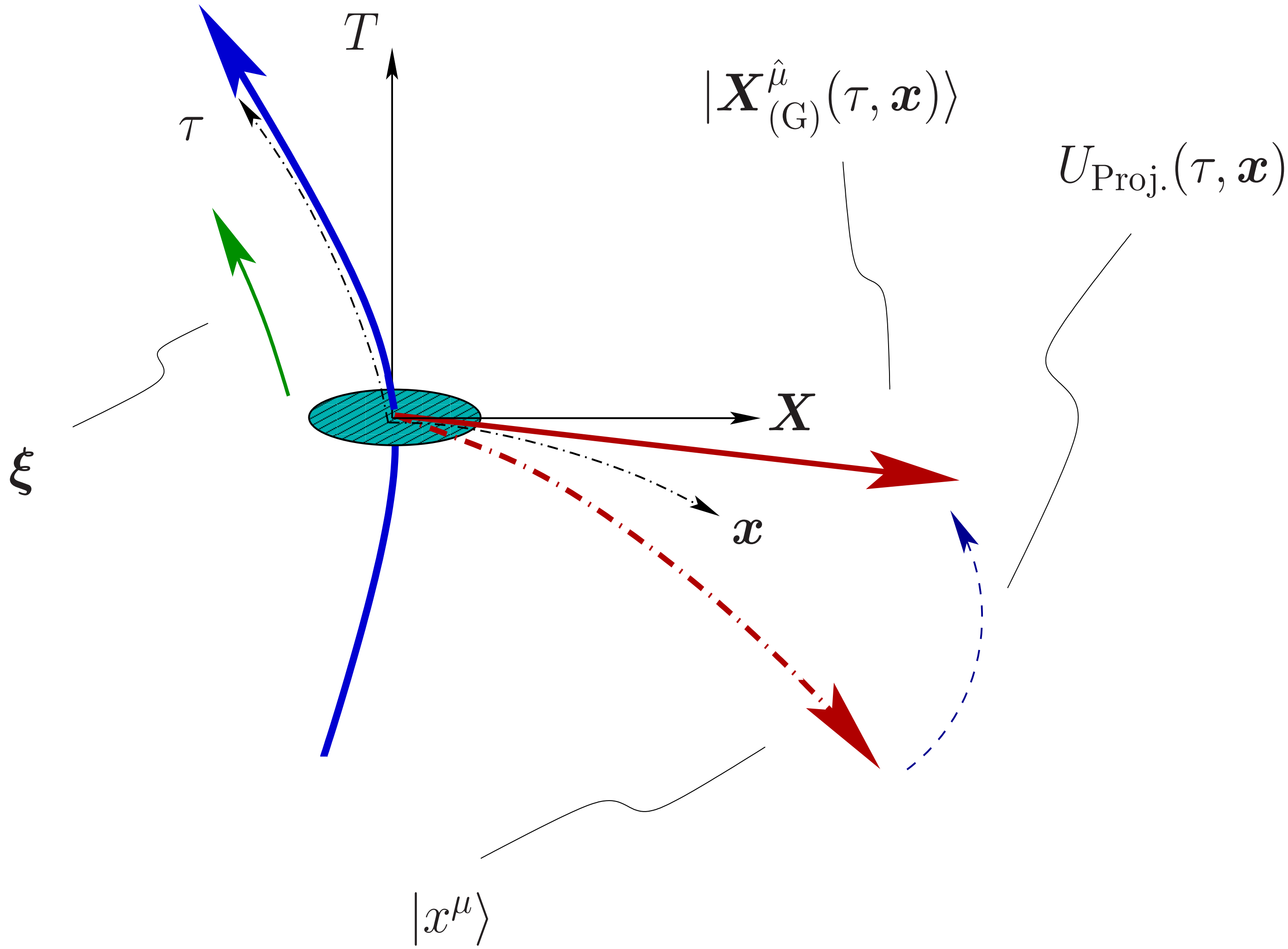

Figure 2: A projection operator $U_{\mathrm{Proj.}}(\tau, \boldsymbol{x})$ transforms the position ket vector $|x^{\mu}\rangle$ into $\left|\boldsymbol{X}^{\hat{\mu}}_{(\mathrm{G})}(\tau, \boldsymbol{x})\right\rangle = U_{\mathrm{Proj.}}(\tau, \boldsymbol{x})\,|x^{\mu}\rangle$ on a local tangent space defined at τ.

3 A New Proper Time Translation Operator in Curved Space-Time

- Consider the geometric description of **Lie transport** as applied to $|\boldsymbol{X}^{\hat{\mu}}(\tau, \boldsymbol{x})\rangle_{\rm G}$.
- For **infinitesimal proper time translation** $\tau \to \tau + \Delta\tau$:

$$\begin{aligned}
&\left|\boldsymbol{X}'^{\hat{\mu}}_{(\rm G)}\left(\tau + \Delta\tau, \boldsymbol{x} + \Delta\boldsymbol{x}\right)\right\rangle \\
&\quad = \left|\boldsymbol{X}^{\hat{\mu}}_{(\rm G)}\left(\tau + \Delta\tau, \boldsymbol{x} + \Delta\boldsymbol{x}\right) - \Delta\tau\left[\left(\mathcal{L}_{\boldsymbol{\xi}}\boldsymbol{X}\right)^{\hat{\mu}}\left(\tau, \boldsymbol{x}\right)\right]\right\rangle \\
&\quad = \left|\boldsymbol{X}^{\hat{\mu}}_{(\rm G)}\left(\tau + \Delta\tau, \boldsymbol{x} + \Delta\boldsymbol{x}\right)\right\rangle \\
&\qquad\qquad - \frac{i}{\hbar}\,\Delta\tau\left(\mathcal{L}_{\boldsymbol{\xi}}\boldsymbol{X}\right)^{\hat{\nu}}\boldsymbol{P}_{\hat{\nu}}\left|\boldsymbol{X}^{\hat{\mu}}_{(\rm G)}\left(\tau, \boldsymbol{x}\right)\right\rangle
\end{aligned}$$

$$\left|\boldsymbol{X}^{\hat{\mu}}_{(\mathrm{G})}(\tau + \Delta\tau, \boldsymbol{x} + \Delta\boldsymbol{x})\right\rangle = \left|\left[\bar{e}^{\hat{\mu}}{}_{\nu}(x^\alpha + \Delta x^\alpha)\right](x^\nu + \Delta x^\nu)\right\rangle$$
$$(\mathcal{L}_{\boldsymbol{\xi}}\boldsymbol{X})^{\hat{\mu}} = \xi^{\hat{\nu}}(\nabla_{\hat{\nu}}\boldsymbol{X}^{\hat{\mu}}) - \boldsymbol{X}^{\hat{\nu}}(\nabla_{\hat{\nu}}\boldsymbol{\xi}^{\hat{\mu}})$$

- Allow for $(\mathcal{L}_{\boldsymbol{\xi}}\boldsymbol{X})^{\hat{\mu}} \to V^0\,(\mathcal{L}_{\boldsymbol{\xi}}\boldsymbol{X})^{\hat{\mu}}$ to demonstrate **path integral invariance under reparametrization** of τ.
- V^0 is the time component of the four-vector $V^\mu = \Delta x^\mu/\Delta\tau$, and serves as the **lapse function**.
- Therefore:

$$\left|\boldsymbol{X}'^{\mu}_{(\mathrm{G})}(\tau + \Delta\tau, \boldsymbol{x} + \Delta\boldsymbol{x})\right\rangle = U_{\Delta\tau}(V^\alpha, \boldsymbol{\xi}^\alpha)\left|\boldsymbol{X}^{\mu}_{(\mathrm{G})}(\tau, \boldsymbol{x})\right\rangle$$

$$
\begin{aligned}
U_{\Delta\tau}(V^\alpha, \boldsymbol{\xi}^\alpha) &= 1 - \frac{i}{\hbar}\, V^0\, \Delta\tau \left\{ \left[\delta^0{}_\alpha - \mathcal{F}(\tilde{R})_{\alpha\beta}\, \frac{V^\beta}{V^0} + (\mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{X})_\alpha \right] \right. \\
&\quad \left. + \frac{i}{\hbar}\, \eta_{\alpha\beta}\, \tilde{R}_{\lambda\sigma}\, x^\sigma\, \frac{V^\beta}{V^0}\, \boldsymbol{P}^\lambda \right\} \boldsymbol{P}^\alpha, \\
\mathcal{F}(\tilde{R})_{\alpha\beta} &= \left(\tilde{R}_{\alpha\beta,\mu} + \frac{1}{2!}\, \tilde{R}_{\alpha\sigma,\beta\mu}\, x^\sigma \right) \Delta x^\mu \\
&\quad + \left(\frac{1}{2!}\, \tilde{R}_{\alpha\beta,\mu\nu} + \frac{1}{3!}\, \tilde{R}_{\alpha\sigma,\beta\mu\nu}\, x^\sigma \right) \Delta x^\mu\, \Delta x^\nu \\
&\quad + O\left((\Delta x)^3 \right).
\end{aligned}
$$

- Note that $U_{\Delta\tau}(V^\alpha, \boldsymbol{\xi}^\alpha)$ is ***not* unitary**:

$$
U^{-1}_{\Delta\tau}(V^\alpha, \boldsymbol{\xi}^\alpha) = U_{-\Delta\tau}(V^\alpha, \boldsymbol{\xi}^\alpha) \neq U^\dagger_{\Delta\tau}(V^\alpha, \boldsymbol{\xi}^\alpha)
$$

4 Configuration Space Path Integral in Curved Space-Time

- Consider determining a **scalar particle propagator** in terms of an **initial position ket vector**: $\left|\boldsymbol{X}^{(\mathrm{G})}_{(\mathrm{i})}(\tau_{\mathrm{i}}, \boldsymbol{x}_{\mathrm{i}})\right\rangle$ and **final position ket vector**:

$$\left|\boldsymbol{X}'^{(\mathrm{G})}_{(\mathrm{f})}(\tau_{\mathrm{f}}, \boldsymbol{x}_{\mathrm{f}})\right\rangle = U^{-1}_{(\tau_{\mathrm{f}}-\tau_{\mathrm{i}})}(V^{\alpha}, \boldsymbol{\xi}^{\alpha}) \left|\boldsymbol{X}^{(\mathrm{G})}_{(\mathrm{f})}(\tau_{\mathrm{i}}, \boldsymbol{x}_{\mathrm{i}})\right\rangle ,$$

$$\tau_{\mathrm{f}} - \tau_{\mathrm{i}} = N\Delta\tau$$

- **Scalar Particle Propagator**:

$$\left\langle \boldsymbol{X}'^{(\mathrm{G})}_{(\mathrm{f})}(\tau_\mathrm{f}, \boldsymbol{x}_\mathrm{f}) \,\middle|\, \boldsymbol{X}^{(\mathrm{G})}_{(\mathrm{i})}(\tau_\mathrm{i}, \boldsymbol{x}_\mathrm{i}) \right\rangle = \left\langle \boldsymbol{X}^{(\mathrm{G})}_{(\mathrm{f})}(\tau_\mathrm{i}, \boldsymbol{x}_\mathrm{i}) \left| U^{-1\dagger}_{N\Delta\tau} \right| \boldsymbol{X}^{(\mathrm{G})}_{(\mathrm{i})}(\tau_\mathrm{i}, \boldsymbol{x}_\mathrm{i}) \right\rangle$$

$$= \left\langle \boldsymbol{X}^{(\mathrm{G})}_{(\mathrm{f})}(\tau_\mathrm{i}, \boldsymbol{x}_\mathrm{i}) \left| \left(\prod_{n=1}^{N} \mathbf{1}_{(n)}\, U^{-1\dagger}_{\Delta\tau} \right) \mathbf{1}_{(0)} \right| \boldsymbol{X}^{(\mathrm{G})}_{(\mathrm{i})}(\tau_\mathrm{i}, \boldsymbol{x}_\mathrm{i}) \right\rangle ,$$

$$\mathbf{1}_{(n)} = \int_{-\infty}^{\infty} \mathrm{d}^3 \boldsymbol{X}_{(n)} \left| \boldsymbol{X}_{(n)}(\tau_n, \boldsymbol{x}_n) \right\rangle \left\langle \boldsymbol{X}_{(n)}(\tau_n, \boldsymbol{x}_n) \right|$$

$$\left\langle \boldsymbol{X}_{(0)} \middle| \boldsymbol{X}^{(\mathrm{G})}_{(\mathrm{i})}(\tau_\mathrm{i}, \boldsymbol{x}_\mathrm{i}) \right\rangle = \int_{-\infty}^{\infty} \frac{\mathrm{d}^3 \boldsymbol{P}_{(0)}}{(2\pi\hbar)^3} \left[1 + \frac{i}{\hbar} \tilde{R}_{ij}(\tau_\mathrm{i}, \boldsymbol{x}_\mathrm{i}) \boldsymbol{X}^j_{(\mathrm{i})} \, \boldsymbol{P}^i_{(0)} \right] \times \exp\left[\frac{i}{\hbar} \left(\boldsymbol{X}_{(0)} - \boldsymbol{X}_{(\mathrm{i})} \right) \cdot \boldsymbol{P}_{(0)} \right] \approx \delta^3 \left(\boldsymbol{X}_{(0)} - \left[\boldsymbol{X}_{(\mathrm{i})} - \tilde{R}_{ij}(\tau_\mathrm{i}, \boldsymbol{x}_\mathrm{i}) \boldsymbol{X}^j_{(\mathrm{i})} \, \hat{\boldsymbol{x}}^i \right] \right),$$

$$\left\langle \boldsymbol{X}^{(\mathrm{G})}_{(\mathrm{f})}(\tau_\mathrm{i}, \boldsymbol{x}_\mathrm{i}) \middle| \boldsymbol{X}_{(N)} \right\rangle \approx \delta^3 \left(\boldsymbol{X}_{(N)} - \left[\boldsymbol{X}_{(\mathrm{f})} - \tilde{R}_{ij}(\tau_\mathrm{i}, \boldsymbol{x}_\mathrm{i}) \boldsymbol{X}^j_{(\mathrm{f})} \, \hat{\boldsymbol{x}}^i \right] \right).$$

- Assume a **Hamiltonian** of the form: $H(\boldsymbol{P}) = \sqrt{m^2 + \boldsymbol{P} \cdot \boldsymbol{P}}$.
- **Reduced Compton Wavelength**: $\lambda\!\!\!^{-} = \hbar/m$.

- It is possible to **integrate out the intermediate momentum states exactly** to first-order in $\tilde{R}_{\mu\nu}$, by using the **integral form of the modified Bessel function**:

$$K_{\pm\nu}\left(\mu\beta\right) = \frac{\beta^{-\nu}}{2}\,e^{-i\nu\pi/2}\int_0^\infty \mathrm{d}N\,N^{\nu-1}\,\exp\left[\frac{i\,\mu}{2}\left(N-\frac{\beta^2}{N}\right)\right],$$

for $\nu = 1/2$, with $\mathrm{Im}(\mu) > 0$ and $\mathrm{Im}(\mu\beta^2) < 0$.

- Assume that $\mathrm{Im}(\Delta\tau) \lesssim 0$ and identify:

$$\begin{aligned}
\mu &= -\frac{V^0\,\Delta\tau}{\lambda}\,, \\
\beta &= -\frac{i}{m}\left[1 - \mathcal{G}(\tilde{R})_{0\alpha}\,\frac{V^\alpha}{V^0} + (\mathcal{L}_{\boldsymbol{\xi}}\boldsymbol{X})_0 + \frac{i}{\hbar}\left(\tilde{R}_{0\alpha}\,x^\alpha\,\frac{V_j}{V^0} - \tilde{R}_{j\alpha}\,x^\alpha\right)\boldsymbol{P}^j\right] \\
&\quad \times \sqrt{\boldsymbol{P}\cdot\boldsymbol{P} + m^2}.
\end{aligned}$$

$$\mathcal{G}(\tilde{R})_{\mu\nu} = \mathcal{F}(\tilde{R})_{\mu\nu} + \tilde{R}_{\mu\nu} + \tilde{R}_{\mu\alpha,\nu}\,x^\alpha + (\tilde{R}^\alpha{}_\alpha)\,\eta_{\mu\nu}.$$

- It is relatively straightforward to demonstrate that when $V^0 \Delta\tau \to \mathrm{d}\tau$, $(V^0 \Delta\tau)^{-1} \to \delta(0)$, and $V^\mu / V^0 \to \dot{x}^\mu(\tau) = (1, \dot{\boldsymbol{x}}(\tau))$, the **integration measure for the skeletonized path integral** becomes

$$\lim_{N\to\infty} \left(\frac{1}{2\pi\lambda\!\!\!^{-}\, i\, V^0 \Delta\tau} \right)^{3N/2} \prod_{n=0}^{N} \int_{-\infty}^{\infty} \mathrm{d}^3 \boldsymbol{X}_{(n)} \quad \to \quad \int \mathcal{D}\left[\boldsymbol{X}(\tau)\right] ,$$

and the **configuration space propagator in curved space-time** becomes

$$\left\langle \boldsymbol{X}'^{(\mathrm{G})}_{(\mathrm{f})}(\tau_{\mathrm{f}}, \boldsymbol{x}_{\mathrm{f}}) \,\middle|\, \boldsymbol{X}^{(\mathrm{G})}_{(\mathrm{i})}(\tau_{\mathrm{i}}, \boldsymbol{x}_{\mathrm{i}}) \right\rangle = \int \mathcal{D}\left[\boldsymbol{X}(\tau)\right] \exp\left[\frac{i}{\hbar} \int_{\tau_{\mathrm{i}}}^{\tau_{\mathrm{f}}} \mathrm{d}\tau\, L^{(\mathrm{G})}_{(\mathrm{Re},0)} \right]$$
$$\times \exp\left[\frac{i}{\hbar} \int_{\tau_{\mathrm{i}}}^{\tau_{\mathrm{f}}} \mathrm{d}\tau \left(L^{(\mathrm{G})}_{(\mathrm{Re},1)} + i \left[L^{(\mathrm{G})}_{(\mathrm{Im},0)} + \lambda\!\!\!^{-}\, \delta(0)\, L^{(\mathrm{G})}_{(\mathrm{Im},1)} \right] \right) \right] .$$

- **Real Contributions to the Free-Particle Lagrangian in Curved Space-Time**:

$$
\begin{aligned}
L^{(\mathrm{G})}_{(\mathrm{Re},0)} &= -m \left[1 - \frac{1}{2} \left\{ 2\,\mathcal{G}(\tilde{R})_{(00)} + 4\,\mathcal{G}(\tilde{R})_{(0j)}\,\dot{x}^j \right.\right. \\
&\quad \left.\left. + \left[\eta_{ij} + 2\,\mathcal{G}(\tilde{R})_{(ij)} \right] \dot{x}^i\,\dot{x}^j \right\} \right] \\
&\approx -m \left[-g^{(\mathrm{eff.})}_{\mu\nu}\,\dot{x}^\mu\,\dot{x}^\nu \right]^{1/2} , \quad g^{(\mathrm{eff.})}_{\mu\nu} = \eta_{\mu\nu} + 2\,\mathcal{G}(\tilde{R})_{(\mu\nu)} ,
\end{aligned}
$$

$$
\begin{aligned}
L^{(\mathrm{G})}_{(\mathrm{Re},1)} &= -m \left\{ (\mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{X})_\mu + 2 \left[(\mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{X})_0\, \delta^0{}_\mu - \mathcal{G}(\tilde{R})_{0\mu} \right] \eta_{ij}\,\dot{x}^i\,\dot{x}^j \right\} \\
&\quad \times \dot{x}^\mu .
\end{aligned}
$$

- **Imaginary Contributions to the Free-Particle Lagrangian in Curved Space-Time**:

$$
\begin{aligned}
L^{(\mathrm{G})}_{(\mathrm{Im},0)} &= m \Big\{ \lambdabar \Big[(\mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{X})^{\alpha}{}_{,\alpha} - \mathcal{H}(\tilde{R})_{\alpha}\, \dot{x}^{\alpha} \Big] \\
&\quad + \frac{1}{2\lambdabar}\, \tilde{R}_{k\alpha}\, x^{\alpha}\, \dot{x}^{k} \left(1 - 3\, \eta_{ij}\, \dot{x}^{i}\, \dot{x}^{j}\right) \\
&\quad - \frac{1}{\lambdabar}\, \tilde{R}_{0\alpha}\, x^{\alpha} \left(1 - 3\, \eta_{ij}\, \dot{x}^{i}\, \dot{x}^{j}\right) \left(1 - \eta_{kl}\, \dot{x}^{k}\, \dot{x}^{l}\right) \Big\} \,,
\end{aligned}
$$

$$
\begin{aligned}
L^{(\mathrm{G})}_{(\mathrm{Im},1)} &= m \Big\{ \frac{3}{2} \Big[(\mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{X})_{0} - \mathcal{G}(\tilde{R})_{00} \Big] + \frac{5}{2} \Big[(\mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{X})_{j} - \mathcal{G}(\tilde{R})_{j0} \Big] \dot{x}^{j} \\
&\quad - \frac{5}{4} \Big[\eta_{ij} + 2\, \mathcal{G}(\tilde{R})_{(ij)} \Big] \dot{x}^{i}\, \dot{x}^{j} \\
&\quad + \Big[(\mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{X})_{0} - \mathcal{G}(\tilde{R})_{0\mu}\, \dot{x}^{\mu} \Big] \eta_{ij}\, \dot{x}^{i}\, \dot{x}^{j} \Big\} \,.
\end{aligned}
$$

$$
\mathcal{H}(\tilde{R})_{\mu} = \mathcal{F}(\tilde{R})^{\alpha}{}_{\mu,\alpha} + \tilde{R}^{\alpha}{}_{\alpha,\mu} \,.
$$

5 Physical Consequences for the Configuration Space Path Integral

- Already at a **purely formal level**, the configuration space path integral correctly yields the **free-particle propagator** in the limit as **space-time curvature vanishes**, while simultaneously revealing some **very interesting predictions**.

- While, all the curvature-dependent terms that correspond to the **conservation of probability** also satisfy the **weak equivalence principle**, at least to leading order in λ, all the **probability violating contributions due to curvature** result in a ***direct coupling*** of λ with the gravitational background.
- This is a ***quantum violation of the weak equivalence principle*** at the **Compton wavelength scale**.
- This detail also indicates a **breakdown of time reversal symmetry** in the scalar propagator under the interchange of $\tau_{\mathrm{i}} \leftrightarrow \tau_{\mathrm{f}}$, providing a **potentially satisfactory explanation** as to why there exists a preference for **time to propagate in the *forward direction* only**.

Regulation of the Path Integral in Cartesian Co-ordinates:

- Explicit evaluation of the **scalar particle propagator in skeletonized form** is a straightforward exercise involving multiple Gaussian integrations with respect to $\prod_{n=1}^{N-1} \mathrm{d}^3 \boldsymbol{X}_{(n)}$.
- However, it is necessary to then **regularize the propagator** in order to remove all of its ***singular* contributions** in the limit as $V^0 \to 0$.

- Normally, this involves describing the skeletonized propagator according to the ansatz

$$\left\langle \boldsymbol{X}'^{(\mathrm{G})}_{(\mathrm{f})}(\tau_{\mathrm{f}}, \boldsymbol{x}_{\mathrm{f}}) \,\middle|\, \boldsymbol{X}^{(\mathrm{G})}_{(\mathrm{i})}(\tau_{\mathrm{i}}, \boldsymbol{x}_{\mathrm{i}}) \right\rangle_{(\mathrm{reg.})} \equiv \left(\frac{1}{2\pi\lambda\, i\, V^0\,(\tau_{\mathrm{f}} - \tau_{\mathrm{i}})} \right)^{3/2} \times \exp\left[\frac{i}{2}\,\frac{m}{\hbar}\, \frac{\left(\eta_{\mu\nu}\, \Delta \boldsymbol{X}^{\mu}_{(\mathrm{i}\to\mathrm{f})}\, \Delta \boldsymbol{X}^{\nu}_{(\mathrm{i}\to\mathrm{f})}\right)}{V^0\,(\tau_{\mathrm{f}} - \tau_{\mathrm{i}})} \right] \sum_{k=0}^{\infty} a_k(x^{\mu}_{\mathrm{i}}, x^{\mu}_{\mathrm{f}})\, \left(i\, V^0\right)^k ,$$

  where $\Delta \boldsymbol{X}^{\mu}_{(\mathrm{i}\to\mathrm{f})} = \boldsymbol{X}^{\mu}_{(\mathrm{f})} - \boldsymbol{X}^{\mu}_{(\mathrm{i})}$ and $a_k(x^{\mu}_{\mathrm{i}}, x^{\mu}_{\mathrm{f}})$ are the curvature-dependent **Seeley-DeWitt coefficients**, whose values are determined from solving the **heat kernel equation**.

- For this skeletonized propagator, it is **unnecessary to compute the Seeley-DeWitt coefficients**, since the propagator prior to the Gaussian integrations can easily be put into **power series form**, such that all inverse powers of V^0 are identified by inspection alone and subsequently **removed by hand**.

$$\left\langle \boldsymbol{X}'^{(\mathrm{G})}_{(\mathrm{f})}(\tau_\mathrm{f}, \boldsymbol{x}_\mathrm{f}) \,\middle|\, \boldsymbol{X}^{(\mathrm{G})}_{(\mathrm{i})}(\tau_\mathrm{i}, \boldsymbol{x}_\mathrm{i}) \right\rangle_{(\mathrm{reg.})} =$$

$$\lim_{N\to\infty} \left(\frac{1}{2\pi\lambda\, i\, V^0\,(\tau_\mathrm{f} - \tau_\mathrm{i})} \right)^{3/2} \exp\left[\frac{i}{2}\,\frac{m}{\hbar}\, \frac{\left(\eta_{\mu\nu}\,\Delta\boldsymbol{X}^{\mu}_{(\mathrm{i}\to\mathrm{f})}\,\Delta\boldsymbol{X}^{\nu}_{(\mathrm{i}\to\mathrm{f})}\right)}{V^0\,(\tau_\mathrm{f} - \tau_\mathrm{i})} \right]$$

$$\times \sum_{k=0}^{\infty}\sum_{n=1}^{N} \frac{1}{k!} \left\{ \left[C^{(k,0)}_{(n)}(x^\mu) + \sum_{l=1}^{4} \frac{k!}{(k+l)!} \left(\frac{-i}{2N}\right)^{l} \frac{C^{(k+l,l)}_{(n)}(x^\mu)}{\lambda^l} \right] \right.$$

$$\left. + \exp\left[\frac{n}{N}\,\Delta\boldsymbol{X}^{\alpha}_{(\mathrm{i}\to\mathrm{f})}\,\frac{\partial}{\partial x^\alpha} \right] \left[C^{(k,0)}_{(\mathcal{L},n)}(x^\mu) + \sum_{l=1}^{2} \frac{k!}{(k+l)!} \left(\frac{-i}{2N}\right)^{l} \frac{C^{(k+l,l)}_{(\mathcal{L},n)}(x^\mu)}{\lambda^l} \right] \right\}$$

$$\times \left(\frac{-i\,V^0\,\Delta\tau}{2\lambda} \right)^{k} \exp\left[-i\,\frac{m}{\hbar} \left(\frac{\tilde{R}_{(ij)}(\tau_\mathrm{i}, \boldsymbol{x}_\mathrm{i})\,\Delta\boldsymbol{X}^{i}_{(\mathrm{i}\to\mathrm{f})}\,\Delta\boldsymbol{X}^{j}_{(\mathrm{i}\to\mathrm{f})}}{V^0\,(\tau_\mathrm{f} - \tau_\mathrm{i})} \right) \right],$$

$$N\Delta\tau = \tau_\mathrm{f} - \tau_\mathrm{i} \equiv 1.$$

- The overall phase can be identified as a **gravitational analogue** of the **Aharonov-Bohm effect** and **Berry's phase**.

- To **leading-order in curvature**, the coefficients $C^{(k,l)}_{(n)}(x^\mu)$ and $C^{(k,l)}_{(\mathcal{L},n)}(x^\mu)$—the latter of which are proportional to $(\mathcal{L}_{\boldsymbol{\xi}}\boldsymbol{X})^\mu(\tau_{\rm i},\boldsymbol{x}_{\rm i})$—are generally ***complex-valued***.
- For $l = 0$:

$$\begin{aligned} C^{(k,0)}_{(n)}(x^\mu) &= 1 - \left(2k - \frac{3}{2}\right)\mathcal{G}(\tilde{R})_{00} + 2i\,k\left(\bar{\lambda}\,\mathcal{H}(\tilde{R})_0 + \frac{1}{\bar{\lambda}}\,\tilde{R}_{00}\,\tau_{(n)}\right. \\ &\quad \left. + \frac{1}{\bar{\lambda}}\,\tilde{R}_{0j}\left[\boldsymbol{X}^j_{(\rm i)} + \frac{n}{N}\,\Delta\boldsymbol{X}^j_{(\rm i\to f)}\right]\right), \end{aligned}$$

$$C^{(k,0)}_{(\mathcal{L},n)}(x^\mu) = \left(2k - \frac{3}{2}\right)(\mathcal{L}_{\boldsymbol{\xi}}\boldsymbol{X})_0(\tau_{\rm i},\boldsymbol{x}_{\rm i}) - 2i\,k\,\bar{\lambda}\,(\mathcal{L}_{\boldsymbol{\xi}}\boldsymbol{X})^\alpha{}_{,\alpha}(\tau_{\rm i},\boldsymbol{x}_{\rm i})\,.$$

6 Conclusion

- This **scalar particle propagator in a locally curved space-time background**, following a **fundamentally different approach** than what currently exists in the literature.
- It reveals what appear to be **very significant physical predictions** with potentially broad implications concerning **quantum mechanical interactions in a non-trivial gravitational field**.
- It is worthwhile to consider **further developments** of this approach when applied to:
  - non-zero integer and half-integer spin particles,
  - many-body particles,
  - quantum field description while approaching a continuum limit.