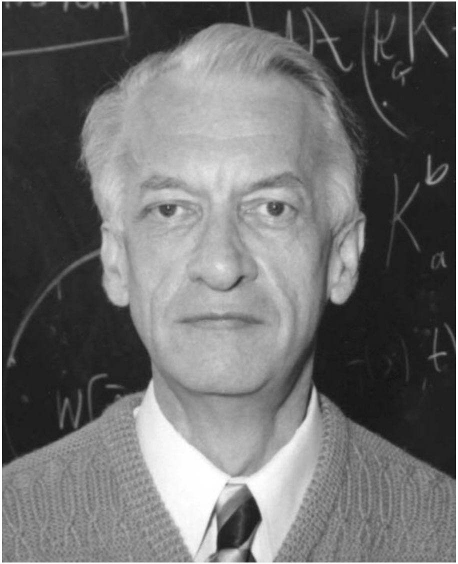
Stuart Dowker, March 18, 1937

- King Edward VII School, Sheffield 1948-1955
- Nottingham University, 1955-1958, BSc
- Birmingham University, 1958-1961, PhD
- Univ. Pennsylvania, Philadelphia, 1961-1963, PostDoc
- Univ. of Manchester, 1963-2002, Lecturer, Reader, Professor
- Univ. Texas, Austin, 1978-79, Visiting Prof.
- Univ. Of Manchester, 2002-, Emeritus Prof.



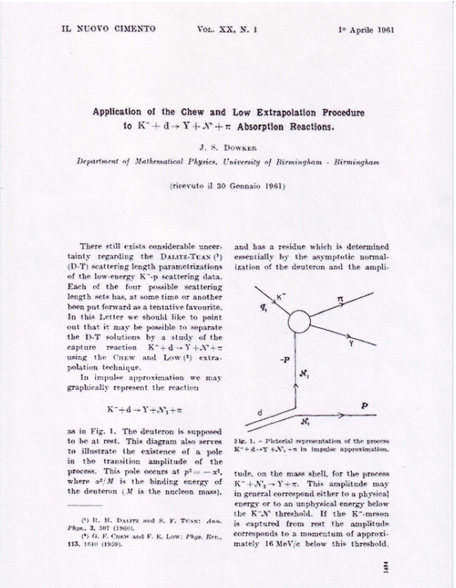
King Edward VII Grammar School, Sheffield, 1948-1955

Nottingham, 1955-1958, BSc

Rudolf Peierls

Birmingham, 1958-1961, PhD

PhD Thesis



IL NUOVO CIMENTO Vol. XX, N. 1 1° Aprile 1961

Application of the Chew and Low Extrapolation Procedure to $K^- + d \to Y + \mathcal{N} + \pi$ Absorption Reactions.

J. S. Dowker
Department of Mathematical Physics, University of Birmingham - Birmingham

(ricevuto il 30 Gennaio 1961)

There still exists considerable uncertainty regarding the Dalitz-Tuan (¹) (D-T) scattering length parametrizations of the low-energy K^--p scattering data. Each of the four possible scattering length sets has, at some time or another been put forward as a tentative favourite. In this Letter we should like to point out that it may be possible to separate the D-T solutions by a study of the capture reaction $K^- + d \to Y + \mathcal{N} + \pi$ using the Chew and Low (²) extrapolation technique.

In impulse approximation we may graphically represent the reaction

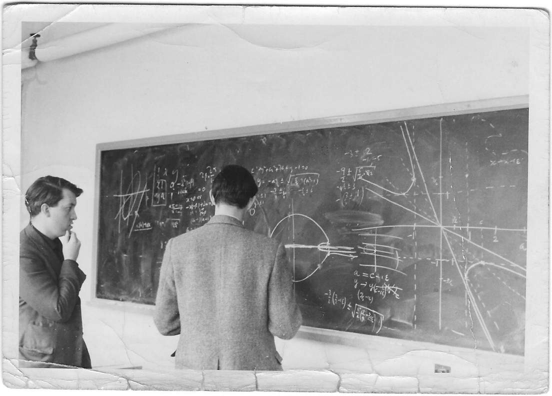
$$K^- + d \to Y + \mathcal{N}_2 + \pi$$

as in Fig. 1. The deuteron is supposed to be at rest. This diagram also serves to illustrate the existence of a pole in the transition amplitude of the process. This pole occurs at $p^2 = -\varkappa^2$, where $\varkappa^2/M$ is the binding energy of the deuteron (M is the nucleon mass) and has a residue which is determined essentially by the asymptotic normalization of the deuteron and the amplitude, on the mass shell, for the process $K^- + \mathcal{N}_2 \to Y + \pi$. This amplitude may in general correspond either to a physical energy or to an unphysical energy below the K^--$\mathcal{N}$ threshold. If the K^--meson is captured from rest the amplitude corresponds to a momentum of approximately 16 MeV/c below this threshold.

Fig. 1. – Pictorial representation of the process $K^- + d \to Y + \mathcal{N}_2 + \pi$ in impulse approximation.

(¹) R. H. Dalitz and S. F. Tuan: *Ann. Phys.*, **3**, 307 (1960).

(²) G. F. Chew and F. E. Low: *Phys. Rev.*, **113**, 1640 (1959).


a = c + iε


MOTTS

Manchester, 1963-2002

Manchester, 1964-1965

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Math master:
"Don't leave your gumption outside the door when you come in!"

"After reading Eddington c. 1960 it was clear to me (and others of course) there is a strong analogy (at least) between gravitation and e&m ... (His work has very strongly influenced me.) So I played a game of asking for the gravitational analogues of existing e&m concepts. The basic analogue is between field strength/charge and curvature/spin."

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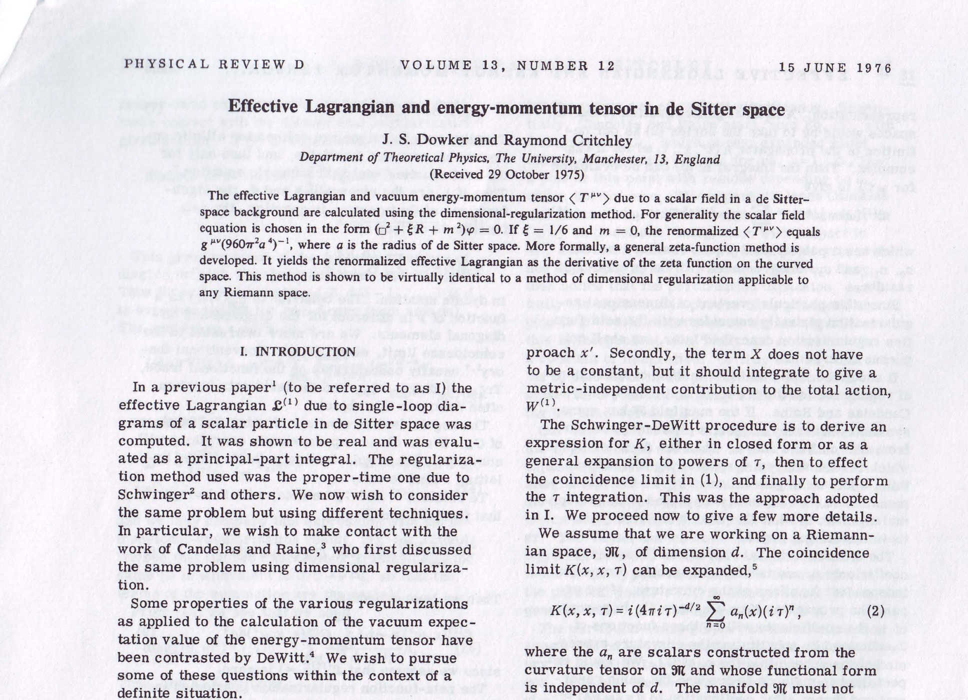
"...spin, in general relativity, plays the passive role that charge plays in electromagnetism in the sense that it is the spin-curvature coupling that knocks a particle off a geodesic."



PHYSICAL REVIEW D VOLUME 13, NUMBER 12 15 JUNE 1976

Effective Lagrangian and energy-momentum tensor in de Sitter space

J. S. Dowker and Raymond Critchley
Department of Theoretical Physics, The University, Manchester, 13, England

(Received 29 October 1975)

The effective Lagrangian and vacuum energy-momentum tensor $\langle T^{\mu\nu}\rangle$ due to a scalar field in a de Sitter-space background are calculated using the dimensional-regularization method. For generality the scalar field equation is chosen in the form $(\Box^2+\xi R+m^2)\varphi=0$. If $\xi=1/6$ and $m=0$, the renormalized $\langle T^{\mu\nu}\rangle$ equals $g^{\mu\nu}(960\pi^2 a^4)^{-1}$, where a is the radius of de Sitter space. More formally, a general zeta-function method is developed. It yields the renormalized effective Lagrangian as the derivative of the zeta function on the curved space. This method is shown to be virtually identical to a method of dimensional regularization applicable to any Riemann space.

I. INTRODUCTION

In a previous paper[1] (to be referred to as I) the effective Lagrangian $\mathcal{L}^{(1)}$ due to single-loop diagrams of a scalar particle in de Sitter space was computed. It was shown to be real and was evaluated as a principal-part integral. The regularization method used was the proper-time one due to Schwinger[2] and others. We now wish to consider the same problem but using different techniques. In particular, we wish to make contact with the work of Candelas and Raine,[3] who first discussed the same problem using dimensional regularization.

Some properties of the various regularizations as applied to the calculation of the vacuum expectation value of the energy-momentum tensor have been contrasted by DeWitt.[4] We wish to pursue some of these questions within the context of a definite situation.

proach x'. Secondly, the term X does not have to be a constant, but it should integrate to give a metric-independent contribution to the total action, $W^{(1)}$.

The Schwinger-DeWitt procedure is to derive an expression for K, either in closed form or as a general expansion to powers of τ, then to effect the coincidence limit in (1), and finally to perform the τ integration. This was the approach adopted in I. We proceed now to give a few more details.

We assume that we are working on a Riemannian space, $\mathfrak{M}$, of dimension d. The coincidence limit $K(x,x,\tau)$ can be expanded,[5]

$$K(x,x,\tau)=i(4\pi i\tau)^{-d/2}\sum_{n=0}^{\infty}a_n(x)(i\tau)^n, \qquad (2)$$

where the a_n are scalars constructed from the curvature tensor on $\mathfrak{M}$ and whose functional form is independent of d. The manifold $\mathfrak{M}$ must not their interesting paper.[3]

The difference is that whereas in (2) and (4) the coefficients a_n are taken to be specific dimension-independent functions of the curvature, if we expand the propagator K on a sphere $S^1_{2\omega}$ in powers of τ, the coefficients will be those functions of 2ω obtained by substituting the curvature expression of the sphere into the a_n of (2). We would expect the two methods to produce the same renormalized theory after continuation to $d = 2\omega = 4$.

We now turn to another regularization method—the zeta-function method. We start from the Feynman Green's function $G_\infty(x'', x')$ expressed in proper-time parametric form

$$\underline{G}_\infty = i\int_0^\infty d\tau\, e^{-im^2\tau}\underline{K}(\tau)\,, \tag{5}$$

with

$$G_\infty(x'', x') = \langle x''|\underline{G}_\infty|x'\rangle$$

and

$$K(x'', x', \tau) = \langle x''|\underline{K}(\tau)|x'\rangle\,,$$

and construct the space-time matrix power $\underline{G}_\infty{}^\nu$. Use of the semigroup property, $\underline{K}(\sigma)\underline{K}(\tau) = \underline{K}(\sigma+\tau)$, rapidly gives

$$(-i\underline{G}_\infty)^\nu = [\Gamma(\nu)]^{-1}\int_0^\infty d\tau\,\tau^{\nu-1}e^{-im^2\tau}\underline{K}(\tau)\,, \tag{6}$$

where we now consider ν to be a complex variable.

If we compare Eq. (6) with one generalization of the Riemann zeta function[5] we are led to call $\underline{G}_\infty{}^\nu$ the zeta function for the manifold $\mathfrak{M}$,

$$\frac{\partial \mathfrak{L}^{(1)}(x, m^2)}{\partial m^2} = \tfrac{1}{2}\, i \lim_{\substack{x'' \to x\\ x' \to x}} G_\infty(x'', x', m^2)\,. \tag{8}$$

Then we have generally

$$\mathfrak{L}^{(1)}_{\text{reg}}(x, m^2) = -\tfrac{1}{2}\, i\int_{m^2}^\infty G^{\text{reg}}_\infty(x, x, \mu^2)\,d\mu^2\,, \tag{9}$$

since we assume that $\mathfrak{L}^{(1)}_{\text{reg}}(x, \infty)$ is zero.

The zeta-function regularization is effected in (9) by replacing G^{reg}_∞ by $\underline{G}_\infty{}^\nu$ and defining

$$\mathfrak{L}^{(1)}_{\text{reg}} = \mathfrak{L}^{(\nu)}, \quad \mathfrak{L}^{(1)} = \lim_{\nu\to 1}\mathfrak{L}^{(\nu)}\,,$$

with

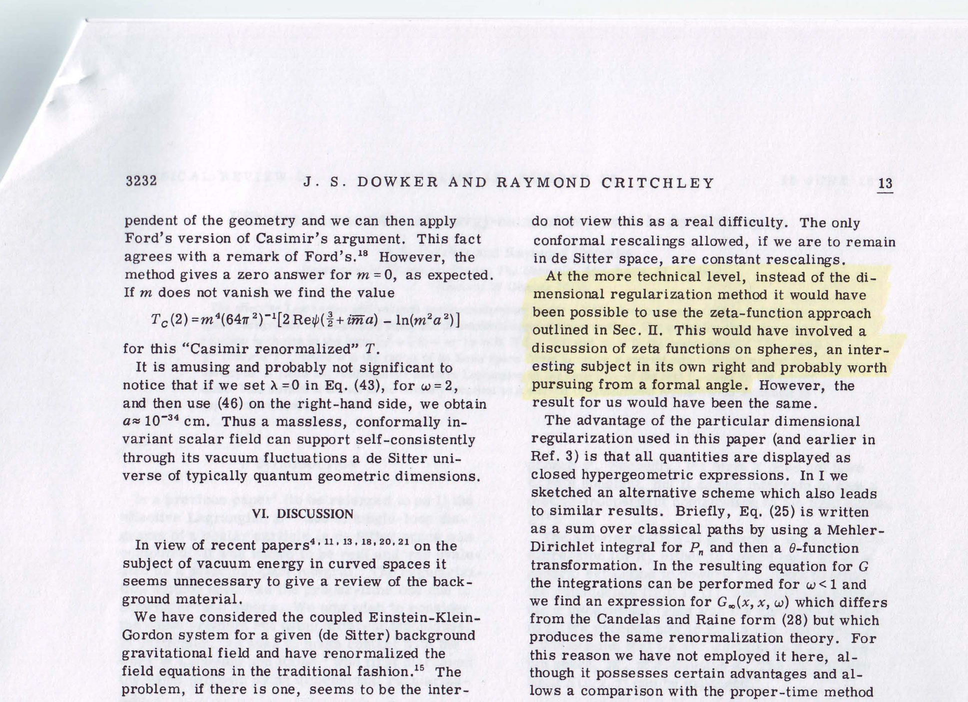
$$\mathfrak{L}^{(\nu)} = -\tfrac{1}{2}\, i\int_{m^2}^\infty \operatorname{diag}\underline{\zeta}_{\mathfrak{M}}(\nu, \mu^2)\,d\mu^2$$
$$= \tfrac{1}{2}\, i(\nu-1)^{-1}\operatorname{diag}\underline{\zeta}_{\mathfrak{M}}(\nu-1, m^2)\,. \tag{10}$$

Then we have

$$\mathfrak{L}^{(1)} = -\tfrac{1}{2}\, i\lim_{\nu\to 1}\,(\nu-1)^{-1}\operatorname{diag}\underline{\zeta}_{\mathfrak{M}}(0, m^2)$$
$$-\tfrac{1}{2}\, i\operatorname{diag}\underline{\zeta}'_{\mathfrak{M}}(0, m^2)\,, \tag{11}$$

where $\underline{\zeta}'(\nu, w) = (d/d\nu)\,\underline{\zeta}(\nu, w)$. The first term in (11) will have to be removed by an infinite renormalization. There may still be finite renormalizations from the ζ' term. It is this term only that is yielded by the method of Salam and Strathdee,[9] which consists of noting that $\ln G = dG^\nu/d\nu|_{\nu=0}$ and then using the formal result $\mathfrak{L}^{(1)} = -\tfrac{1}{2}\, i\operatorname{diag}\ln\underline{G}_\infty$.

In a general space-time we will not know ζ in closed form and we must have recourse to the



pendent of the geometry and we can then apply Ford's version of Casimir's argument. This fact agrees with a remark of Ford's.[18] However, the method gives a zero answer for $m = 0$, as expected. If m does not vanish we find the value

$$T_C(2) = m^4(64\pi^2)^{-1}[2\,\mathrm{Re}\,\psi(\tfrac{3}{2} + i\bar{m}a) - \ln(m^2a^2)]$$

for this "Casimir renormalized" T.

It is amusing and probably not significant to notice that if we set $\lambda = 0$ in Eq. (43), for $\omega = 2$, and then use (46) on the right-hand side, we obtain $a \approx 10^{-34}$ cm. Thus a massless, conformally invariant scalar field can support self-consistently through its vacuum fluctuations a de Sitter universe of typically quantum geometric dimensions.

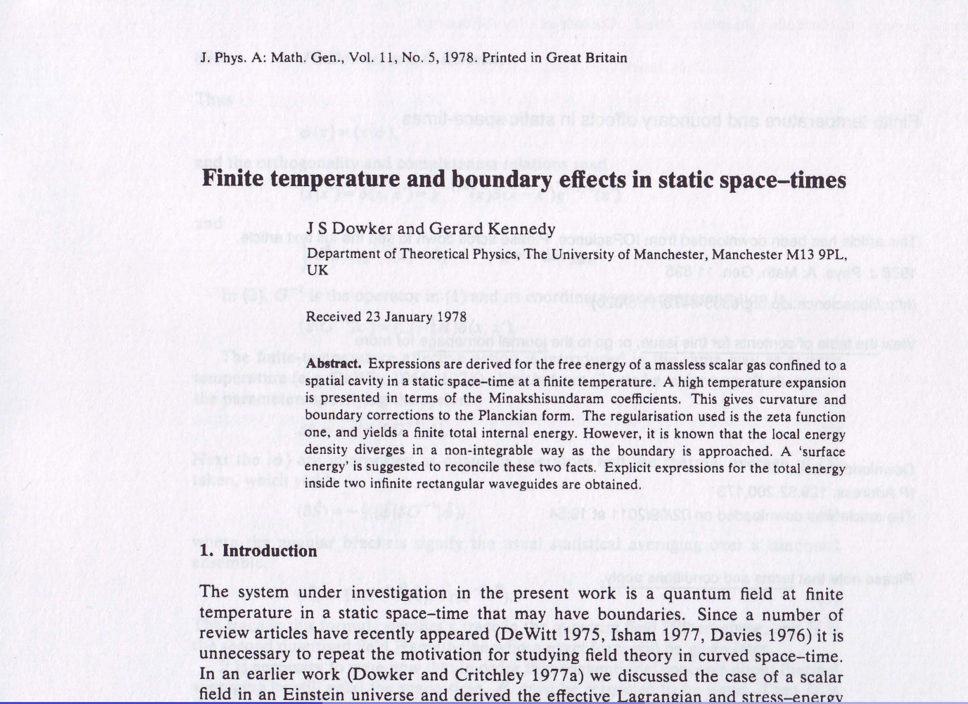
VI. DISCUSSION

In view of recent papers[4, 11, 13, 18, 20, 21] on the subject of vacuum energy in curved spaces it seems unnecessary to give a review of the background material.

We have considered the coupled Einstein-Klein-Gordon system for a given (de Sitter) background gravitational field and have renormalized the field equations in the traditional fashion.[15] The problem, if there is one, seems to be the interdo not view this as a real difficulty. The only conformal rescalings allowed, if we are to remain in de Sitter space, are constant rescalings.

At the more technical level, instead of the dimensional regularization method it would have been possible to use the zeta-function approach outlined in Sec. II. This would have involved a discussion of zeta functions on spheres, an interesting subject in its own right and probably worth pursuing from a formal angle. However, the result for us would have been the same.

The advantage of the particular dimensional regularization used in this paper (and earlier in Ref. 3) is that all quantities are displayed as closed hypergeometric expressions. In I we sketched an alternative scheme which also leads to similar results. Briefly, Eq. (25) is written as a sum over classical paths by using a Mehler-Dirichlet integral for P_n and then a θ-function transformation. In the resulting equation for G the integrations can be performed for $\omega < 1$ and we find an expression for $G_\infty(x, x, \omega)$ which differs from the Candelas and Raine form (28) but which produces the same renormalization theory. For this reason we have not employed it here, although it possesses certain advantages and allows a comparison with the proper-time method



J. Phys. A: Math. Gen., Vol. 11, No. 5, 1978. Printed in Great Britain

Finite temperature and boundary effects in static space–times

J S Dowker and Gerard Kennedy

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Received 23 January 1978

Abstract. Expressions are derived for the free energy of a massless scalar gas confined to a spatial cavity in a static space–time at a finite temperature. A high temperature expansion is presented in terms of the Minakshisundaram coefficients. This gives curvature and boundary corrections to the Planckian form. The regularisation used is the zeta function one, and yields a finite total internal energy. However, it is known that the local energy density diverges in a non-integrable way as the boundary is approached. A 'surface energy' is suggested to reconcile these two facts. Explicit expressions for the total energy inside two infinite rectangular waveguides are obtained.

1. Introduction

The system under investigation in the present work is a quantum field at finite temperature in a static space–time that may have boundaries. Since a number of review articles have recently appeared (DeWitt 1975, Isham 1977, Davies 1976) it is unnecessary to repeat the motivation for studying field theory in curved space–time. In an earlier work (Dowker and Critchley 1977a) we discussed the case of a scalar field in an Einstein universe and derived the effective Lagrangian and stress–energy

Applied firstly to the spatial sections, $\bar{\mathcal{M}}_3$, (39) gives

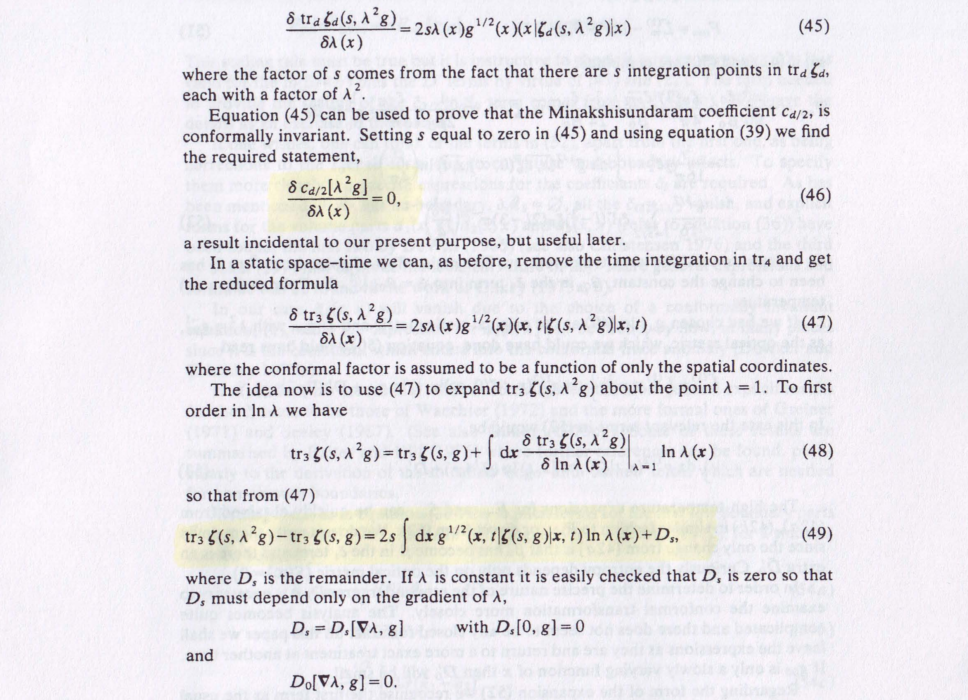
$$\overline{\mathrm{tr}}_3\, \bar{\zeta}_3(0,\infty)=\tfrac{1}{8}\pi^{-3/2}\bar{c}_{3/2}$$

which shows that the $\bar{c}_{3/2}$ pole cancels the $\bar{\zeta}_3(0,\infty)$ term in (38). When applied to the space–time, $\bar{\mathcal{M}}$, for which $d=4$, we can remove the time integration in the tr_4 to give tr_3 since the integrands are time independent. Thus

$$\overline{\mathrm{tr}}_3\, \bar{\zeta}(0,\infty)=\mathrm{i}(16\pi^2)^{-1}\bar{c}_2, \tag{40}$$

where the factor of i comes from the continuation from the Euclideanised space $\bar{\mathcal{M}}_{\mathrm{E}}$. This result shows that the subtraction (34) cancels the $\bar{c}_2$ pole in (38) and we finally arrive at the 'renormalised' free energy (equivalently, the negative of the effective Lagrangian),

$$\begin{aligned}\bar{F}_{\mathrm{ren}}=&-\frac{\pi^2}{90}\frac{\bar{c}_0}{\beta_0^4}-\frac{\zeta_{\mathrm{R}}(3)}{4\pi^{3/2}}\frac{\bar{c}_{1/2}}{\beta_0^3}-\frac{1}{24}\frac{\bar{c}_1}{\beta_0^2}-\overline{\mathrm{tr}}_3\,\bar{\zeta}_3'(0,\infty)\frac{1}{2\beta_0}\\&+\frac{\bar{c}_{3/2}}{8\pi^{3/2}}\frac{1}{\beta_0}\ln\beta_0-\frac{\bar{c}_2}{16\pi^2}\left[\ln\left(\frac{\beta_0}{4\pi}\right)+\gamma\right]\\&-\frac{\pi^{3/2}}{16}\sum_{l=5/2}^{\infty}\bar{c}_l\Gamma(l-\tfrac{3}{2})\zeta_{\mathrm{R}}(2l-3)\pi^{-2l}\left(\frac{\beta_0^2}{4}\right)^{l-2}.\end{aligned}\tag{41}$$



$$\frac{\delta\,\mathrm{tr}_d\,\zeta_d(s,\lambda^2 g)}{\delta\lambda(x)} = 2s\lambda(x)g^{1/2}(x)(x|\zeta_d(s,\lambda^2 g)|x) \tag{45}$$

where the factor of s comes from the fact that there are s integration points in $\mathrm{tr}_d\,\zeta_d$, each with a factor of λ^2.

Equation (45) can be used to prove that the Minakshisundaram coefficient $c_{d/2}$, is conformally invariant. Setting s equal to zero in (45) and using equation (39) we find the required statement,

$$\frac{\delta\,c_{d/2}[\lambda^2 g]}{\delta\lambda(x)} = 0, \tag{46}$$

a result incidental to our present purpose, but useful later.

In a static space–time we can, as before, remove the time integration in tr_4 and get the reduced formula

$$\frac{\delta\,\mathrm{tr}_3\,\zeta(s,\lambda^2 g)}{\delta\lambda(x)} = 2s\lambda(x)g^{1/2}(x)(x,t|\zeta(s,\lambda^2 g)|x,t) \tag{47}$$

where the conformal factor is assumed to be a function of only the spatial coordinates.

The idea now is to use (47) to expand $\mathrm{tr}_3\,\zeta(s,\lambda^2 g)$ about the point $\lambda = 1$. To first order in $\ln\lambda$ we have

$$\mathrm{tr}_3\,\zeta(s,\lambda^2 g) = \mathrm{tr}_3\,\zeta(s,g) + \int \mathrm{d}x\,\frac{\delta\,\mathrm{tr}_3\,\zeta(s,\lambda^2 g)}{\delta\ln\lambda(x)}\bigg|_{\lambda=1}\ln\lambda(x) \tag{48}$$

so that from (47)

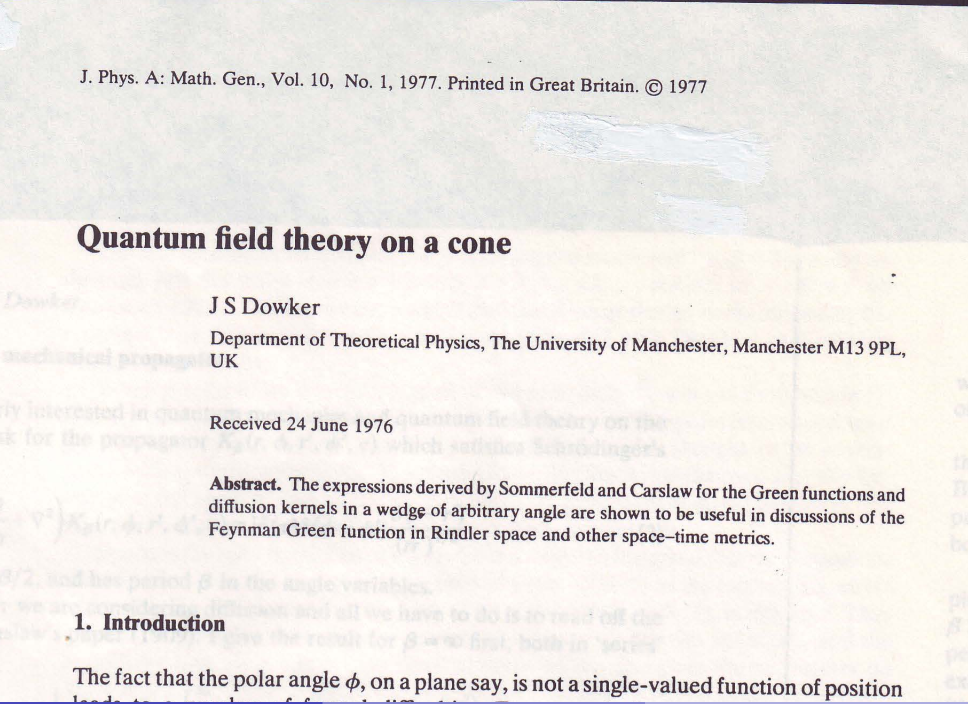
$$\mathrm{tr}_3\,\zeta(s,\lambda^2 g) - \mathrm{tr}_3\,\zeta(s,g) = 2s\int \mathrm{d}x\,g^{1/2}(x,t|\zeta(s,g)|x,t)\ln\lambda(x) + D_s, \tag{49}$$

where D_s is the remainder. If λ is constant it is easily checked that D_s is zero so that D_s must depend only on the gradient of λ,

$$D_s = D_s[\nabla\lambda, g] \qquad \text{with } D_s[0,g] = 0$$

and

$$D_0[\nabla\lambda, g] = 0.$$



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Quantum field theory on a cone

J S Dowker

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Received 24 June 1976

Abstract. The expressions derived by Sommerfeld and Carslaw for the Green functions and diffusion kernels in a wedge of arbitrary angle are shown to be useful in discussions of the Feynman Green function in Rindler space and other space–time metrics.

1. Introduction

The fact that the polar angle ϕ, on a plane say, is not a single-valued function of position

"I have always been interested in exact solutions, even if unphysical, so long as they are pretty. They seem to be working mechanisms that fit together, complete in themselves, like a watch."

Casimir energies and forces in the presence of background potentials

Klaus Kirsten

Baylor University

Benasque, Sept. 22, 2011

(Supported by the NSF under PHY-0757791)

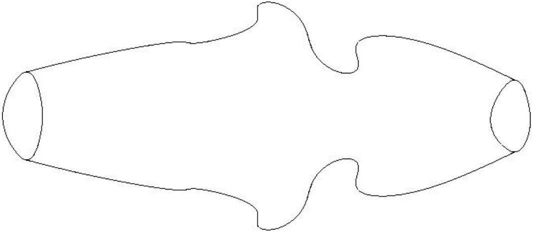
Joint work with Matthew Beauregard, Guglielmo Fucci, Pedro Morales (Baylor University)

Outline

1. Motivations
2. Basic ideas in one dimensions
3. Compactly supported potentials
4. Spherically symmetric potentials
5. Surfaces of revolution
6. Outlook

Motivations

- Casimir energy for surfaces of revolution



Motivations

Eigenvalue problem for a suitable differential operator P:

$$Pu_\ell(x) = \lambda_\ell u_\ell(x), \qquad 0 < \lambda_1 \leq \lambda_2 \ldots, \quad \lambda_\ell \to \infty \quad \text{as} \quad \ell \to \infty.$$

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- Heat kernel:

$$K_P(\tau) \quad = \quad \sum_{\ell=1}^{\infty} e^{-\tau\lambda_\ell}$$

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- Heat kernel:

$$\begin{aligned} K_P(\tau) &= \sum_{\ell=1}^{\infty} e^{-\tau \lambda_\ell} \\ &\overset{\tau \to 0}{\sim} \sum_{\ell=0,1/2,1,\ldots}^{\infty} a_\ell(P, \mathcal{B}) \ \tau^{\ell - D/2} \end{aligned}$$

Motivations

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- Zeta function:

$$\zeta_P(s) = \sum_{\ell=0}^{\infty} \lambda_\ell^{-s} \qquad \Re s > \frac{D}{2}$$

Zeta function $\zeta_P(s) = \sum_{\ell=0}^{\infty} \lambda_\ell^{-s}$ as best organization of the spectrum

- Casimir energy:

$$\text{"}\;\; E_P = \frac{1}{2}\sum_{\ell=0}^{\infty} \lambda_\ell^{1/2} \;\to\; \frac{1}{2}\zeta_P\left(s = -\frac{1}{2}\right) \;\;\text{"}$$

Zeta function $\zeta_P(s) = \sum_{\ell=0}^{\infty} \lambda_\ell^{-s}$ as best organization of the spectrum

- Casimir energy:

$$\text{"}\ \ E_P = \frac{1}{2}\sum_{\ell=0}^{\infty} \lambda_\ell^{1/2} \ \rightarrow\ \frac{1}{2}\zeta_P\left(s=-\frac{1}{2}\right)\ \text{"}$$

More precisely:

$$\zeta_P\left(-\frac{1}{2}+\epsilon\right) = -\frac{1}{\epsilon}\frac{1}{\sqrt{4\pi}}a_{\frac{D+1}{2}}(P,\mathcal{B}) + \text{FP}\ \zeta_P\left(-\frac{1}{2}\right) + \mathcal{O}(\epsilon)$$

Basic ideas in one dimensions

- Let:

$$P = -\frac{d^2}{dx^2} + V(x)$$

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$$Pu_\lambda(x) = \left(-\frac{d^2}{dx^2} + V(x)\right) u_\lambda(x) = k^2 u_k(x) \qquad k \in \mathbb{C}$$

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- Impose the initial condition:

$$u_k(0) = 0 \ , \qquad u_k'(0) = 1$$

- This defines a unique solution:

$$u_k(x)$$

Basic ideas in one dimensions

Eigenvalues for boundary value problem determined by: $u_k(L) = 0$

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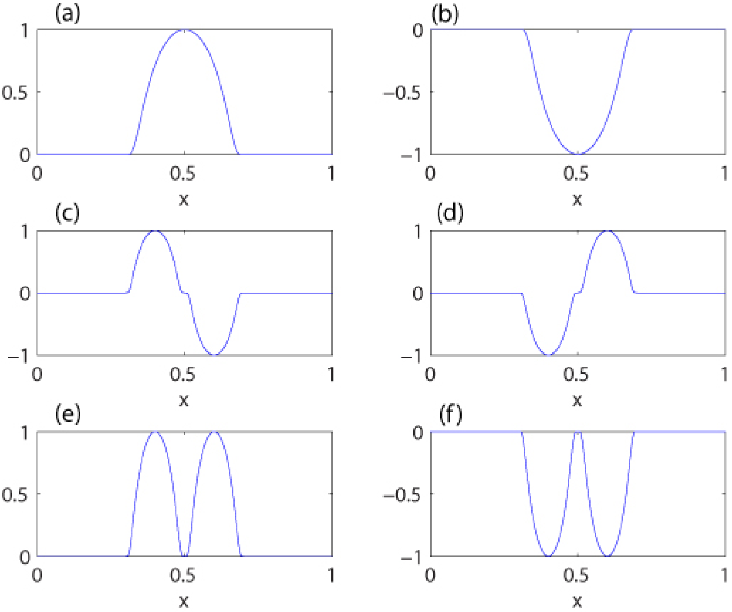
Basic ideas in one dimensions

Eigenvalues for boundary value problem determined by: $u_k(L) = 0$

- Zeta function:

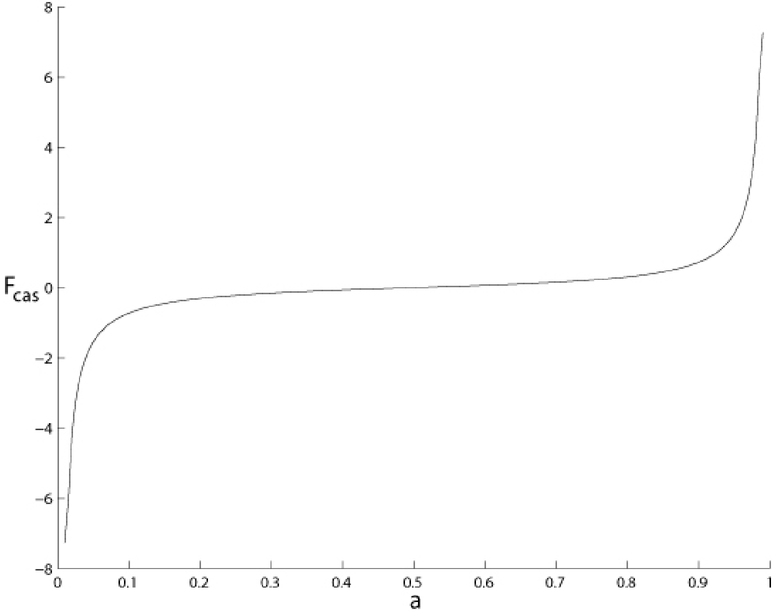
$$\begin{aligned}\zeta_P(s) &= \frac{1}{2\pi i}\int_\gamma dk\ k^{-2s}\ \frac{d}{dk}\ln u_k(L)\\ &= \frac{\sin \pi s}{\pi}\int_0^\infty dk\ k^{-2s}\frac{d}{dk}\ln u_{ik}(L)\end{aligned}$$


(a)
1
0.5
0
0
0.5
1
x
(b)
0
−0.5
−1
0
0.5
1
x
(c)
1
0
−1
0
0.5
1
x
(d)
1
0
−1
0
0.5
1
x
(e)
1
0.5
0
0
0.5
1
x
(f)
0
−0.5
−1
0
0.5
1
x

Compactly supported potentials

- Casimir force:

$$\begin{aligned} F_{Cas} &= -\frac{1}{2}\frac{\partial}{\partial a}\zeta\left(-\frac{1}{2}\right) \\ &= \frac{1}{2\pi}\int\limits_0^\infty dkk\frac{\partial}{\partial a}\frac{\partial}{\partial k}\ln u_{ik}(L). \end{aligned}$$


8
6
4
2
F_{cas} 0
−2
−4
−6
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0
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0.3
0.4
0.5
0.6
0.7
0.8
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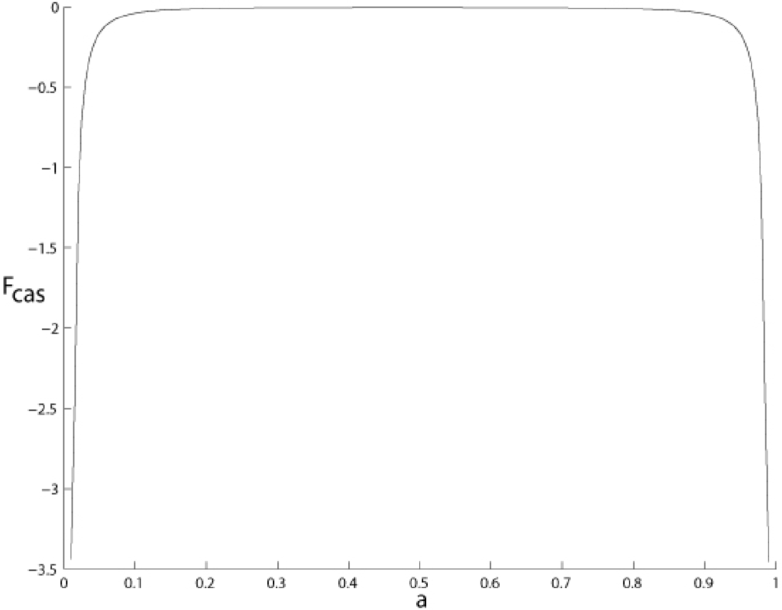
1
a


0
−0.5
−1
−1.5
−2
−2.5
−3
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F_{cas}
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0.1
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0.3
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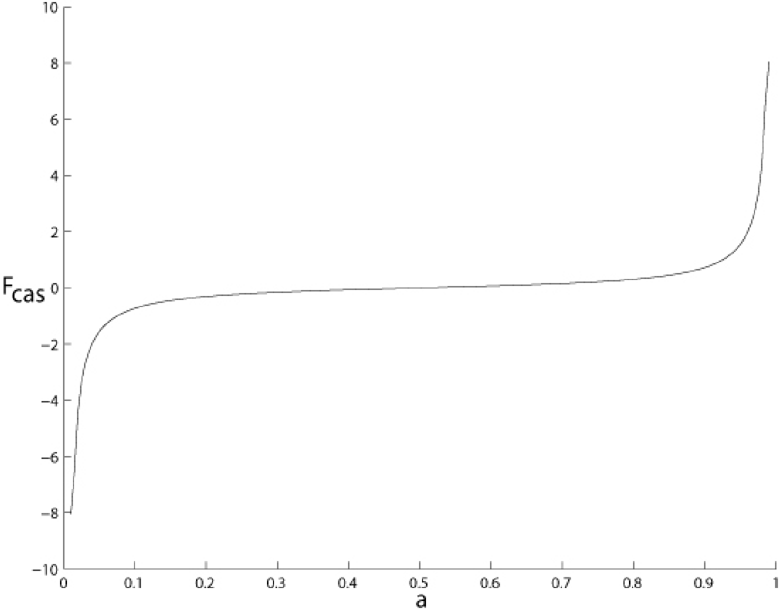

10
8
6
4
2
F_{cas} 0
−2
−4
−6
−8
−10
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0.1
0.2
0.3
0.4
0.5
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a

Spherically symmetric potentials

- Let:

$$P = -\Delta + V(r)$$

- Eigenvalue problem:

$$P\psi_{n,\ell}(r,\Omega) = \lambda_{n,\ell}^2 \psi_{n,\ell}(r,\Omega), \quad \psi_{n,\ell}(a,\Omega) = \psi_{n,\ell}(b,\Omega) = 0$$

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- Radial differential equation $\nu = \ell + \frac{d-1}{2}$:

$$\left(\frac{d^2}{dr^2} + \frac{d}{r}\frac{d}{dr} - \frac{\nu^2 - \left(\frac{d-1}{2}\right)^2}{r^2} - V(r) + \lambda^2 \right) \phi_{\lambda,\nu}(r) \quad = \quad 0,$$

$$\phi_{\lambda,\nu}(a) = 0, \quad \phi'_{\lambda,\nu}(a) = 1.$$

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Eigenvalues for boundary value problem determined by: $\phi_{\lambda,\nu}(b) = 0$

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- Zeta function:

$$\zeta_P(s) = \frac{\sin \pi s}{\pi} \sum_\nu d_\nu \int_0^\infty dk \; k^{-2s} \frac{d}{dk} \ln \phi_{ik,\nu}(b).$$

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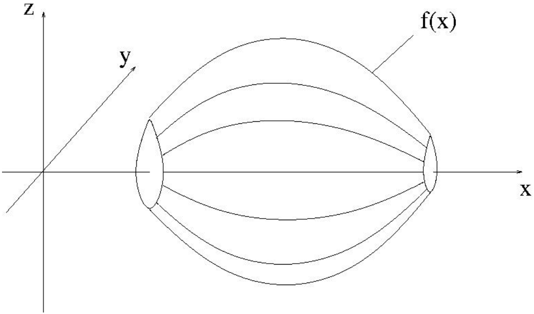
- Example for asymptotic term:

$$A_{-1}(s) = \frac{\Gamma\left(s - \frac{1}{2}\right)}{4\sqrt{\pi} s \Gamma(s)} \zeta_{\mathcal{N}} \left(s - \frac{1}{2}\right) \left(b^{2s} - a^{2s}\right).$$

Surfaces of revolution



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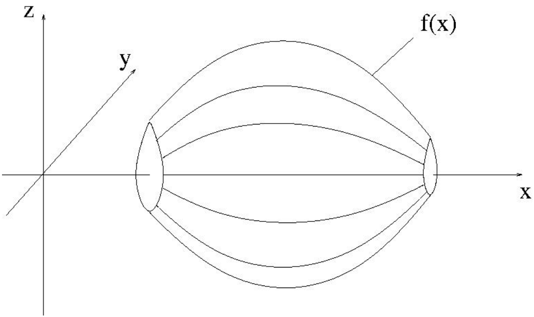

- Parameterization and metric of the surface

$$S = \begin{pmatrix} x \\ f(x)\cos\theta \\ f(x)\sin\theta \end{pmatrix},$$

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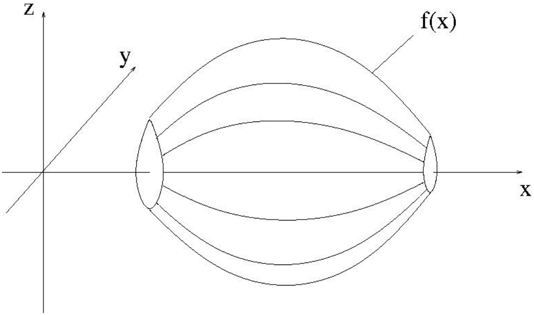

- Parameterization and metric of the surface

$$S = \begin{pmatrix} x \\ f(x)\cos\theta \\ f(x)\sin\theta \end{pmatrix}, \qquad g(x) = \begin{pmatrix} 1 + (f'(x))^2 & 0 \\ 0 & f^2(x) \end{pmatrix}$$

Surfaces of revolution

- Laplacian on the surface

$$\begin{aligned}\Delta\varphi &= \frac{1}{\sqrt{\det g}}\partial_\mu\left(g^{\mu\nu}\sqrt{\det g}\,\partial_\nu\right)\varphi \\ &= \frac{1}{1+(f')^2}\left(\frac{\partial^2\varphi}{\partial x^2}-\frac{f'f''}{1+(f')^2}\,\frac{\partial\varphi}{\partial x}+\frac{f'}{f}\,\frac{\partial\varphi}{\partial x}\right)+\frac{1}{f^2}\,\frac{\partial^2\varphi}{\partial\theta^2}\end{aligned}$$

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- Eigenvalue problem for the Laplacian

$$\varphi_{n,k}(x,\theta)=\psi_{n,k}(x)\;e^{ik\theta},\quad k\in\mathbb{Z}$$

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- Eigenvalue problem for the Laplacian

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$$\psi''_{n,k}(x)+\psi'_{n,k}(x)\left(\frac{f'}{f}-\frac{f'f''}{1+(f')^2}\right)+\left(\lambda_{n,k}-\frac{k^2}{f^2}\right)\left(1+(f')^2\right)\psi_{n,k}(x)=0$$

$$\psi_{n,k}(a)=\psi_{n,k}(b)=0$$

Surfaces of revolution

- Let

$$u = \frac{f'}{f} - \frac{f'f''}{1+(f')^2}, \qquad v = \left(\lambda - \frac{k^2}{f^2}\right)\left(1+(f')^2\right)$$

- Consider

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- Zeta function

$$\zeta(s) = \frac{1}{2\pi i} \sum_{k=-\infty}^{\infty} \int_\gamma d\lambda\, \lambda^{-s} \frac{d}{d\lambda} \ln \psi_{k,\lambda}(b), \qquad \Re s > 1$$

- Relative Casimir energy for surfaces of revolution:

$$E^{rel}_{Cas,asym} = \frac{1}{2\pi}\zeta_R'(-2)\int_a^b dx\left(\frac{\sqrt{1+f'^2}}{f^2}-\frac{1}{c^2}\right)$$

$$+\frac{1}{48\pi}\int_a^b dx\,\frac{12ff''-11f'^2(1+f'^2)+3\ln(4\pi f)\left[f'^2(1+f'^2)-2ff''\right]}{f^2(1+f'^2)^{3/2}}$$

$$+\frac{1}{256}\int_a^b dx\,\frac{3f'^2(1+f'^2)^2-6ff'f''(1+f'^2)-64ff'f''^2+8f^2f'''(1+f'^2)}{f^2(1+f'^2)^2}$$

$$+\frac{1}{4\pi}\int_a^b dx\left[\sqrt{1+f'^2}-1\right].$$

$$
\begin{aligned}
E^{rel}_{Cas,fin} = & -\frac{1}{\pi}\sum_{k=1}^{\infty} k \int_0^{\infty} dz z^{1/2} \frac{d}{dz} \\
& \Bigg\{ \ln \frac{2\sqrt{1+zc^2}\varphi_{k,-k^2z}(b)}{1-\exp\left(-\frac{2(b-a)k}{c}\sqrt{1+zc^2}\right)} \\
& -\frac{1}{16k}\int_a^b dx \frac{-zff'^2(-4+zf^2)(1+f'^2)+2zf^2f''(1+zf^2)}{(1+zf^2)^{5/2}(1+f'^2)^{3/2}} \\
& -\frac{1}{16k^2}\int_a^b dx \Big[zf \Big\{ f' \Big[-(4+zf^2(-10+zf^2))f'^2(1+f'^2)^2 \\
& +f(1+zf^2)(-7+2zf^2)(1+f'^2)f'' + 4f^2(1+zf^2)^2 f''^2 \Big] \\
& -f^2(1+zf^2)^2(1+f'^2)f''' \Big\} \frac{1}{(1+zf^2)^4(1+f'^2)^3} \Big] \Bigg\}
\end{aligned}
$$

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- Casimir energies and forces for 'separable situations' computable by simple numerics; other boundary conditions easily obtained by changing initial conditions.

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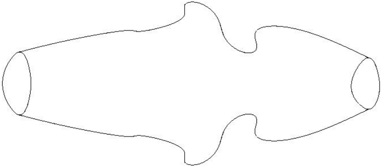

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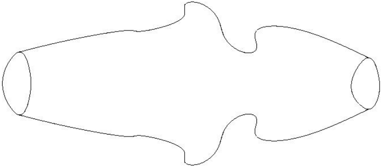

- Cusp-like singularities?