The Casimir Effect for Conical Pistons

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The Conical Piston

The generalized cone is defined as the $D = d + 1$ dimensional compact manifold $\mathscr{M} = \mathcal{I} \times \mathscr{N}$, with $\mathcal{I} \subseteq [0, b]$, and with $\mathscr{N}$ representing a smooth, compact Riemannian d-dimensional base manifold. $\mathscr{M}$ is locally described by the line element

$$ds^2 = dr^2 + r^2 d\Sigma^2_{\mathscr{N}} \quad , \quad r \in \mathcal{I} \ .$$

Piston Configuration

- $\mathscr{N}_a$ is a cross section of $\mathscr{M}$ at $r = a \in (0, b)$.
- $\mathscr{N}_a$ naturally divides $\mathscr{M}$ in two regions
  - $M_I = [0, a] \times \mathscr{N}$, with $\partial M_I = \{0\} \cup \mathscr{N}_a$,
  - $M_{II} = (a, b] \times \mathscr{N}$, with $\partial M_{II} = \mathscr{N}_a \cup \mathscr{N}_b$,
- The piston configuration is $M_I \cup_{\mathscr{N}_a} M_{II}$, where the piston itself is modelled by the cross section $\mathscr{N}_a$.

Remark: M_I and M_{II} have *different geometry* unlike standard Casimir pistons.

Analysis on the Conical Piston

Let $\varphi_i \in \mathcal{L}^2(\mathscr{M})$ with $i = (I, II)$, we consider the eigenvalue equation

$$\left(-\Delta_{\mathscr{M}} + m^2\right)\varphi_i = \alpha_i^2 \varphi_i \ ,$$

where, on the generalized cone, $\Delta_{\mathscr{M}}$ is an operator of Bessel type

$$\Delta_{\mathscr{M}} = \frac{\partial^2}{\partial r^2} + \frac{d}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{\mathscr{N}} \ .$$

Solutions: By setting $\alpha_i^2 = \gamma_i^2 + m^2$,

- For Region I; $\varphi_I = r^{\frac{1-d}{2}} J_\nu(\gamma_I r)\Phi(\Omega)$.
- For Region II; $\varphi_{II} = r^{\frac{1-d}{2}} \Big[A\, J_\nu(\gamma_{II} r) + B\, Y_\nu(\gamma_{II} r)\Big]\Phi(\Omega)$,

where

$$\Delta_{\mathscr{N}}\Phi(\Omega) = -\lambda^2\Phi(\Omega) \ , \quad \nu^2 = \lambda^2 + \frac{(1-d)^2}{4} \ .$$

Spectral Zeta Function and Casimir Energy

The spectral zeta function associated with the conical piston can be written as

$$\zeta(s) = \zeta_I(s) + \zeta_{II}(s) \ , \quad \text{where} \quad \zeta_i(s) = \sum_{\gamma_i} \left(\gamma_i^2 + m^2\right)^{-s} \ .$$

In this framework, by setting $s = -1/2 + \alpha$, the Casimir energy is found when $\alpha \to 0$

$$E_{\rm Cas}(a) = \frac{1}{2}{\rm FP}\zeta\left(-\frac{1}{2}, a\right) + \frac{1}{2}\left(\frac{1}{\alpha} + \ln\mu^2\right){\rm Res}\,\zeta\left(-\frac{1}{2}, a\right) + O(\alpha) \ ,$$

and the corresponding force on the piston is

$$F_{\rm Cas}(a) = -\frac{\partial}{\partial a}E_{\rm Cas}(a) \ .$$

Remark: An unambiguous prediction of the force can be obtained only if $\frac{\partial}{\partial a}{\rm Res}\,\zeta\left(-\frac{1}{2}, a\right) = 0$.

Dirichlet Boundary Conditions

Boundary conditions will provide implicit equations for the eigenvalues γ_i which are used to explicitly compute $\zeta_I(s)$ and $\zeta_{II}(s)$.

Dirichlet Boundary Conditions

- Dirichlet BC's on ∂M_I lead to

$$J_\nu(\gamma_I a) = 0 \; .$$

- Dirichlet BC's on ∂M_{II} lead to

$$\begin{cases} A\, J_\nu(\gamma_{II} a) + B\, Y_\nu(\gamma_{II} a) = 0 \\ A\, J_\nu(\gamma_{II} b) + B\, Y_\nu(\gamma_{II} b) = 0 \; , \end{cases}$$

which has a non-trivial solution for (A, B) if

$$\mathcal{P}_\nu(\gamma_{II}, a, b) = J_\nu(\gamma_{II} a) Y_\nu(\gamma_{II} b) - J_\nu(\gamma_{II} b) Y_\nu(\gamma_{II} a) = 0 \; .$$

Analytic Continuation of the Spectral Zeta Function

We represent the spectral zeta function in terms of a contour integral. In region I we have

$$\zeta_I(s,a) = \sum_{\nu} d(\nu) \frac{1}{2\pi i} \int_{\Gamma} dk \left[k^2 + m^2\right]^{-s} \frac{\partial}{\partial k} \ln\left[k^{-\nu} J_\nu(ka)\right] \ ,$$

which is valid for $\Re(s) > D/2$.
The analytic continuation to the domain $\Re(s) \leq D/2$, can be performed and leads to the result

$$\zeta_I(s) = Z_I(s,a) + \sum_{i=-1}^{D} A_i^{(I)}(s,a) \ .$$

- $Z_I(s,a) \sim a^{2s} f(s)$ is an analytic function for $-1 < \Re(s) < 1/2$.
- $A_i^{(I)}(s,a) \sim a^{2s} g_i(s)$ are meromorphic functions of s expressed in terms of $\zeta_{\mathcal{N}}(s) = \sum_\nu d(\nu)\nu^{-2s}$.

Analytic Continuation of the Spectral Zeta Function

A similar argument can be used for the analytic continuation of the spectral zeta function in region II

$$\zeta_{II}(s,a,b)=\sum_{\nu}d(\nu)\frac{1}{2\pi i}\int_{\Gamma'}d\kappa\left[\kappa^2+m^2\right]^{-s}\frac{\partial}{\partial\kappa}\ln\mathcal{P}_{\nu}(\kappa,a,b)\,,$$

and leads to the result

$$\zeta_{II}(s,a,b)=Z_{II}(s,a,b)+\mathscr{F}_{\mathcal{D}}(s,a,b)+\sum_{i=-1}^{D}A_i^{(II)}(s,a,b)\,.$$

- $Z_{II}(s,a,b)\sim a^{2s}\tilde{f}(s)+b^{2s}\tilde{g}(s)$ is an analytic function for $-1<\Re(s)<1/2$.
- $\mathscr{F}_{\mathcal{D}}(s,a,b)$ is an analytic function for $\Re(s)<1/2$.
- $A_i^{(II)}(s,a,b)=(-1)^iA_i^{(I)}(s,a)+A_i^{(I)}(s,b)$.

Casimir Force for Dirichlet Boundary Conditions

By taking the limit as $s \to -1/2$ in $\zeta_I(s,a)$ and $\zeta_{II}(s,a,b)$ and by differentiating with respect to a we obtain the following expression for the Casimir force on the piston

$$F_{\text{Cas}}^{\text{Dir}}(a,b) = \frac{1}{a^2} H_{\mathcal{D}}[\mathscr{N}] - \frac{1}{2}\mathscr{F}'_{\mathcal{D}}(-1/2,a,b) + \frac{1}{a^2}\left(\frac{1}{\alpha} + \ln\mu^2\right) G_{\mathcal{D}}[\mathscr{N}]\,.$$

Limiting Behavior

- **Large a and b**. In this situation $q = b/a - 1 \to 0$ and we obtain

$$F_{\text{Cas}}^{\text{Dir}}(q) = \frac{\Gamma(D+1)\zeta_R(D+1)}{2^{D+1}\sqrt{\pi}\,\Gamma\left(\frac{D}{2}\right)}\frac{\mathscr{A}_0^{\mathscr{N}}}{q^{D+1}} + O\left(q^{-D}\right)\,.$$

- **Limit a → 0**. In this case $\mathscr{F}'_{\mathcal{D}}(-1/2,a,b)$ is subleading and

$$F_{\text{Cas}}^{\text{Dir}}(a) \sim \frac{1}{a^2} H_{\mathcal{D}}[\mathscr{N}] + \frac{1}{a^2}\left(\frac{1}{\alpha} + \ln\mu^2\right) G_{\mathcal{D}}[\mathscr{N}] + O(a^{-1})\,.$$

Neumann Boundary Conditions

Neumann Boundary Conditions: by denoting $\beta = (1-d)/2$

- Neumann BC's on ∂M_I lead to

$$\beta J_\nu(a\gamma_I) + a\gamma_I J'_\nu(a\gamma_I) = 0 \ ,$$

- Neumann BC's on ∂M_{II} lead to

$$\begin{cases} A\left[\beta J_\nu(a\gamma_{II}) + a\gamma_{II} J'_\nu(a\gamma_{II})\right] + B\left[\beta Y_\nu(a\gamma_{II}) + a\gamma_{II} Y'_\nu(a\gamma_{II})\right] = 0 \\ A\left[\beta J_\nu(b\gamma_{II}) + b\gamma_{II} J'_\nu(b\gamma_{II})\right] + B\left[\beta Y_\nu(b\gamma_{II}) + b\gamma_{II} Y'_\nu(b\gamma_{II})\right] = 0 \ . \end{cases}$$

which possesses a non-trivial solution for (A, B) if

$$\begin{aligned} &\left[\beta J_\nu(a\gamma_{II}) + a\gamma_{II} J'_\nu(a\gamma_{II})\right]\left[\beta Y_\nu(b\gamma_{II}) + b\gamma_{II} Y'_\nu(b\gamma_{II})\right] \\ &- \quad \left[\beta Y_\nu(a\gamma_{II}) + a\gamma_{II} Y'_\nu(a\gamma_{II})\right]\left[\beta J_\nu(b\gamma_{II}) + b\gamma_{II} J'_\nu(b\gamma_{II})\right] = 0 \ . \end{aligned}$$

Analytic Continuation

The analytic continuation in this case proceeds along the same lines. For region I we have

$$\zeta_I^{\mathcal{N}}(s,a) = W_I(s,a) + \sum_{i=-1}^{D} A_i^{(\mathcal{N},I)}(s,a) \,.$$

- $W_I(s,a) \sim a^{2s} h(s)$ is an analytic function for $-1 < \Re(s) < 1/2$.
- $A_i^{(\mathcal{N},I)}(s,a) \sim a^{2s} l(s)$ are meromorphic functions of s expressed in terms of $\zeta_{\mathcal{N}}(s)$.

For region II we obtain

$$\zeta_{II}^{\mathcal{N}}(s,a,b) = W_{II}(s,a,b) + \mathscr{F}_{\mathcal{N}}(s,a,b) + \sum_{i=-1}^{D} A_i^{(\mathcal{N},II)}(s,a,b) \,,$$

- $W_{II}(s,a,b) \sim a^{2s}\tilde{h}(s) + b^{2s}\tilde{l}(s)$ is an analytic function for $-1 < \Re(s) < 1/2$.
- $\mathscr{F}_{\mathcal{N}}(s,a,b)$ is an analytic function for $\Re(s) < 1/2$.
- $A_i^{(\mathcal{N},II)}(s,a,b) = (-1)^i A_i^{(\mathcal{N},I)}(s,a) + A_i^{(\mathcal{N},I)}(s,b)$.

Casimir Force for Neumann Boundary Conditions

From the expressions for $\zeta_I^{\mathcal{N}}(s,a)$ and $\zeta_{II}^{\mathcal{N}}(s,a,b)$ one obtains the following expression for the Casimir force on the piston

$$F_{\rm Cas}^{\rm Neu}(a,b) = \frac{1}{a^2} H_{\mathcal{N}}[\mathscr{N}] - \frac{1}{2}\mathscr{F}'_{\mathcal{N}}(-1/2,a,b) + \frac{1}{a^2}\left(\frac{1}{\alpha} + \ln\mu^2\right) G_{\mathcal{N}}[\mathscr{N}]\,.$$

Limiting Behavior

- **Large a and b**. In this situation $q = b/a - 1 \to 0$ and we obtain

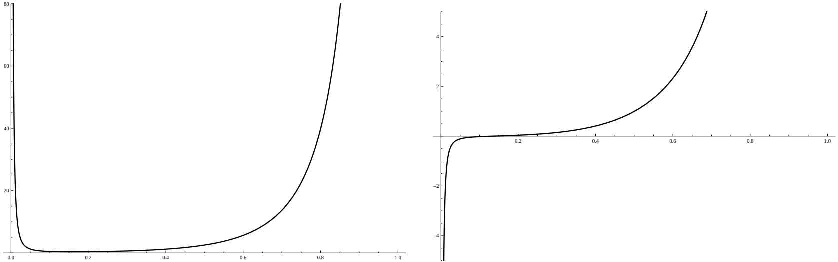
$$F_{\rm Cas}^{\rm Neu}(q) \sim F_{\rm Cas}^{\rm Dir}(q)\,.$$

- **Limit $\mathbf{a \to 0}$**. In this case $\mathscr{F}'_{\mathcal{N}}(-1/2,a,b)$ is subleading and

$$F_{\rm Cas}^{\rm Neu}(a) \sim \frac{1}{a^2} H_{\mathcal{N}}[\mathscr{N}] + \frac{1}{a^2}\left(\frac{1}{\alpha} + \ln\mu^2\right) G_{\mathcal{N}}[\mathscr{N}] + O(a^{-1})\,.$$

Dirichlet Boundary Conditions

d-dimensional sphere as base manifold $\mathscr{N}$.



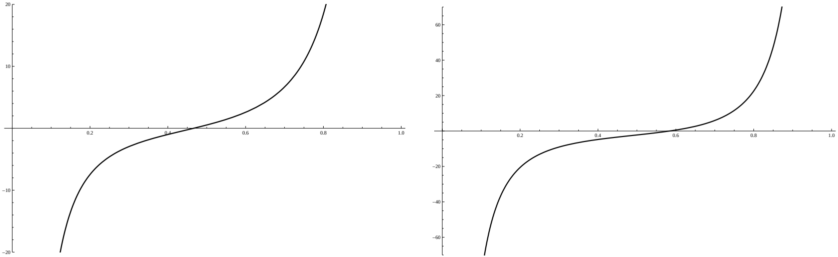
(a) $d = 2$, and $D = 3$

(b) $d = 4$, and $D = 5$

Figure: Plots of the Casimir force, $F_{\text{Cas}}^{\text{Dir}}(a, 1)$, on the piston $\mathscr{N}$ for Dirichlet boundary conditions as a function of the position a.

Neumann Boundary Conditions

d-dimensional sphere as base manifold $\mathscr{N}$.



(a) $d = 2$, and $D = 3$ (b) $d = 4$, and $D = 5$

Figure: Plots of the Casimir force, $F_{\text{Cas}}^{\text{Neu}}(a, 1)$, on the piston $\mathscr{N}$ for Neumann boundary conditions as function of the position a.

Hybrid Boundary Conditions

First Type: Dirichlet BC's at $r = a$, Neumann BC's at $r = b$

$$J_\nu(\gamma_I a) = 0 \ ,$$

$$\begin{cases} AJ_\nu(\gamma_{II} a) + BY_\nu(\gamma_{II} a) = 0 \\ A\left[\beta J_\nu(\gamma_{II} b) + \gamma_{II} b J'_\nu(\gamma_{II} b)\right] + B\left[\beta Y_\nu(\gamma_{II} b) + \gamma_{II} b Y'_\nu(\gamma_{II} b)\right] = 0 \ , \end{cases}$$

from the second condition we have

$$J_\nu(\gamma_{II} a)\left[\beta Y_\nu(\gamma_{II} b) + \gamma_{II} b Y'_\nu(\gamma_{II} b)\right] - Y_\nu(\gamma_{II} a)\left[\beta J_\nu(\gamma_{II} b) + \gamma_{II} b J'_\nu(\gamma_{II} b)\right] = 0 \ .$$

Second Type: Neumann BC's at $r = a$, Dirichlet BC's at $r = b$

$$\beta J_\nu(\gamma_I a) + a\gamma_I J'_\nu(\gamma_I a) = 0 \ ,$$

$$\begin{cases} A\left[\beta J_\nu(\gamma_I a) + a\gamma_I J'_\nu(\gamma_I a)\right] + B\left[\beta Y_\nu(\gamma_I a) + a\gamma_I Y'_\nu(\gamma_I a)\right] = 0 \\ AJ_\nu(\gamma_{II} b) + BY'_\nu(\gamma_{II} b) = 0 \ , \end{cases}$$

from the second condition we have

$$Y_\nu(\gamma_{II} b)\left[\beta J_\nu(\gamma_{II} a) + \gamma_{II} a J'_\nu(\gamma_{II} a)\right] - J_\nu(\gamma_{II} b)\left[\beta Y_\nu(\gamma_{II} a) + \gamma_{II} a Y'_\nu(\gamma_{II} a)\right] = 0.$$

Casimir Force for Hybrid Boundary Conditions

By setting $j = (1, 2)$ to denote HBC's of first and second type we have

$$F_{\rm Cas}^{\mathcal{H}_j}(a,b) = \frac{1}{a^2} P_{\mathcal{H}_j}[\mathscr{N}] - \frac{1}{2}\mathscr{F}'_{\mathcal{H}_j}(-1/2, a, b) + \frac{1}{a^2}\left(\frac{1}{\alpha} + \ln\mu^2\right) Q_{\mathcal{H}_j}[\mathscr{N}]\;.$$

Limiting Behavior

- **Large a and b**. In this situation $q = b/a - 1 \to 0$ and we obtain

$$F_{\rm Cas}^{\mathcal{H}_j}(q) \sim -F_{\rm Cas}^{\rm Dir}(q)\;.$$

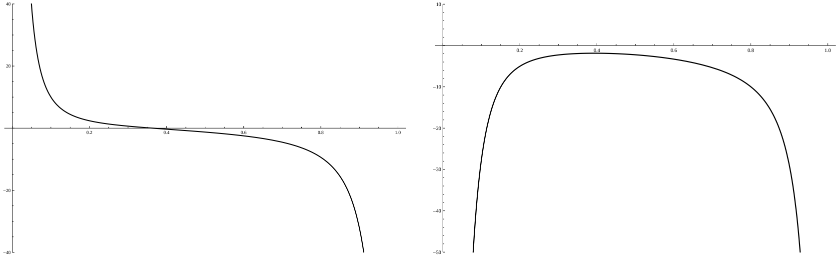
- **Limit a → 0**. In this case $\mathscr{F}'_{\mathcal{H}_j}(-1/2, a, b)$ is subleading and

$$F_{\rm Cas}^{\mathcal{H}_j}(a) \sim \frac{1}{a^2} P_{\mathcal{H}_j}[\mathscr{N}] + \frac{1}{a^2}\left(\frac{1}{\alpha} + \ln\mu^2\right) Q_{\mathcal{H}_j}[\mathscr{N}] + O(a^{-1})\;.$$

Hybrid Boundary Conditions of First Type

Dirichlet BC's at $r = a$, and Neumann BC's at $r = b$

d-dimensional sphere as base manifold $\mathscr{N}$.



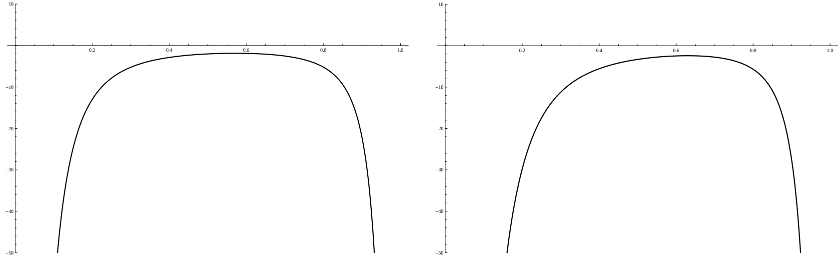
(a) $d = 2$, and $D = 3$ (b) $d = 4$, and $D = 5$

Figure: Plots of the Casimir force, $F^{\mathcal{H}_1}_{\text{Cas}}(a, 1)$, on the piston $\mathscr{N}$ for hybrid boundary conditions of first type as a function of the position a.

Hybrid Boundary Conditions of Second Type

Neumann BC's at $r = a$, and Dirichlet BC's at $r = b$

d-dimensional sphere as base manifold $\mathscr{N}$.



(a) $d = 2$, and $D = 3$ (b) $d = 4$, and $D = 5$

Figure: Plots of the Casimir force, $F^{\mathcal{H}_2}_{\text{Cas}}(a, 1)$, on the piston $\mathscr{N}$ for hybrid boundary conditions of second type as a function of the position a.

Summary of the Main Points

Differences between conical pistons and standard Casimir pistons.

- Presence of a singularity at the origin. Furthermore, the conical piston is a *curved* manifold.
- The two chambers of the conical piston have *different* geometry.
- Hybrid BC's of first and second type lead to *different* results for the Casimir force on the piston.
- The Casimir force is, in general, *not* symmetric with respect to the point $r_0 = b/2$.

Outlook and Generalizations

- Study of Casimir pistons modelled after the spherical suspension (or Riemann cap). This is a singular Riemannian manifold with line element

$$ds^2 = d\theta^2 + \sin^2\theta d\Sigma^2_{\mathcal{N}} \; , \quad \theta \in [0, \pi) \; .$$

- More generally, one can consider Casimir pistons modelled after compact manifolds described by the line element

$$ds^2 = dr^2 + f^2(r) d\Sigma^2_{\mathcal{N}} \; .$$

  - **Case 1**: $f(r) \sim r^\delta$, $\delta > 0$ and $\delta \neq 1$, as $r \to 0$. Would allow the analysis of singularities other than the conical one.
  - **Case 2**: $f(r) \sim C$ as $r \to 0$ (*Warped Product Manifolds*). The two chambers have a different geometry but none of them contains a singularity.

- Extend the analysis of the conical Casimir piston to include finite temperature effects.

References

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