# QUANTUM BACKREACTION (CASIMIR) EFFECT WITHOUT INFINITIES ALGEBRAIC ANALYSIS

Andrzej Herdegen

Jagiellonian University
Kraków

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- Casimir effect, in most general terms, is the backreaction of a quantum system responding to an adiabatic change of external conditions. This backreaction is expected to be quantitatively measured by a change in the expectation value of a certain energy observable of the system.
- However, for this concept to be applicable, the system has to retain its identity in the process. Most prevailing tendencies in the analysis of the effect seem to overlook this question.

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- However, for this concept to be applicable, the system has to retain its identity in the process. Most prevailing tendencies in the analysis of the effect seem to overlook this question.

- In general, a quantum theory is defined by an algebra of observables, whose representations by operators in a Hilbert space define concrete physical systems described by the theory. A quantum system retains its identity if both the algebra as well as its representation do not change.
- I shall discuss the resulting restrictions for admissible models of changing external conditions. These ideas are applied to quantum field models. No infinities arise, if the algebraic demands are respected.

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- I shall discuss the resulting restrictions for admissible models of changing external conditions. These ideas are applied to quantum field models. No infinities arise, if the algebraic demands are respected.

# SYSTEM Q-M



- Q – relatively simple quantum system (e.g. a quantum field)
- M – complex macroscopic system (say, conducting plates) with collective effective coordinates $a$
- Full closed theory of Q-M out of reach
- Approximation:
  M is 'heavy' – characterized by very large inertia; thus:
- variables $a$ are classical
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# Isolated system Q (HP)



- Basic quantum variables at a fixed time form an abstract algebra $\mathcal{A}$, e.g. CCR algebra.
- Algebra is represented by operators in a Hilbert space $\mathcal{H}$:

$$\pi : \mathcal{A} \mapsto \pi(\mathcal{A}) , \quad A \mapsto \pi(A);$$

  Density operators in $\mathcal{H}$ represent states of the system $Q$.
- Intrinsic dynamics of $Q$ defined by an automorphism of $\mathcal{A}$:

$$\alpha_t : \mathcal{A} \mapsto \mathcal{A} , \quad A \mapsto \alpha_t A$$

  implemented by a unitary evolution in the Hilbert space $\mathcal{H}$:

$$\pi(\alpha_t A) = U(t)\pi(A)U(t)^* , \quad U(t) = \exp(itH) ,$$

  where $H$ – the energy operator of the system, with nonnegative spectrum and a ground state, represented by a unit eigenvector; energy may be normalized to be zero in that state.

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# Introducing M



- Add part $M$ into the system: characterized by classical variables $a$; no quantum degrees added.
- System $Q$ should retain its identity: algebra $\mathcal{A}$ must remain unaffected.
- States to be considered must be physically comparable: the representation $\pi$ of $\mathcal{A}$ must remain unaffected.

# Dynamics of Q with frozen M



- Degrees $a$ frozen – system Q still a closed system in interaction with conditions created by $M$; for each $a$ evolution: an automorphism of $\mathcal{A}$:

$$\alpha_{at} : \mathcal{A} \mapsto \mathcal{A}\,, \quad A \mapsto \alpha_{at} A\,.$$

- Evolution implemented in representation $\pi$: for each $a$

$$\pi(\alpha_{at} A) = U_a(t)\pi(A)U_a(t)^*\,, \quad U_a(t) = \exp(itH_a)\,.$$

- For each $a$ the generator $H_a$ defined by this up to:

$$H_a \to H_a + \lambda_a \,\text{id}\,,$$

where $\lambda_a$ is any real function of parameters $a$.

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# Coupled system Q-M (SP)



- Unitary evolution of Q in SP (Q not closed – evolution on algebraic level: to restrictive).
- Suppose that $a(t)$ is known as a 'slow' function of time (system $M$ is 'heavy'). Adiabatic approximation with initial ($t = 0$) eigenstate of $H_{a(0)}$:

$$\psi(t) = e^{i\varphi(t)}\psi_{a(t)} ,$$

where $H_a\psi_a = E_a\psi_a$ and $\varphi(t)$ is a real function depending functionally on $E_a$ and $\psi_a$.
- Evolution of expectation value of an observable $B$ given by

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# Backreaction – determination of $a(t)$



- Intrinsic energy stored in part Q of the system represented by $H$, which in the coupled system is not a constant of motion any more; its expectation value

$$\mathcal{E}_a := (\psi_a, H\psi_a)\,,$$

depends on time through variables $a$.

- Changes in $\mathcal{E}_a$ correspond to the energy which has been transferred from $Q$ to the rest of the system, which (with the suppression of all microscopic details of $M$) is described by the variables $a$. Thus $\mathcal{E}_a$ plays the role of a potential energy with respect to these variables. We assume that the rest of the total energy of the coupled system is supplied by the kinetic energy of $M$, thus we obtain a potential system, with the generalized force given by

$$\mathcal{F}_a = -\frac{\partial \mathcal{E}_a}{\partial a}\,.$$

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# Classical system



- Symplectic space (phase space):
  $\mathcal{L} \subset \mathcal{R} \oplus \mathcal{R} \ni V \equiv (v \oplus u)$, $\mathcal{R}$ – real Hilbert space
  symplectic form

$$\sigma(V_1, V_2) = (v_2, u_1) - (v_1, u_2)$$

- Hamiltonian function

$$\mathcal{H}(v, u) = \frac{1}{2}[(u, u) + (hv, hv)],$$

  $h$ – positive selfadjoint operator on $\mathcal{R}$,
- Symplectic evolution

$$T_t(v \oplus u) = (\cos(ht)v + \sin(ht)h^{-1}u) \oplus (-\sin(ht)hv + \cos(ht)u).$$

- Denote

$$V'(V) = (v', u) + (u', v),$$

  then

$$(T_t V')(V) = V'(T_t V).$$

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# Quantum system



- 'Quantization': $V'(V) \to \Phi(V)$ – algebraic elements satisfying CCR

$$[\Phi(V_1), \Phi(V_2)] = i\sigma(V_1, V_2)\,\mathrm{id}\,, \qquad V \in \mathcal{L}$$

evolution

$$\alpha_t(\Phi(V)) = \Phi(T_t V)$$

- Vacuum representation $\Phi(V) \to \Phi_0(V)$ – operators in a Fock space; rep. defined by demands:

$$\Phi_0(T_t V) = U(t)\Phi_0(V)U^*(t)\,, \qquad U(t) = \exp(itH)\,.$$

$H$ – positive, with ground state vector $\Omega$

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# Examples



- Free massless scalar field – initial value formulation:

$$\mathcal{R} = L^2_{\mathbb{R}}(\mathbb{R}^3)\,, \quad h = \sqrt{-\Delta}\,, \quad \mathcal{L} = \mathcal{D}_{\mathbb{R}}(\mathbb{R}^3) \oplus \mathcal{D}_{\mathbb{R}}(\mathbb{R}^3)$$

- Scalar field with boundary conditions on surfaces coordinated by parameters $a$:

$$\mathcal{R} = L^2_{\mathbb{R}}(\mathbb{R}^3)\,,\ [h^B_a]^2 = -\Delta_{\mathrm{b.c.}}\,,\ \mathcal{L}^B_a = \mathcal{D}_{\mathcal{R}}(h^B_a) \oplus \mathcal{D}_{\mathcal{R}}([h^B_a]^{-1/2})$$

- 'Momentum'-regularized boundary conditions: $h_a = f(h, h^B_a)$, such that $h_a \simeq h^B_a$ for small momentum transfer, and $h_a \simeq h$ for large momentum transfer
- Scalar field with external static interaction depending on parameters $a$: $h^2_a = -\Delta + V_a$, $V_a$ – perturbation

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# Examples – quantization

- Algebra CCR

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# Conditions



- Stability of algebras: $\mathcal{L} = \mathcal{L}_a$
  Not satisfied in the sharp boundary conditions case!
  (and no way to satisfy the condition by any extension of symplectic spaces)
- Relation between representations: when stability of algebras is ensured then annihilation/creation operators of representations determined by $h$ and $h_a$ are related by a Bogoljubov transformation

$$a_a(f) = a(T_a f) + a^*(S_a f)\,, \; a_a^*(f) = a^*(T_a f) + a(S_a f)$$

with $T_a$ linear and $S_a$ antilinear, determined by $h$ and $h_a$.

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# Conditions



- Representations are equivalent iff

  $$\mathcal{N}_a \equiv \operatorname{Tr}[S_a S_a^*] = \frac{1}{4}\operatorname{Tr}[h^{-1/2}(h_a - h)h_a^{-1}(h_a - h)h^{-1/2}] < \infty$$

  Then $\mathcal{N}_a = (\Omega_a, N\Omega_a)$, $N$ – particle (or 'excitation') number

- Casimir energy for ground state

  $$\mathcal{E}_a = (\Omega_a, H\Omega_a) = \frac{1}{4}\operatorname{Tr}[(h_a - h)h_a^{-1}(h_a - h)]$$

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# Planar symmetry



- $\mathcal{R} = L^2_{\mathbb{R}}(\mathbb{R}^2) \otimes L^2_{\mathbb{R}}(\mathbb{R})$,

$$h^2 = (h_\perp \otimes \mathrm{id})^2 + (\mathrm{id} \otimes h_z)^2 \,, \quad h_a^2 = (h_\perp \otimes \mathrm{id})^2 + (\mathrm{id} \otimes h_{za})^2$$
$$h_\perp^2 = -\Delta_\perp \,, \quad h_z^2 = -\partial_z^2$$

- $h_{za}$ will model parallel planes at a distance $a$
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# Two models



- 1. Transparency to large momenta:

$$h_{za} = h_z + G(h_z)\left[F(h^B_{za}) - F(h_z)\right]G(h_z)\,,$$

$h^B_{za}$ – sharp Dirichlet or Neumann bound. cond. at points separated by $a$,
$F, G(p) \to 0$ $(p \to \infty)$, $\quad F(p) = p$, $G(p) = 1$ $(p \leq p_0)$

- 2. Nonlocality control:

$$h_{za} = h_z + V_a\,, \quad V_a(z, z') = g(z-b)\overline{g(z'-b)} + g(z+b)\overline{g(z'+b)}$$

$g$ of compact support, even (D) or odd (N) $(b = a/2)$

- Both classes of models meet the admissibility criteria, so the Casimir energy per area $\varepsilon_a$ is well defined and finite for all $a$

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# Approximating boundary conditions



- For each model – a one parameter ($\lambda \in (0, 1\rangle$) family of rescaled models, such that for fixed $a$ and $\lambda \to 0$ the sharp boundary conditions are recovered
- The Casimir energy per area scales:

$$\varepsilon_{a,\lambda} = \lambda^{-3} \varepsilon_{a/\lambda}$$

Thus to find scaling behavior of $\varepsilon_{a,\lambda}$ for scaled models: expand $\varepsilon_a$ in powers of $1/a$ up to third order

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# Expanding $\varepsilon_a$

- Casimir energy $\varepsilon_a$ given by a complex integral expression depending functionally on functions defining the models and on $a$
- Expansion

$$\varepsilon_a = \varepsilon_\infty + \frac{\gamma}{a^2} - \frac{\pi^2}{1440 a^3} + \text{higher terms}$$

$\varepsilon_\infty$ energy needed to produce field configuration around two independent plates

$\gamma$ $= 0$ for Dirichlet case, and
model dependent for Neumann case
– contribution from inside the walls

$a^{-3}$-term – universal for large class of models

- Models for conducting plates and electromagnetic field – sum of the D and N terms

# Expanding $\varepsilon_a$



- Casimir energy $\varepsilon_a$ given by a complex integral expression depending functionally on functions defining the models and on $a$
- Expansion

$$\varepsilon_a = \varepsilon_\infty + \frac{\gamma}{a^2} - \frac{\pi^2}{1440a^3} + \text{higher terms}$$

$\varepsilon_\infty$ energy needed to produce field configuration around two independent plates

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- Proper formulation of the problem removes the usual sources of infinities
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