The Euler-Heisenberg Effective Action:
History and Scientific Legacy

celebrating the 75th anniversary of the paper

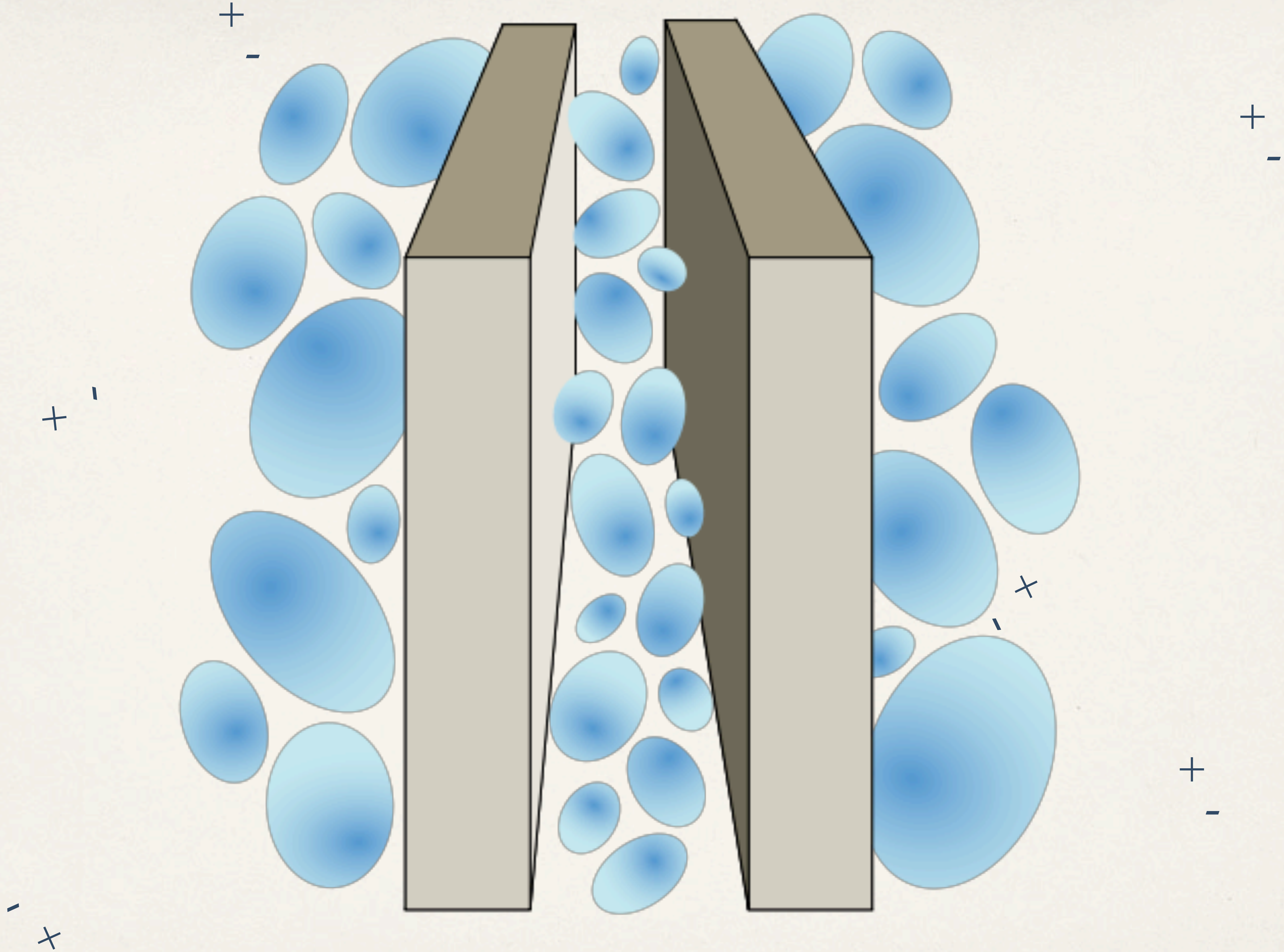
W. Heisenberg & H. Euler, *Consequences of Dirac's theory of the positron*,
Zeitschr. Phys., **98**, 714 (1936)

Gerald Dunne
University of Connecticut

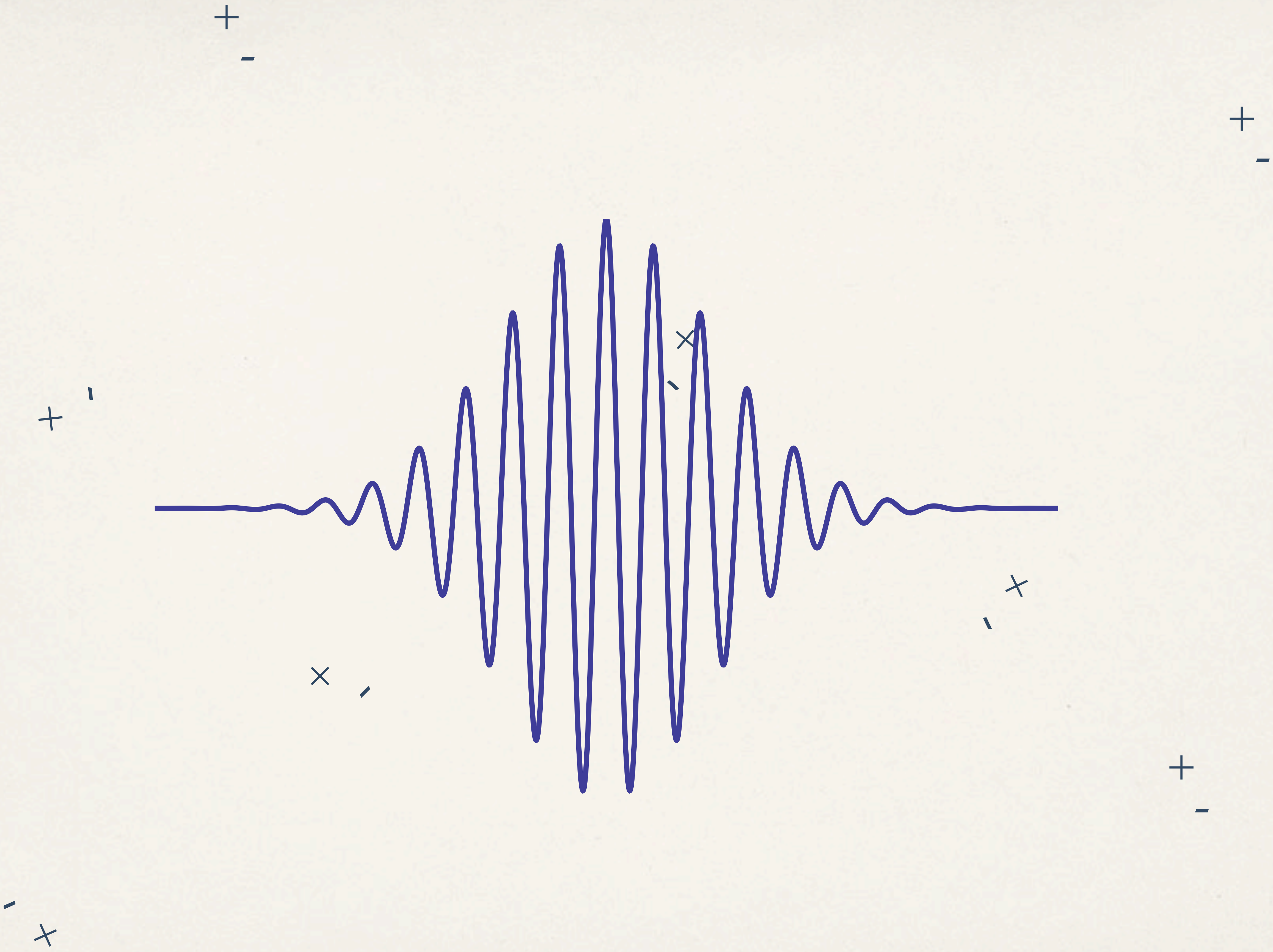
[also with thanks to Walter Dittrich, Tübingen]

QFEXT11, Benasque

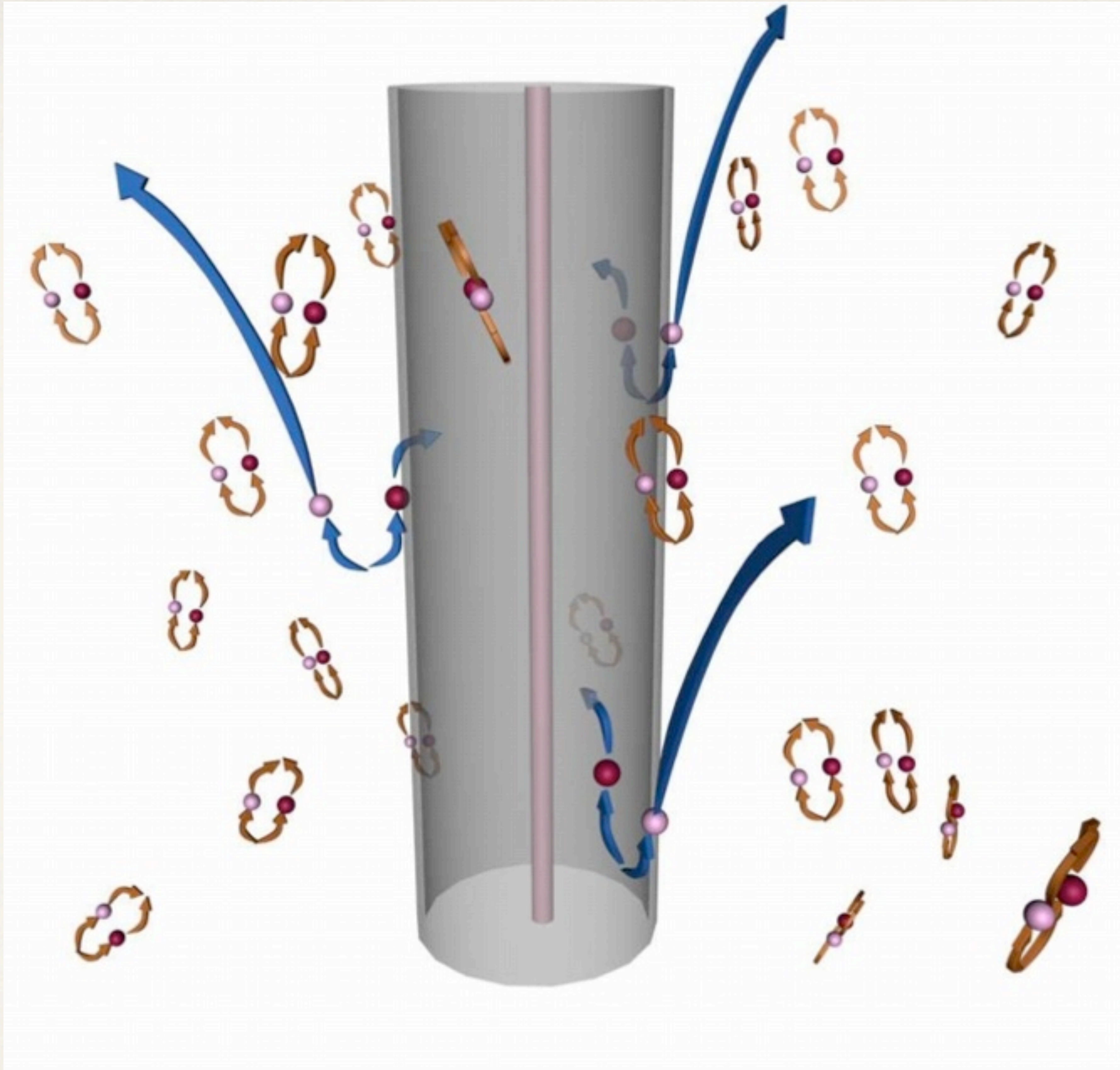
the quantum vacuum



Casimir effect



“vacuum polarization”



Hawking radiation

Euler-Heisenberg Effective Action

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Folgerungen aus der Diracschen Theorie des Positrons.

Von **W. Heisenberg** und **H. Euler** in Leipzig.

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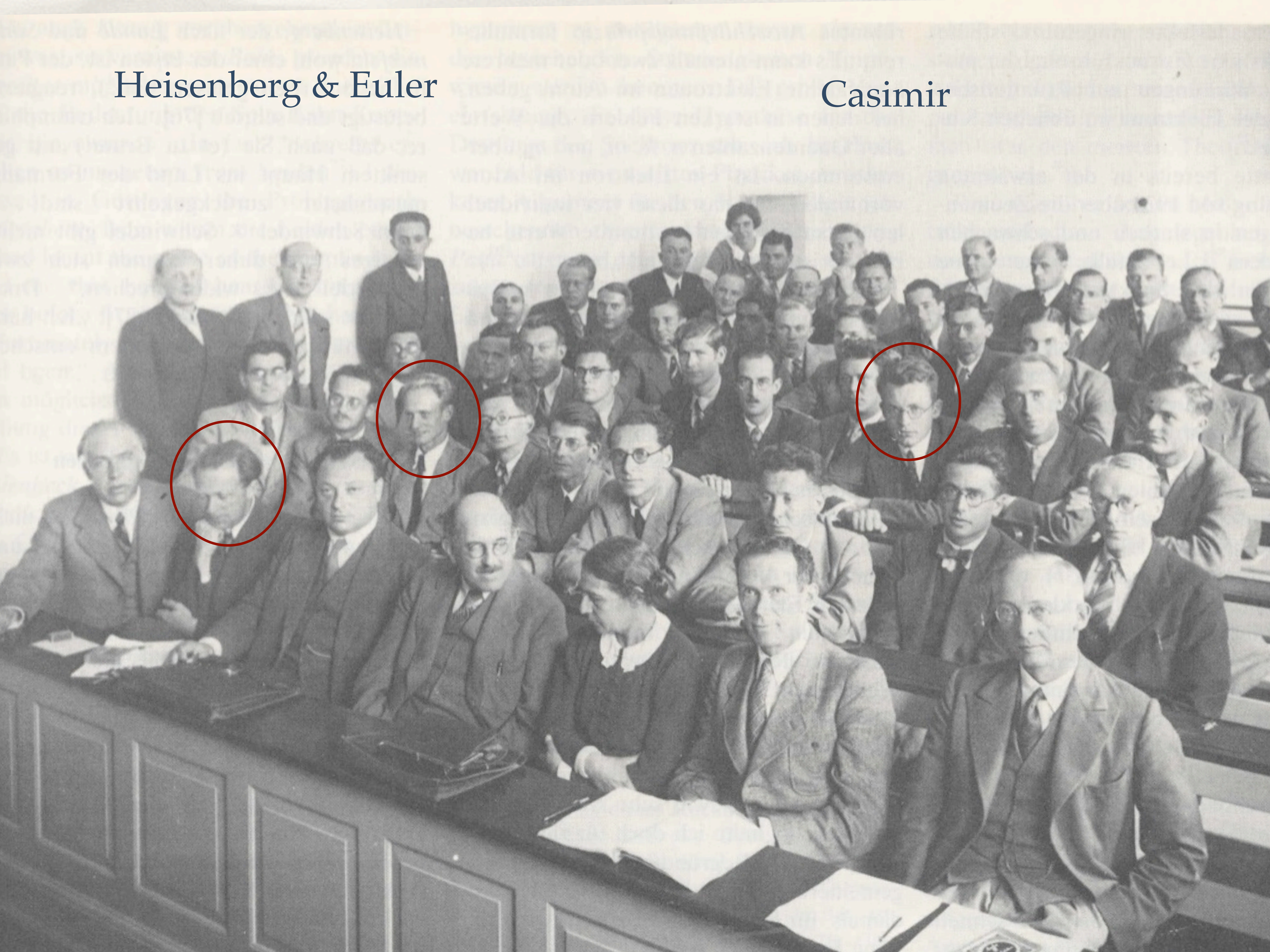
$$\mathfrak{L} = \frac{1}{2}(\mathfrak{E}^2-\mathfrak{B}^2) + \frac{e^2}{hc}\int_0^\infty e^{-\eta}\frac{d\eta}{\eta^3}\left\{ i\eta^2(\mathfrak{E}\mathfrak{B})\cdot \frac{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2-\mathfrak{B}^2+2i(\mathfrak{E}\mathfrak{B})}\right)+\text{konj}}{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2-\mathfrak{B}^2+2i(\mathfrak{E}\mathfrak{B})}\right)-\text{konj}} + |\mathfrak{E}_k|^2 + \frac{\eta^2}{3}(\mathfrak{B}^2-\mathfrak{E}^2)\right\}.$$

($\mathfrak{E}$, $\mathfrak{B}$ Kraft auf das Elektron.

$$|\mathfrak{E}_k| = \frac{m^2c^3}{e\hbar} = \frac{1}{„137“}\frac{e}{(e^2/mc^2)^2} = \text{„Kritische Feldstärke“.})$$

some historical background and context

Heisenberg & Euler


Heisenberg & Euler
Casimir

brief timeline

- 1928: Dirac equation, relativistic electron
- 1929: Dirac sea, hole theory
- 1931: Dirac: hole = positive-charge-electron

"A hole, if there were one, would be a new kind of particle, unknown to experimental physics, having the same mass and opposite charge to an electron"

- 1932: Anderson: discovered (& named) the "positron"
- 1933 Solvay conference: Dirac: hole = positron,
  vacuum polarization and charge renormalization

- 1933/34: Heisenberg: vacuum polarization in Dirac theory
- 1936: Heisenberg's student Hans Euler: PhD at Leipzig
- 1936: Heisenberg & Euler effective action

the road from 1928 to 1936 was not easy ...

Heisenberg and Pauli correspondence

"The saddest chapter of modern physics is and remains the Dirac theory"
H to P, 1928

"That the hole theory will lead to many kinds of horrors as long as the self-energy cannot be put in order, that I quite believe" H to P, 1934

"With regard to quantum electrodynamics, we are still at the stage in which we were in 1922 with regard to quantum mechanics. We know that everything is wrong." H to P, 1935

Positron theory

Dirac: *Theory of the positron*, Solvay Conference, 1933

"Any state of negative energy which is not occupied represents a lack of uniformity and this must be shown by observation as a kind of hole. It is possible to assume that the positrons are these holes."

Heisenberg: *The fluctuations of charge connected with the formation of matter from radiation*, 1934

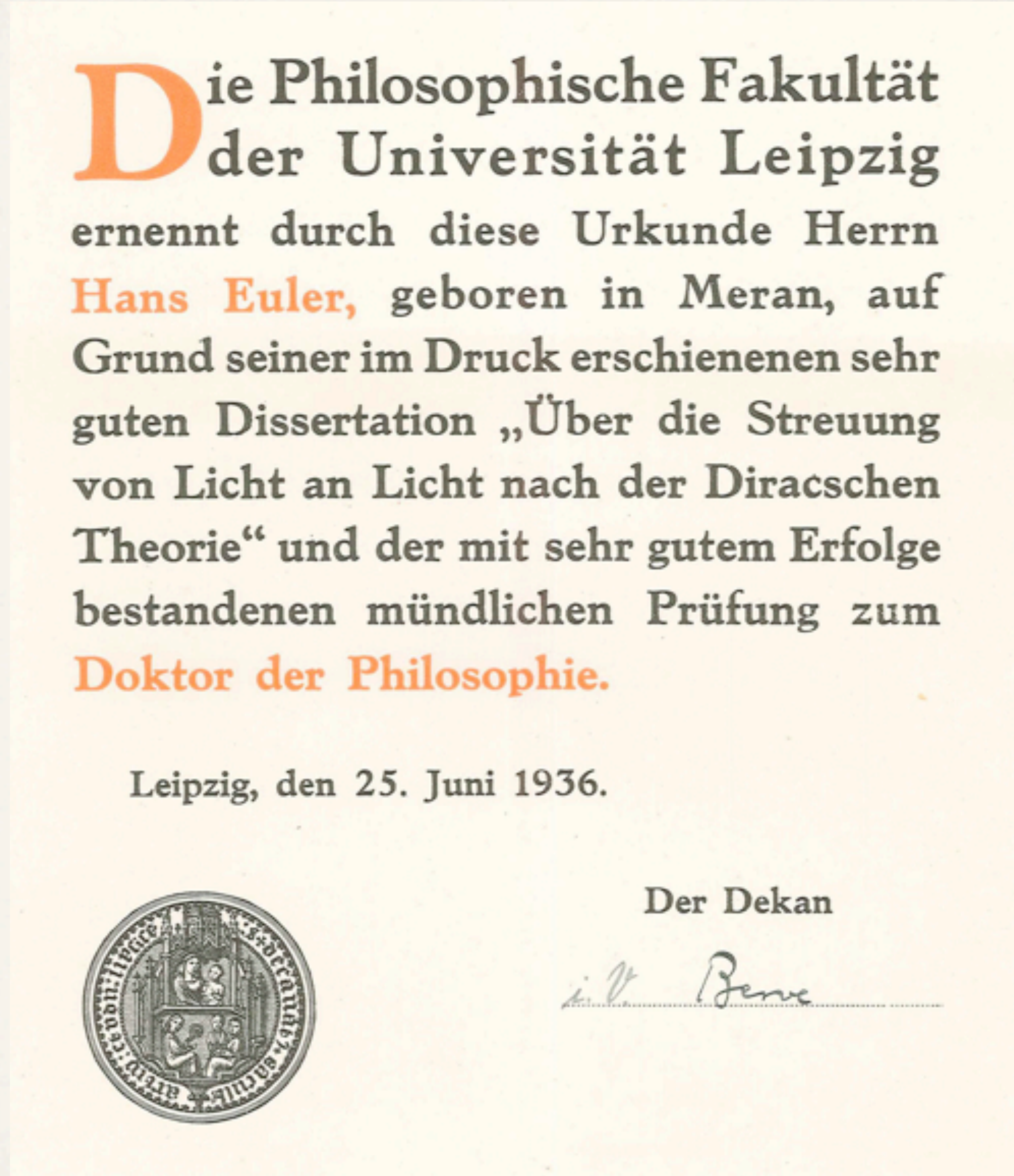
Heisenberg: *Comments on the Dirac theory of the positron*, 1934

"Halpern and Debye have already independently drawn attention to the fact that the Dirac theory of the positron leads to the scattering of light by light - even when the energy of the photons is not sufficient to create pairs."

Euler & Kockel: *The scattering of light by light in Dirac's theory*, 1934

"The connection between the quantities B and D, on the one hand, and E and H, on the other, is therefore nonlinear in this theory, since the scattering of light implies a deviation from the superposition principle."

61. Hans Euler (1909–1941),
Assistent bei Werner Heisenberg,
Leipzig Mai/Juni 1936



Die Philosophische Fakultät der Universität Leipzig ernennt durch diese Urkunde Herrn Hans Euler, geboren in Meran, auf Grund seiner im Druck erschienenen sehr guten Dissertation „Über die Streuung von Licht an Licht nach der Diracschen Theorie" und der mit sehr gutem Erfolge bestandenen mündlichen Prüfung zum Doktor der Philosophie.

Leipzig, den 25. Juni 1936.

Der Dekan

i. V. Berve

"Über die Streuung von Licht an Licht nach der Diracschen Theorie," Ann. Phys. (Leipzig) **26** (1936), 398-448.

On the scattering of light by light in Dirac's theory [1])

By Hans Euler

Translated by D. H. Delphenich

effective Lagrangian:

$$L = \frac{\mathfrak{E}^2 - \mathfrak{B}^2}{2} + \frac{1}{90\pi}\frac{\hbar c}{e^2}\frac{1}{E_0^2}[(\mathfrak{E}^2 - \mathfrak{B}^2)^2 + 7(\mathfrak{E}\mathfrak{B})^2]$$

vacuum polarization:

$$\mathfrak{D} = \mathfrak{E} + \frac{1}{90\pi}\frac{\hbar c}{e^2}\frac{1}{E_0^2}[4(\mathfrak{E}^2 - \mathfrak{B}^2)\mathfrak{E} - 14(\mathfrak{E}\mathfrak{B})\mathfrak{B}]$$

$$\mathfrak{h} = \mathfrak{B} + \frac{1}{90\pi}\frac{\hbar c}{e^2}\frac{1}{E_0^2}[4(\mathfrak{E}^2 - \mathfrak{B}^2)\mathfrak{B} - 14(\mathfrak{E}\mathfrak{B})\mathfrak{E}]$$

$E_0 = \dfrac{e}{\left(\dfrac{e^2}{mc^2}\right)^2}$ = "field strength at the electron radius."

"Über die Streuung von Licht an Licht nach der Diracschen Theorie," Naturwiss. **23** (1935), 246-247.

The scattering of light by light in Dirac's theory

By H. Euler and B. Köckel

Translated by D. H. Delphenich

The experimental test of the deviation from the MAXWELL theory is difficult since the noteworthy effects are extraordinarily small. The interaction cross-section for the scattering of light by light with the mean wavelength λ is, from (1), of the order of magnitude:

$$Q \sim \left(\frac{e^2}{mc^2}\right)^4 \left(\frac{\hbar}{mc}\right)^4 \cdot \frac{1}{\lambda^2},$$

in DIRAC's theory, hence, about 10^{-28} cm^2 for γ-rays and 10^{-76} cm^2 for visible light.

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It is interesting to compare this supplementary term to the MAXWELL energy, which arises from the quantum-mechanical possibility of pair creation, with the one that BORN[1] obtained in the framework of classical theory, and the first term in its development is:

$$\frac{(1.2361)^4}{32\pi} \cdot \frac{1}{E_0^2} \int [(\mathfrak{B}^2 - \mathfrak{D}^2)^2 + 4(\mathfrak{B}\mathfrak{D})^2]\, dV \,.$$

[1] M. BORN, Proc. Roy. Soc. Lond., M. BORN and L. INFELD, ibid., A143 (1933), 410, A144, 423; A147 (1934), 522.

Scattering of Light by Light

In a recent paper[1] Euler and Kockel have calculated the effective cross-section for the scattering of light by light. The calculation was carried out for the case of small frequencies ($\hbar\omega << mc^2$), the frequencies being taken in a frame of reference, where the total momentum of the colliding quanta vanishes.

We have calculated the cross-section for the opposite case of large frequencies ($\hbar\omega >> mc^2$). For the integral cross-section we get an expression of the form:

$$\sigma = a\,\alpha^4 \left(\frac{c}{\omega}\right)^2,$$

where $\alpha = \frac{e^2}{\hbar c}$ with a constant a, which is difficult to compute. According to Euler and Kockel, for small frequencies σ is proportional to ω^6. Consequently σ has a maximum value in a region $\hbar\omega \sim mc^2$.

It is also difficult to compute the dependence of the differential cross-section on the angle of scattering. We find that for the small angles the polarization of the light quanta is not altered. The differential cross-section for small angles is

$$d\sigma = 8\pi\,\alpha^4 \left(\frac{c}{\omega}\right)^2 \log^4 \Theta\, d\theta,$$

Θ being the angle of scattering and $d\theta$ the solid angle. This formula is valid for small angles, but not essentially small compared with $mc^2/\hbar\omega$. In the latter case it is necessary to insert into the logarithm $mc^2/\hbar\omega$ in place of Θ.

The formula has a relative accurateness of $1/\log\Theta$. The cross-section increases with decreasing angles, but not very rapidly, and it is impossible to affirm that this region plays the main role in the integral cross-section.

The detailed calculations will appear elsewhere.

A. Akhieser.
L. Landau.
I. Pomeranchook.

Ukrainian Physico-Technical Institute,
Kharkov, U.S.S.R.

[1] *Naturwiss.*, 23, 246 (1935).

high frequency limit

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$|\mathfrak{E}_k| = \frac{m^2 c^3}{e\hbar} = \frac{1}{„137“}\,\frac{e}{(e^2/mc^2)^2}$ = „Kritische Feldstärke“.)

what did Heisenberg & Euler actually do?

vacuum polarization due to slowly varying [constant] fields

$$\mathfrak{D} = \mathfrak{E} + 4\pi\mathfrak{P}, \qquad \mathfrak{H} = \mathfrak{B} - 4\pi\mathfrak{M},$$

$$\mathfrak{D}_i = \frac{\partial\mathfrak{L}}{\partial\mathfrak{E}_i}, \quad \mathfrak{H}_i = -\frac{\partial\mathfrak{L}}{\partial\mathfrak{B}_i},$$

$$U(\mathfrak{E},\mathfrak{B}) = \frac{1}{4\pi}\Big[\sum_i \mathfrak{D}_i\mathfrak{E}_i - \mathfrak{L}\Big] = \frac{1}{4\pi}\Big(\sum_i \mathfrak{E}_i\frac{\partial\mathfrak{L}}{\partial\mathfrak{E}_i} - \mathfrak{L}\Big)$$

"Due to relativistic invariance, the Lagrangian can only depend on the two invariants E^2-B^2 and (E.B). The calculation of U(E, B) can be reduced to the question of how much energy density is associated with the matter fields in a background of constant fields E and B."

Linear Modifications in the Maxwell Field Equations

ROBERT SERBER,* *University of California, Berkeley*

(Received April 24, 1935)

Expressions, accurate to the first order in e^2, are obtained for the charge and current densities which, according to positron theory, are induced in vacuum by an electromagnetic field. Because the corresponding correction terms in the Maxwell field equations involve integral operators, it does not seem possible to treat the modified field equations by Hamiltonian methods.

$$\operatorname{div} \mathbf{E}(\mathbf{s}') - \int \Lambda(s)\square\square \operatorname{div} \mathbf{E}(\mathbf{s}'')ds'' = 4\pi j_0(\mathbf{s}'), \quad (20)$$

JULY 1, 1935 PHYSICAL REVIEW VOLUME 48

Polarization Effects in the Positron Theory

E. A. UEHLING,* *University of California*

(Received April 24, 1935)

Some of the consequences of the positron theory for the special case of impressed electrostatic fields are investigated. By imposing a restriction only on the maximum value of the field intensity, which must always be assumed much smaller than a certain critical value, but with no restrictions on the variation of this intensity, a formula for the charge induced by a charge distribution is obtained. The existence of an induced charge corresponds to a polarization of the vacuum, and as a consequence, to deviations from Coulomb's law for the mutual potential energy of point charges. Consequences of these deviations which are investigated are the departures from the Coulombian scattering law for heavy particles and the displacement in the energy levels for atomic electrons moving in the field of the nucleus.

One thus expects that along with the Maxwellian energy of the individual light quanta there is a mutual interaction between the light quanta of the form:

$$\bar{U}_1 = \frac{\hbar c}{e^2} \frac{1}{E_0^2} \int \left[FFFF + \left(\frac{\hbar}{mc} \frac{\partial}{\partial x} F \right) \left(\frac{\hbar}{mc} \frac{\partial}{\partial x} F \right) FF + \cdots \right] dV \ . \tag{1.3}$$

H. Euler, thesis 1936

Euler-Heisenberg and Casimir: two sides of a coin

classical electromagnetic field

quantized virtual e+e-

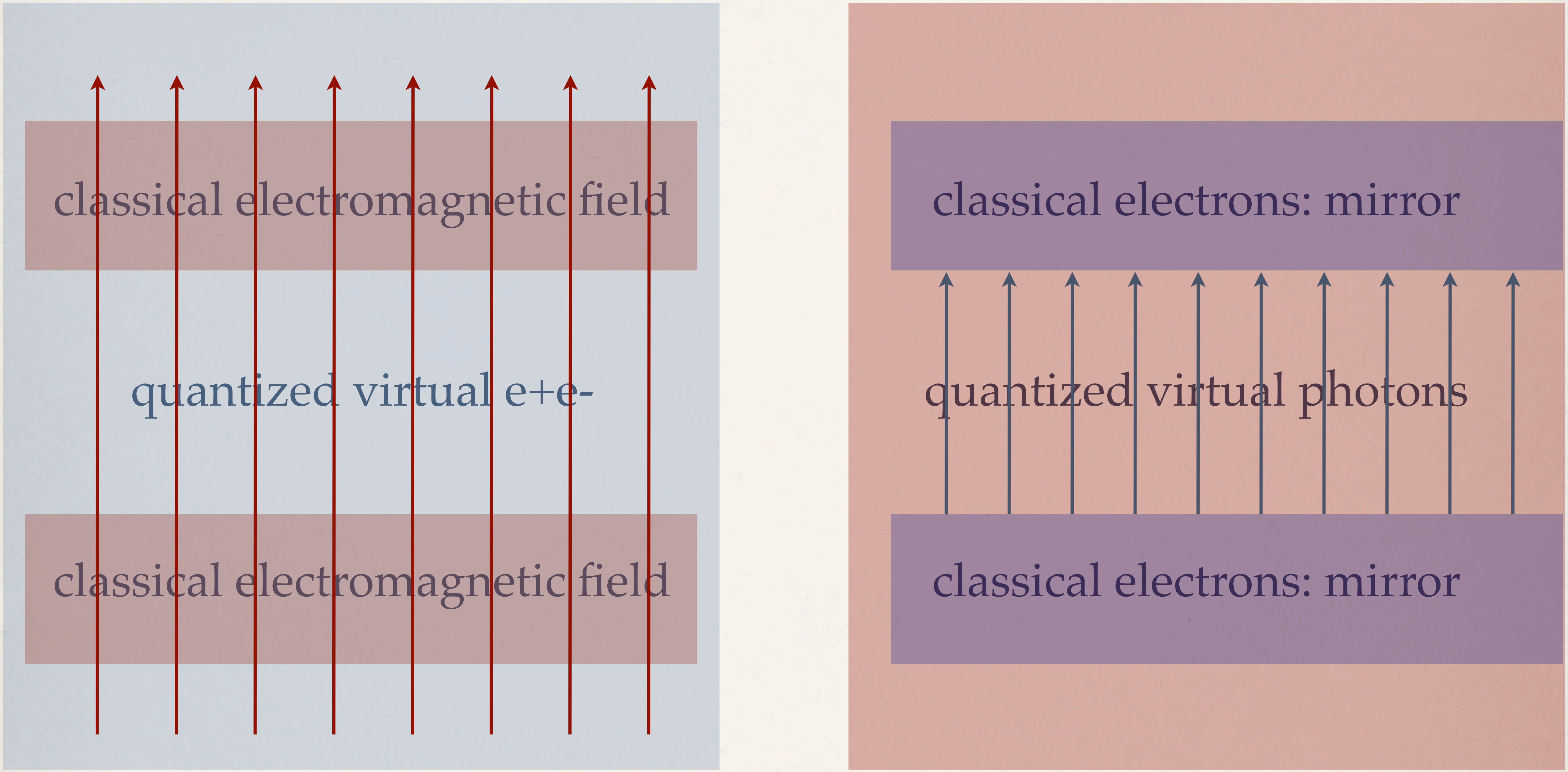
classical electromagnetic field

classical electrons: mirror

quantized virtual photons

classical electrons: mirror

Euler-Heisenberg and Casimir: two sides of a coin



what did Heisenberg & Euler actually do?

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"In the presence of only a magnetic field, the stationary states can be divided into those of negative and positive energy.
... The situation is different in an electric field.
... This difficulty is physically related to the fact that in an electric field, pairs of positrons and electrons are created. The exact analysis of this problem was performed by Sauter."

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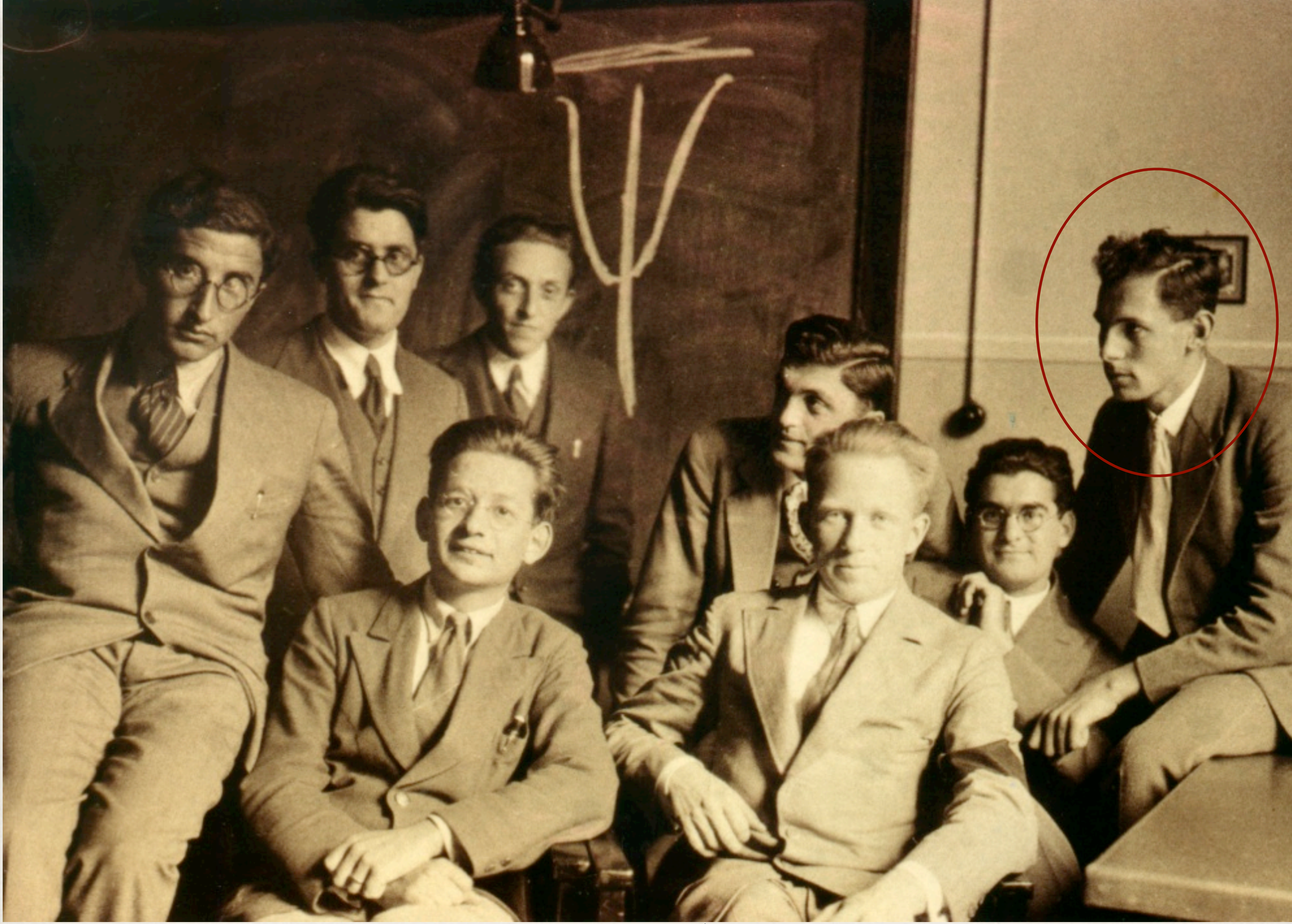
On the behavior of an electron in a homogeneous electric field in Dirac's relativistic theory

By **Fritz Sauter** in Munich

Translated by D. H. Delphenich



rear: ?, Placzek, Wick, Bloch, Weisskopf, Sauter
front: Peierls, Heisenberg; 1930/31, Leipzig

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The solutions of the Dirac equation with the potential $V = vx$ will be obtained and their behavior will be discussed. Along with the region of the function that also appears in the non-relativistic calculations, there is a region in the Dirac theory in which the impulse and velocity of the electron possess opposite signs. In conjunction with that, the probability will be computed for an electron to go from the “positive impulse” region to the “negative impulse” region. This yields the result that transition probability first takes on finite values when the magnitude of the potential ramp over a distance that is equal to the Compton wavelength is comparable to the rest energy of the electron. The large values for the transition probability that were computed by O. Klein for a potential well whose order of magnitude is twice the rest energy are understood to be limiting values in the case of an infinitely steep potential ramp.

$$D = e^{-k^2\pi} \qquad k^2 = \frac{2\pi}{hc}\frac{(mc^2)^2}{v} \sim 1 \qquad \frac{vh}{mc} \sim mc^2$$

“This case would correspond to around 10^16 volt/cm.”

This agrees with the conjecture of N. Bohr that was given in the introduction, that one first obtains the finite probability for the transition of an electron into the region of negative impulse when the potential ramp *vh*/*mc* over a distance of the Compton wavelength *h*/*mc* has the order of magnitude of the rest energy.

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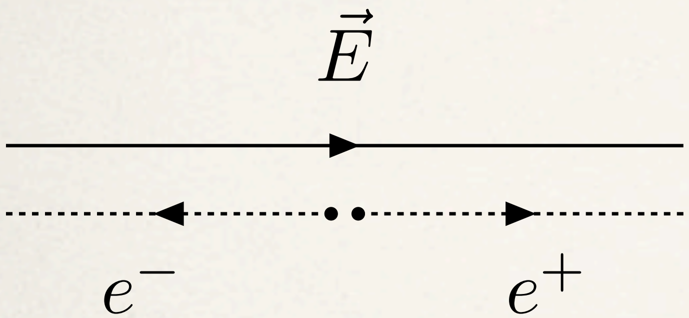

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$$2\,e\,\mathcal{E}\,\frac{\hbar}{m\,c} \sim 2\,m\,c^2$$

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what did Heisenberg & Euler actually do?

electric field: tunnelling from Dirac sea

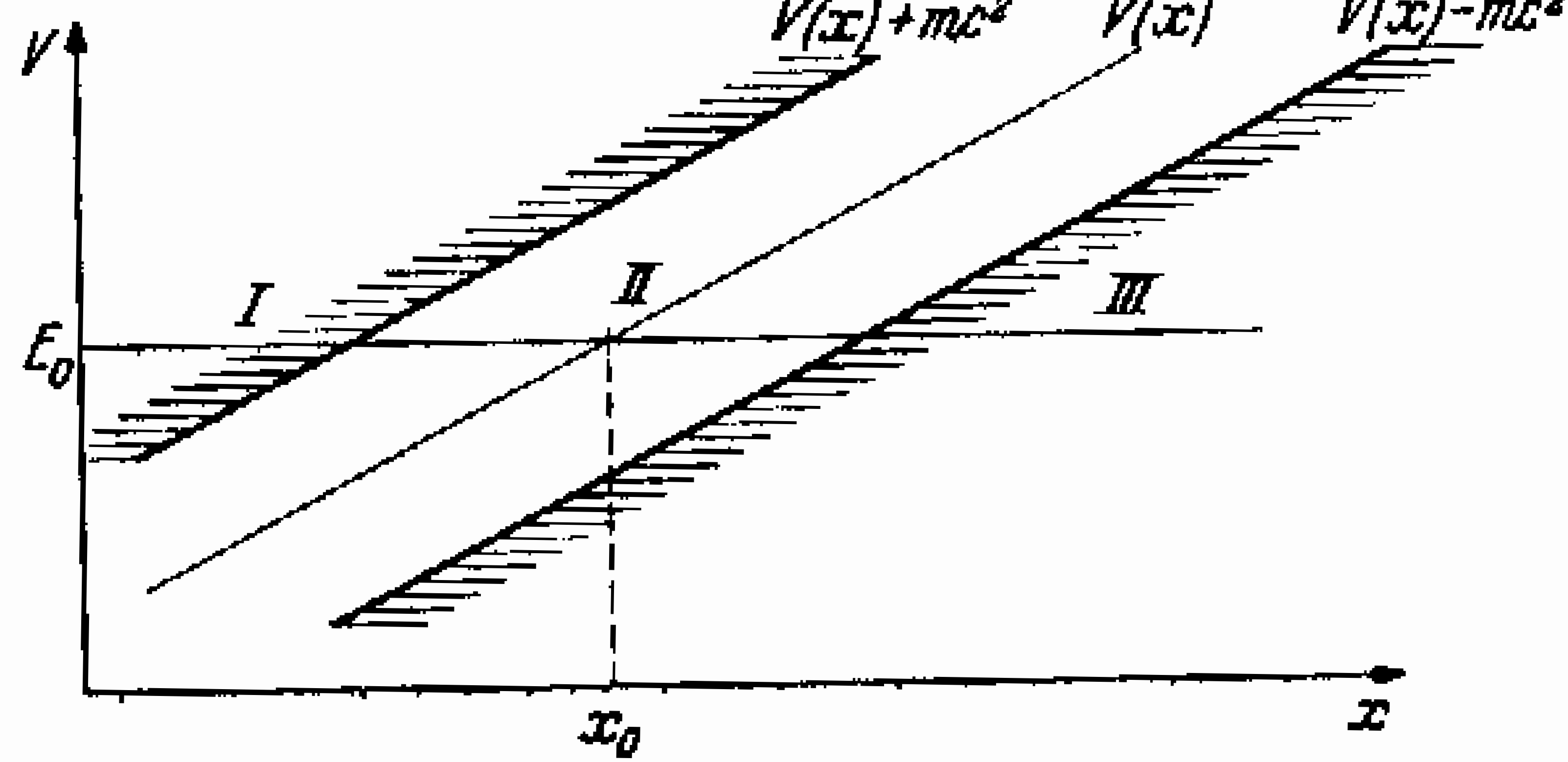

Fig. 1.

"a wavefunction that begins large in region I decreases slowly in region III, where the transmission coefficient through region II (which plays the role of a Gamow-wall) calculated by Sauter has the order of magnitude $e^{-\frac{m^2c^3}{\hbar e|\mathcal{E}|}\pi}$."

what did Heisenberg & Euler actually do?

Dirac equation for constant EM fields:
harmonic oscillators; parabolic cylinder functions; Euler-Maclaurin summation; integral representations ...

$$\mathfrak{L} = \frac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2) + 4\pi^2 m c^2 \left(\frac{m c}{h}\right)^3 \int_0^\infty e^{-\eta} \frac{d\eta}{\eta^3} \left\{ -a\eta \operatorname{ctg} a\eta \cdot b\eta \operatorname{Ctg} b\eta + 1 + \frac{\eta^2}{3}(b^2 - a^2) \right\}$$

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"The integral around the pole η=pi/a has the value

$$-\frac{2i}{\pi} \cdot 4a^2 mc^2 \left(\frac{mc}{h}\right)^3 \cdot e^{-\pi/a}$$

(for b=0). This is the order of the terms which are associated with the pair creation in an electric field."

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Dirac equation for constant EM fields:
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$$\mathfrak{L} \approx \frac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2) + 16\,\pi^2\, m\, c^2 \left(\frac{m\,c}{h}\right)^3 \Big[\frac{(a^2 - b^2)^2 + 7\,(a\,b)^2}{180} + \frac{13\,(a\,b)^2\,(a^2 - b^2) + 2\,(a^2 - b^2)^3}{630} \cdots \Big].$$

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$$\mathfrak{L} = \tfrac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2) + \frac{e^2}{h\,c}\int_0^\infty e^{-\eta}\,\frac{d\eta}{\eta^3}\left\{ i\,\eta^2\,(\mathfrak{E}\,\mathfrak{B})\,\frac{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2 - \mathfrak{B}^2 + 2\,i\,(\mathfrak{E}\,\mathfrak{B})}\right) + \text{konj.}}{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2 - \mathfrak{B}^2 + 2\,i\,(\mathfrak{E}\,\mathfrak{B})}\right) - \text{konj.}} + |\mathfrak{E}_k|^2 + \frac{\eta^2}{3}(\mathfrak{B}^2 - \mathfrak{E}^2)\right\}. \quad (45\,\text{a})$$

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field-free subtraction

charge renormalization term

V. Weisskopf, *The electrodynamics of the vacuum on the basis of the quantum theory of the electron*, 1936

$$D_i = \sum_k \epsilon_{ik} E_k \qquad , \qquad H_i = \sum_k \mu_{ik} B_k$$

$$\epsilon_{ik} = \delta_{ik} + \frac{e^4 \hbar}{45 \pi m^4 c^7} \left[2 \left(\vec{E}^2 - \vec{B}^2 \right) \delta_{ik} + 7 B_i B_k \right] \qquad \mu_{ik} = \delta_{ik} + \frac{e^4 \hbar}{45 \pi m^4 c^7} \left[2 \left(\vec{E}^2 - \vec{B}^2 \right) \delta_{ik} - 7 E_i E_k \right]$$

"the electromagnetic properties of the vacuum can be described by a field-dependent electric and magnetic polarisability of empty space, which leads, for example, to refraction of light in electric fields or to a scattering of light by light."

V. Weisskopf, *The electrodynamics of the vacuum on the basis of the quantum theory of the electron*, 1936

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"the electromagnetic properties of the vacuum can be described by a field-dependent electric and magnetic polarisability of empty space, which leads, for example, to refraction of light in electric fields or to a scattering of light by light."

$$U' = \frac{1}{8\pi^2} mc^2 \left(\frac{mc}{\hbar} \right)^3 \int_0^\infty \frac{d\eta}{\eta^3} e^{-\eta} \left\{ \eta \mathcal{B} \, \mathrm{Ctg} \, \eta \mathcal{B} - 1 - \frac{\eta^2}{3} \mathcal{B}^2 \right\}$$

spinor QED

$$U'_{\mathrm{skal}} = -\frac{1}{16\pi^2} mc^2 \left(\frac{mc}{\hbar} \right)^3 \int_0^\infty \frac{d\eta}{\eta^3} e^{-\eta} \left\{ \eta \mathcal{B} \frac{1}{\mathrm{Sin} \, \eta \mathcal{B}} - 1 + \frac{\eta^2}{6} \mathcal{B}^2 \right\}$$

scalar QED

the proper-time formalism

PHYSICAL REVIEW VOLUME 76, NUMBER 6 SEPTEMBER 15, 1949

The Theory of Positrons

R. P. FEYNMAN
Department of Physics, Cornell University, Ithaca, New York

(Received April 8, 1949)

The problem of the behavior of positrons and electrons in given external potentials, neglecting their mutual interaction, is analyzed by replacing the theory of holes by a reinterpretation of the solutions of the Dirac equation. It is possible to write down a complete solution of the problem in terms of boundary conditions on the wave function, and this solution contains automatically all the possibilities of virtual (and real) pair formation and annihilation together with the ordinary scattering processes, including the correct relative signs of the various terms.

In this solution, the "negative energy states" appear in a form which may be pictured (as by Stückelberg) in space-time as waves traveling away from the external potential backwards in time. Experimentally, such a wave corresponds to a positron approaching the potential and annihilating the electron. A particle moving forward in time (electron) in a potential may be scattered forward in time (ordinary scattering) or backward (pair annihilation). When moving backward (positron) it may be scattered backward in time (positron scattering) or forward (pair production). For such a particle the amplitude for transition from an initial to a final state is analyzed to any order in the potential by considering it to undergo a sequence of such scatterings.

The amplitude for a process involving many such particles is the product of the transition amplitudes for each particle. The exclusion principle requires that antisymmetric combinations of amplitudes be chosen for those complete processes which differ only by exchange of particles. It seems that a consistent interpretation is only possible if the exclusion principle is adopted. The exclusion principle need not be taken into account in intermediate states. Vacuum problems do not arise for charges which do not interact with one another, but these are analyzed nevertheless in anticipation of application to quantum electrodynamics.

The results are also expressed in momentum-energy variables. Equivalence to the second quantization theory of holes is proved in an appendix.

PHYSICAL REVIEW VOLUME 80, NUMBER 3 NOVEMBER 1, 1950

Mathematical Formulation of the Quantum Theory of Electromagnetic Interaction

R. P. FEYNMAN*
Department of Physics, Cornell University, Ithaca, New York

(Received June 8, 1950)

The validity of the rules given in previous papers for the solution of problems in quantum electrodynamics is established. Starting with Fermi's formulation of the field as a set of harmonic oscillators, the effect of the oscillators is integrated out in the Lagrangian form of quantum mechanics. There results an expression for the effect of all virtual photons valid to all orders in $e^2/\hbar c$. It is shown that evaluation of this expression as a power series in $e^2/\hbar c$ gives just the terms expected by the aforementioned rules.

In addition, a relation is established between the amplitude for a given process in an arbitrary unquantized potential and in a quantum electrodynamical field. This relation permits a simple general statement of the laws of quantum electrodynamics.

A description, in Lagrangian quantum-mechanical form, of particles satisfying the Klein-Gordon equation is given in an Appendix. It involves the use of an extra parameter analogous to proper time to describe the trajectory of the particle in four dimensions.

Feynman's worldline representation

"We try to represent the amplitude for a particle to get from one point to another as a sum over all trajectories of an amplitude exp(i S) where S is the classical action for a given trajectory. To maintain the relativistic invariance in evidence the idea suggests itself of describing a trajectory in space-time by giving the four variables x_mu(u) as functions of some fifth parameter u … (somewhat analogous to proper time) …"

$$i\partial\varphi/\partial u = -\tfrac{1}{2}(i\partial/\partial x_\mu - A_\mu)^2\varphi$$

[19] The physical ideas involved in such a description are discussed in detail by Y. Nambu, Prog. Theor. Phys. **5**, 82 (1950). An equation of type (2A) extended to the case of Dirac electrons has been studied by V. Fock, Physik Zeits. Sowjetunion **12**, 404 (1937).

$$\varphi(x_{\nu,n}, u_0) = \int \exp -\frac{i\epsilon}{2}\sum_{i=1}^{n}\Big[\Big(\frac{x_{\mu,i}-x_{\mu,i-1}}{\epsilon}\Big)^2 + \epsilon^{-1}(x_{\mu,i}-x_{\mu,i-1})(A_\mu(x_i)+A_\mu(x_{i-1}))\Big] \cdot \varphi(x_{\nu,0}, 0) \prod_{i=0}^{n-1}(d^4\tau_i/4\pi^2\epsilon^2 i). \quad (4A)$$

That is, roughly, the amplitude for getting from one point to another with a given value of u_0 is the sum over all trajectories of $\exp(iS)$ where

$$S = -\int_0^{u_0}[\tfrac{1}{2}(dx_\mu/du)^2 + (dx_\mu/du)A_\mu(x)]du, \quad (5A)$$

Progress of Theoretical Physics, Vol. V, No. 1, Jan.~Feb., 1950.

The Use of the Proper Time in Quantum Electrodynamics I.

Yôichirô NAMBU

*Department of Physics, University of City Osaka**

(Received November 8, 1949)

whole of it at glance. The time itself loses sense as the indicator of the development of phenomena; there are particles which flow down as well as up the stream of time; the eventual creation and annihilation of pairs that may occur now and then, is no creation nor annihilation, but only a change of directions of moving particles, from past to future, or from future to past; a virtual pair, which, according to the ordinary view, is foredoomed to exist only for a limited interval of time, may also be regarded as a single particle that is circulating round a closed orbit in the four-dimensional theatre; a real particle is then a particle whose orbit is not closed but reaches to infinity. . .

Worldline QED effective action

PHYSICAL REVIEW VOLUME 80, NUMBER 3 NOVEMBER 1, 1950

Mathematical Formulation of the Quantum Theory of Electromagnetic Interaction

R. P. FEYNMAN*
Department of Physics, Cornell University, Ithaca, New York

(Received June 8, 1950)

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effective action expressed as a 4d QM path integral

$$\Gamma[A] = -\int_0^\infty \frac{dT}{T}\, e^{-m^2 T} \int d^4x \int_{x(T)=x(0)=x} \mathcal{D}x \; e^{-S[x]}$$

$$S[x] \equiv \int_0^T d\tau \left(\frac{1}{4}\dot{x}_\mu^2 + e A_\mu \dot{x}_\mu \right)$$

τ: proper time

PHYSICAL REVIEW VOLUME 82, NUMBER 5 JUNE 1, 1951

On Gauge Invariance and Vacuum Polarization

JULIAN SCHWINGER
Harvard University, Cambridge, Massachusetts

(Received December 22, 1950)

This paper is based on the elementary remark that the extraction of gauge invariant results from a formally gauge invariant theory is ensured if one employs methods of solution that involve only gauge covariant quantities. We illustrate this statement in connection with the problem of vacuum polarization by a prescribed electromagnetic field. The vacuum current of a charged Dirac field, which can be expressed in terms of the Green's function of that field, implies an addition to the action integral of the electromagnetic field. Now these quantities can be related to the dynamical properties of a "particle" with space-time coordinates that depend upon a proper-time parameter. The proper-time equations of motion involve only electromagnetic field strengths, and provide a suitable gauge invariant basis for treating problems. Rigorous solutions of the equations of motion can be obtained for a constant field, and for a plane wave field. A renormalization of field strength and charge, applied to the modified lagrange function for constant fields, yields a finite, gauge invariant result which implies nonlinear properties for the electromagnetic field in the vacuum. The contribution of a zero spin charged field is also stated. After the same field strength renormalization, the modified physical quantities describing a plane wave in the vacuum reduce to just those of the maxwell field; there are no nonlinear phenomena for a single plane wave, of arbitrary strength and spectral composition. The results obtained for constant (that is, slowly varying fields), are then applied to treat the two-photon disintegration of a spin zero neutral meson arising from the polarization of the proton vacuum. We obtain approximate, gauge invariant expressions for the effective interaction between the meson and the electromagnetic field, in which the nuclear coupling may be scalar, pseudoscalar, or pseudovector in nature. The direct verification of equivalence between the pseudoscalar and pseudovector interactions only requires a proper statement of the limiting processes involved. For arbitrarily varying fields, perturbation methods can be applied to the equations of motion, as discussed in Appendix A, or one can employ an expansion in powers of the potential vector. The latter automatically yields gauge invariant results, provided only that the proper-time integration is reserved to the last. This indicates that the significant aspect of the proper-time method is its isolation of divergences in integrals with respect to the proper-time parameter, which is independent of the coordinate system and of the gauge. The connection between the proper-time method and the technique of "invariant regularization" is discussed. Incidentally, the probability of actual pair creation is obtained from the imaginary part of the electromagnetic field action integral. Finally, as an application of the Green's function for a constant field, we construct the mass operator of an electron in a weak, homogeneous external field, and derive the additional spin magnetic moment of $\alpha/2\pi$ magnetons by means of a perturbation calculation in which proper-mass plays the customary role of energy.

$$\mathfrak{L}^{(1)}(x)=\tfrac{1}{2}i\int_0^\infty ds\,s^{-1}\exp(-im^2s)\,\mathrm{tr}(x|U(s)|x),$$

$$U(s)=\exp(-i\mathfrak{H}s),\qquad \mathfrak{H}=-(\gamma\Pi)^2=\Pi_\mu{}^2-\tfrac{1}{2}e\sigma_{\mu\nu}F_{\mu\nu},$$

The latter notation emphasizes that $U(s)$ may be regarded as the operator describing the development of a system governed by the "hamiltonian," $\mathfrak{H}$, in the "time" s, the matrix element of $U(s)$ being the transformation function from a state in which $x_\mu(s=0)$ has the value x_μ'' to a state in which $x_\mu(s)$ has the value x_μ'.

constant field U(s) found by Fock (1937) and Nambu (1950)

$$\mathfrak{L}=-\mathfrak{F}-\frac{1}{8\pi^2}\int_0^\infty ds\,s^{-3}\exp(-m^2s)$$
$$\times\left[(es)^2\mathfrak{G}\frac{\mathrm{Re}\cosh esX}{\mathrm{Im}\cosh esX}-1-\tfrac{2}{3}(es)^2\mathfrak{F}\right]$$
$$=\tfrac{1}{2}(\mathbf{E}^2-\mathbf{H}^2)+\frac{2\alpha^2}{45}\frac{(\hbar/mc)^3}{mc^2}$$
$$\times[(\mathbf{E}^2-\mathbf{H}^2)^2+7(\mathbf{E}\cdot\mathbf{H})^2]+\cdots.$$

$$X^2=(\mathbf{H}+i\mathbf{E})^2.$$

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$$\times[(\mathbf{E}^2-\mathbf{H}^2)^2+7(\mathbf{E}\cdot\mathbf{H})^2]+\cdots.$$

$$\mathbf{X}^2=(\mathbf{H}+i\mathbf{E})^2.$$

plane wave Dirac eqn solved by Volkov (1935)

Thus, there are no nonlinear vacuum phenomena for a single plane wave, of arbitrary strength and spectral composition.

“Schwinger pair production”

$$\mathcal{L} = \tfrac{1}{2}\mathcal{E}^2 - (1/8\pi^2)\int_0^\infty ds\, s^{-3} \exp(-m^2 s)$$

$$\times [e\mathcal{E}s \cot(e\mathcal{E}s) - 1 + \tfrac{1}{3}(e\mathcal{E}s)^2], \quad (6.39)$$

has singularities at

$$s = s_n = n\pi/e\mathcal{E}, \quad n = 1, 2, \cdots. \quad (6.40)$$

If the integration path is considered to lie above the real axis, which is an alternative version of the device embodied in Eq. (6.32), we obtain a positive imaginary contribution to $\mathcal{L}$,

$$2\,\mathrm{Im}\mathcal{L} = \frac{1}{4\pi}\sum_{n=1}^{\infty} s_n^{-2} \exp(-m^2 s_n)$$

$$= \frac{\alpha^2}{\pi^2}\mathcal{E}^2 \sum_{n=1}^{\infty} n^{-2} \exp\left(\frac{-n\pi m^2}{e\mathcal{E}}\right). \quad (6.41)$$

This is the probability, per unit time and per unit volume, that a pair is created by the constant electric field.

Euler-Heisenberg effective action and functional determinants

PHYSICAL REVIEW VOLUME 90, NUMBER 4 MAY 15, 1593

Fredholm Theory of Scattering in a Given Time-Dependent Field

ABDUS SALAM, *St. John's College, Cambridge, England, and Government College, Lahore, Pakistan*

AND

P. T. MATTHEWS,* *Cavendish Laboratory, Cambridge, England*

(Received October 27, 1952)

It is shown that Feynman's relativistic solution for the scattering of an electron (or pair creation) by a given external field is the Fredholm resolvent of the related integral equation and is thus the unique and absolutely convergent solution for any strength of field.

PHYSICAL REVIEW VOLUME 93, NUMBER 3 FEBRUARY 1, 1954

The Theory of Quantized Fields. V

JULIAN SCHWINGER
Harvard University, Cambridge, Massachusetts

(Received October 26, 1953)

$$(0\sigma_1|0\sigma_2)=\exp(i\mathcal{W}_0).$$

$$\det(1+\lambda K)=\exp[Tr\log(1+\lambda K)]$$

$$e^{iw}=\det(1-e\gamma AG_+{}^0)$$

$$W=-i\,\mathrm{tr}\,\ln\left(1-e\gamma AG_+^0\right)$$

Euler-Heisenberg effective action and functional determinants

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Scientific legacy of Heisenberg & Euler’s paper

- pair production from vacuum
- light-light scattering
- vacuum polarization physics
- effective field theory
- gravitational effective actions
- zeta functions

Scientific legacy of Heisenberg & Euler’s paper

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beyond uniform fields

this is the analogue of probing geometric and temperature effects in Casimir physics

SOVIET PHYSICS JETP VOLUME 20, NUMBER 5 MAY, 1965

IONIZATION IN THE FIELD OF A STRONG ELECTROMAGNETIC WAVE

L. V. KELDYSH

P. N. Lebedev Physics Institute, Academy of Sciences, U.S.S.R.

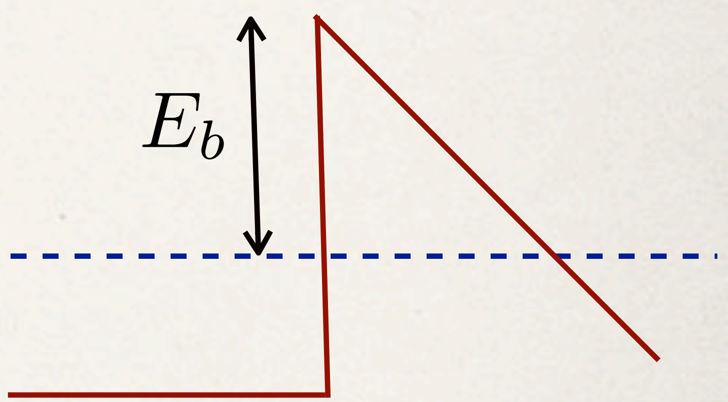
$$\mathcal{E}(t) = \mathcal{E}\,\cos(\omega\,t)$$

Submitted to JETP editor May 23, 1964

J. Exptl. Theoret. Phys. (U.S.S.R.) 47, 1945-1957 (November, 1964)

Expressions are obtained for the probability of ionization of atoms and solid bodies in the field of a strong electromagnetic wave whose frequency is lower than the ionization potential. In the limiting case of low frequencies these expressions change into the well known formulas for the probability of tunnel auto-ionization; at high frequencies they describe processes in which several photons are absorbed simultaneously. The ionization probability has a number

adiabaticity parameter : $\gamma \equiv \dfrac{\omega}{\omega_t} = \dfrac{\omega\sqrt{2mE_b}}{e\mathcal{E}}$



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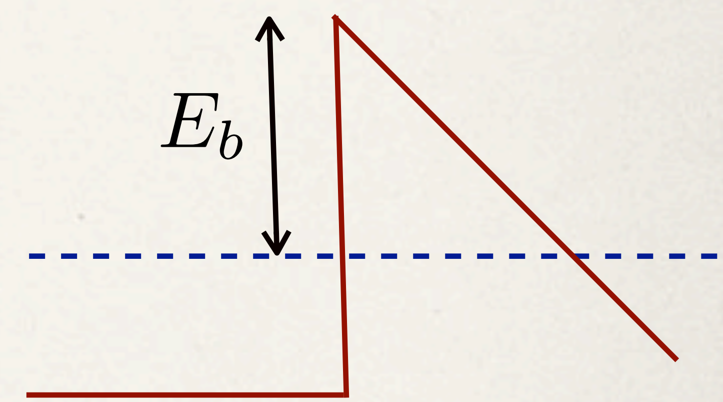

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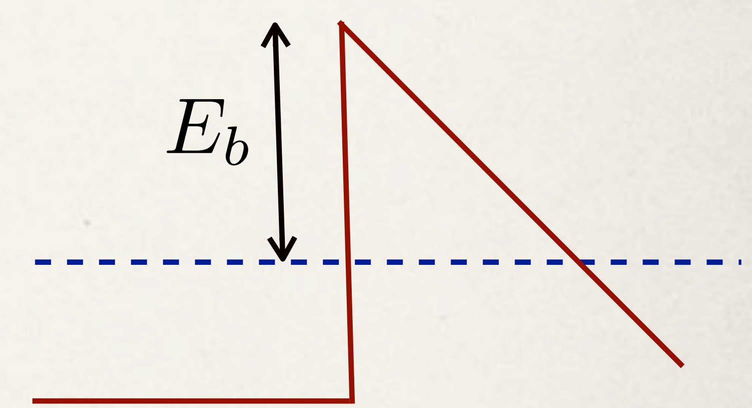

$\omega \ll \omega_t$ "instantaneous"; non-perturbative; tunnelling

$\omega \gg \omega_t$ "no time to tunnel"; perturbative; multi-photon

PHYSICAL REVIEW D

PARTICLES AND FIELDS

THIRD SERIES, VOL. 2, NO. 7 1 OCTOBER 1970

$$\gamma = \frac{m\omega}{eE}$$

Pair Production in Vacuum by an Alternating Field

E. BREZIN AND C. ITZYKSON

Service de Physique Théorique, Centre d'Etudes Nucléaires de Saclay, 91, Gif-sur-Yvette, France

(Received 15 April 1970)

We discuss the creation of pairs of charged particles in an alternating electric field. The dependence on the frequency is computed and found negligible. We obtain a formula for the field intensities required in order to observe the effect $E \gtrsim m\omega_0 c/e \sinh(\hbar\omega_0/4mc^2)$.

$$\gamma \ll 1, \quad w \simeq \frac{\alpha E^2}{2\pi} \exp\left(-\frac{\pi m^2}{eE}\right)$$

$$\gamma \gg 1, \quad w \simeq \frac{\alpha E^2}{8}\left(\frac{eE}{2m\omega_0}\right)^{4m/\omega_0}.$$

SOVIET PHYSICS JETP VOLUME 35, NUMBER 4 OCTOBER, 1972

Pair Production in a Variable and Homogeneous Electric Field as an Oscillator Problem

V. S. Popov

Institute of Theoretical and Experimental Physics

Submitted November 2, 1971

Zh. Eksp. Teor. Fiz. 62, 1248–1262 (April, 1972)

The calculation of the probability w of pair production in a strong electric field E(t) which is homogeneous in space reduces to the problem of the parametric excitation of a quantum oscillator (or, what is equivalent, to the calculation of the reflection coefficient for a plane wave incident on a one-dimensional potential barrier). Because of this relation, the calculation of the pair-production probability does not require the determination of the exact solutions of the relativistic wave equations and the Green's function. In the particular case when $E(t) = E(\cosh\omega t)^{-2}$ the probability w coincides with the coefficient of reflection from the Eckart potential. The relation between the exact formulas for w and the quasiclassical approximation (the "imaginary time" method) is investigated in detail, and the limits of applicability of the quasiclassical approximation are indicated.

Positron Production in Multiphoton Light-by-Light Scattering

D. L. Burke, R. C. Field, G. Horton-Smith, J. E. Spencer, and D. Walz
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94309

S. C. Berridge, W. M. Bugg, K. Shmakov, and A. W. Weidemann
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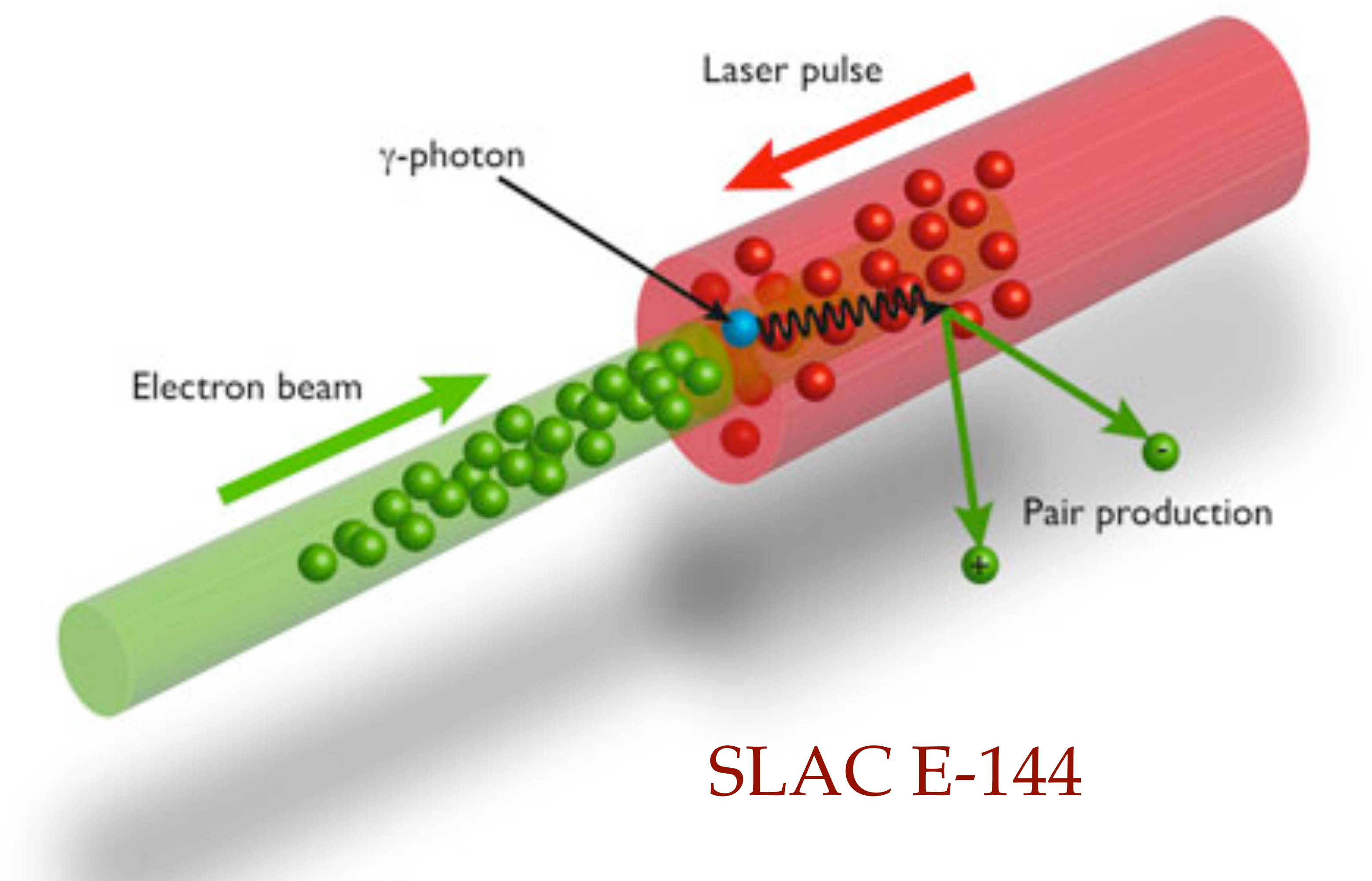
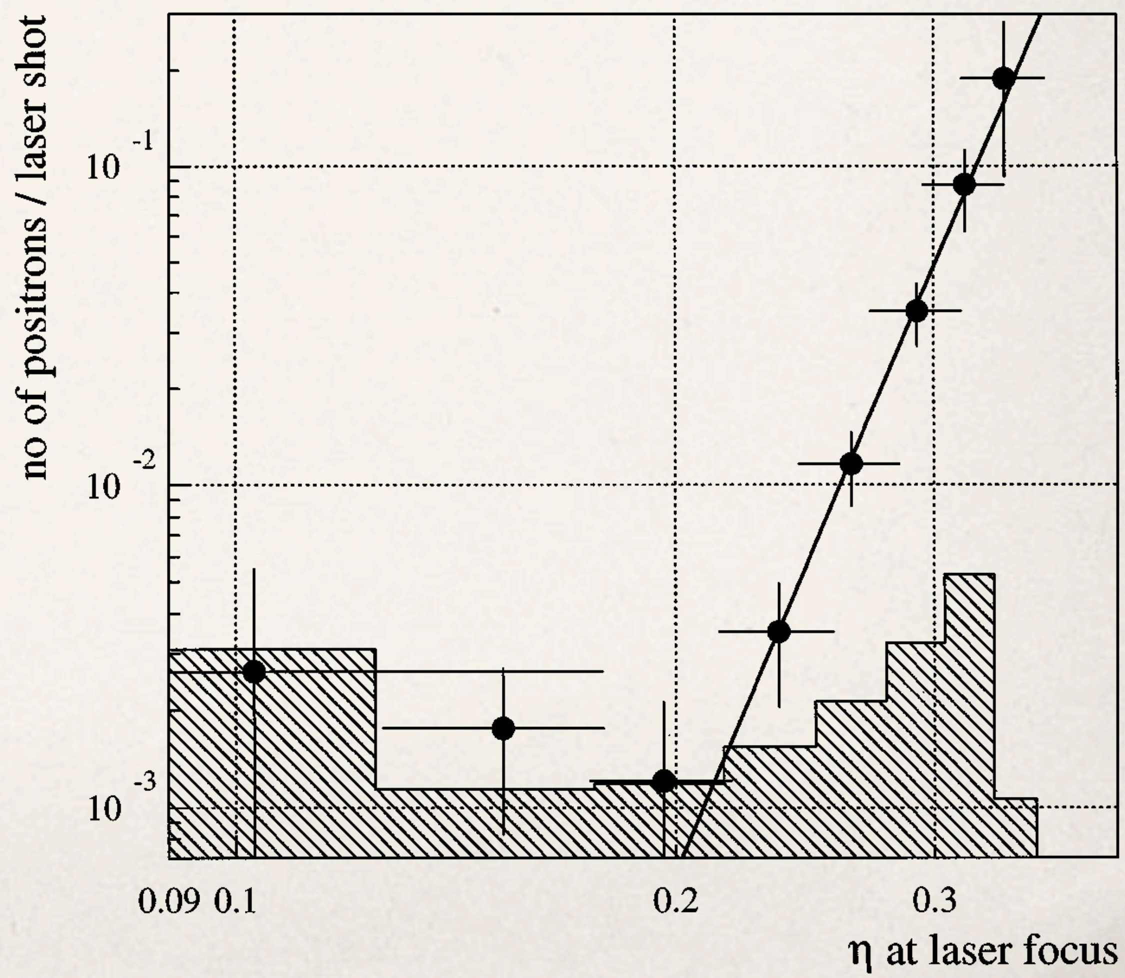
Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627
(Received 2 June 1997)





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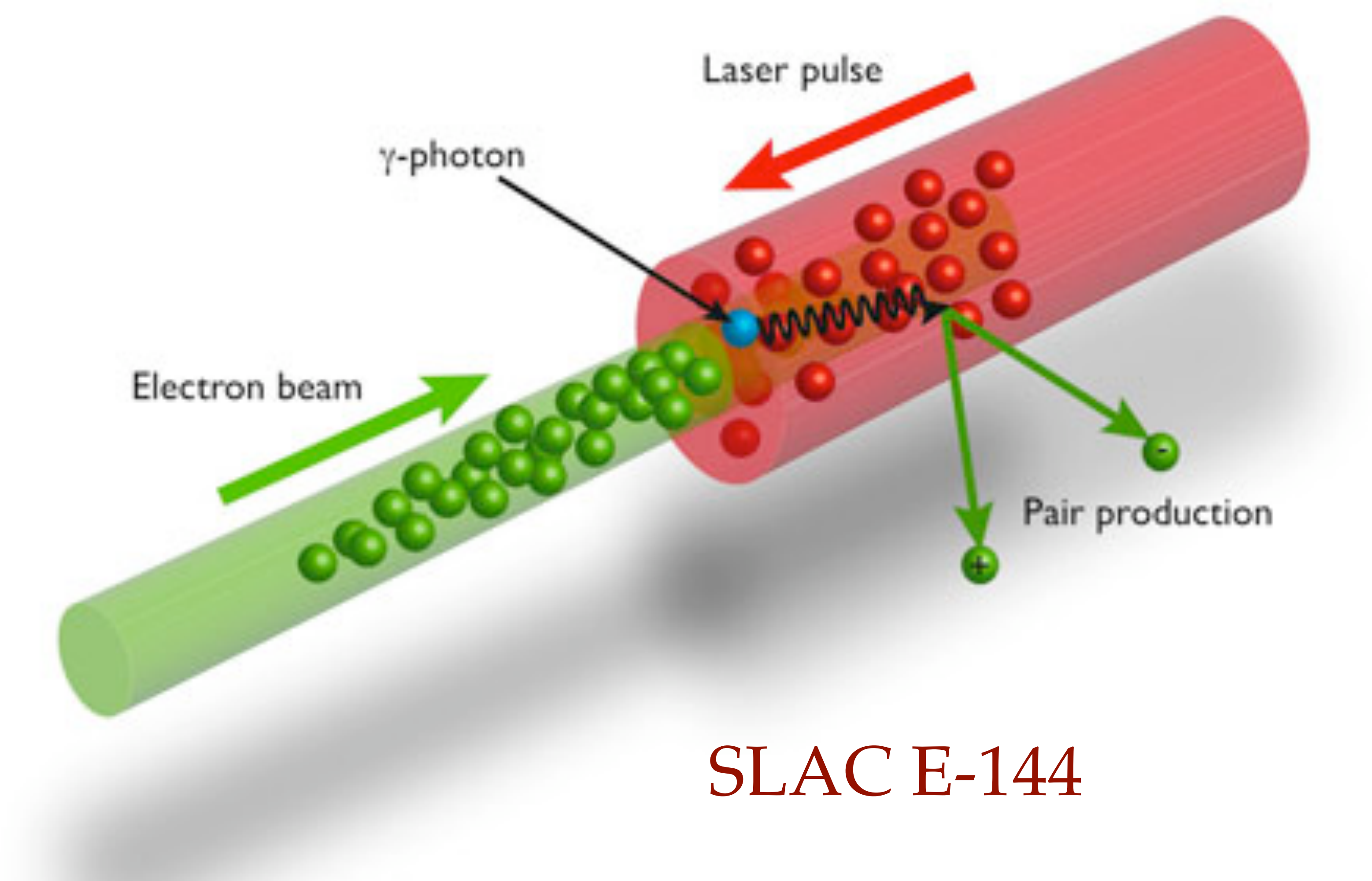
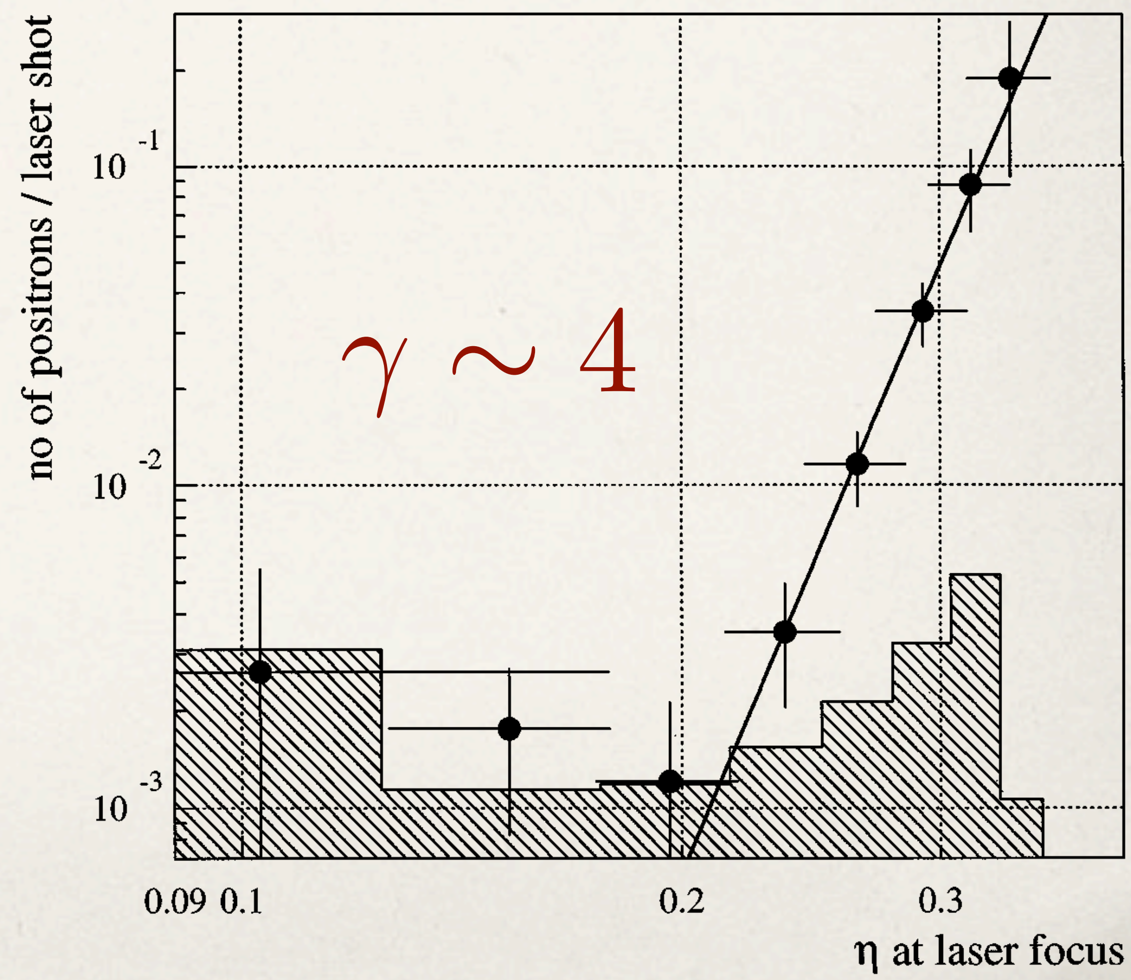


SLAC E-144



$\gamma \sim 4$

experimental goal

probe pair production from vacuum
in the nonperturbative regime $\gamma \ll 1$

SLAC E-144: $\gamma \sim 4$

XFEL: $\gamma \sim 0.1$

HiPER: $\gamma \sim 0.02$

ELI: $\gamma \sim 0.0002$

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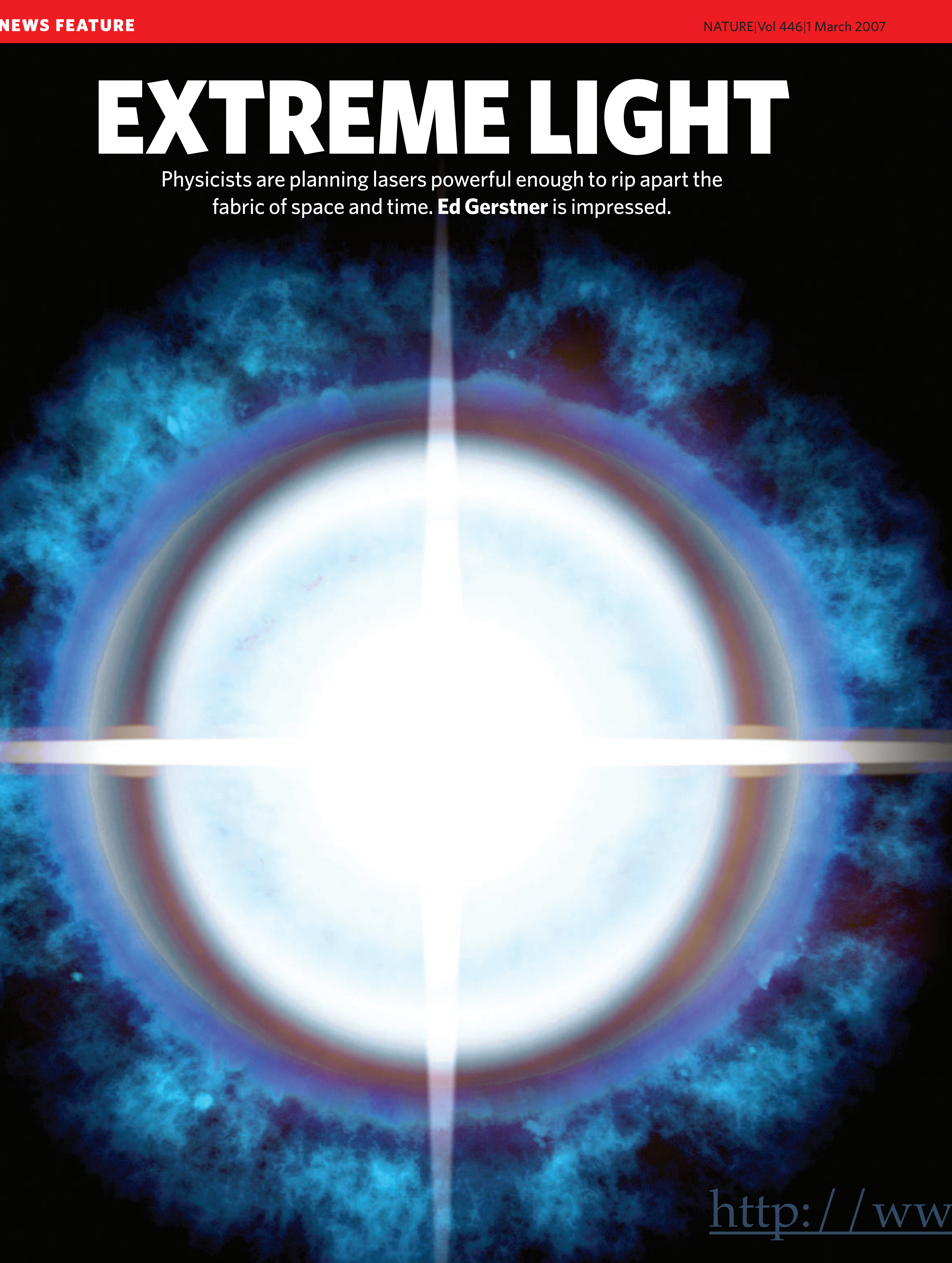
ELI: $\gamma \sim 0.0002$

theoretical goal

reliable computation of vacuum pair production rate
and spectrum in realistic short-pulse focussed laser fields

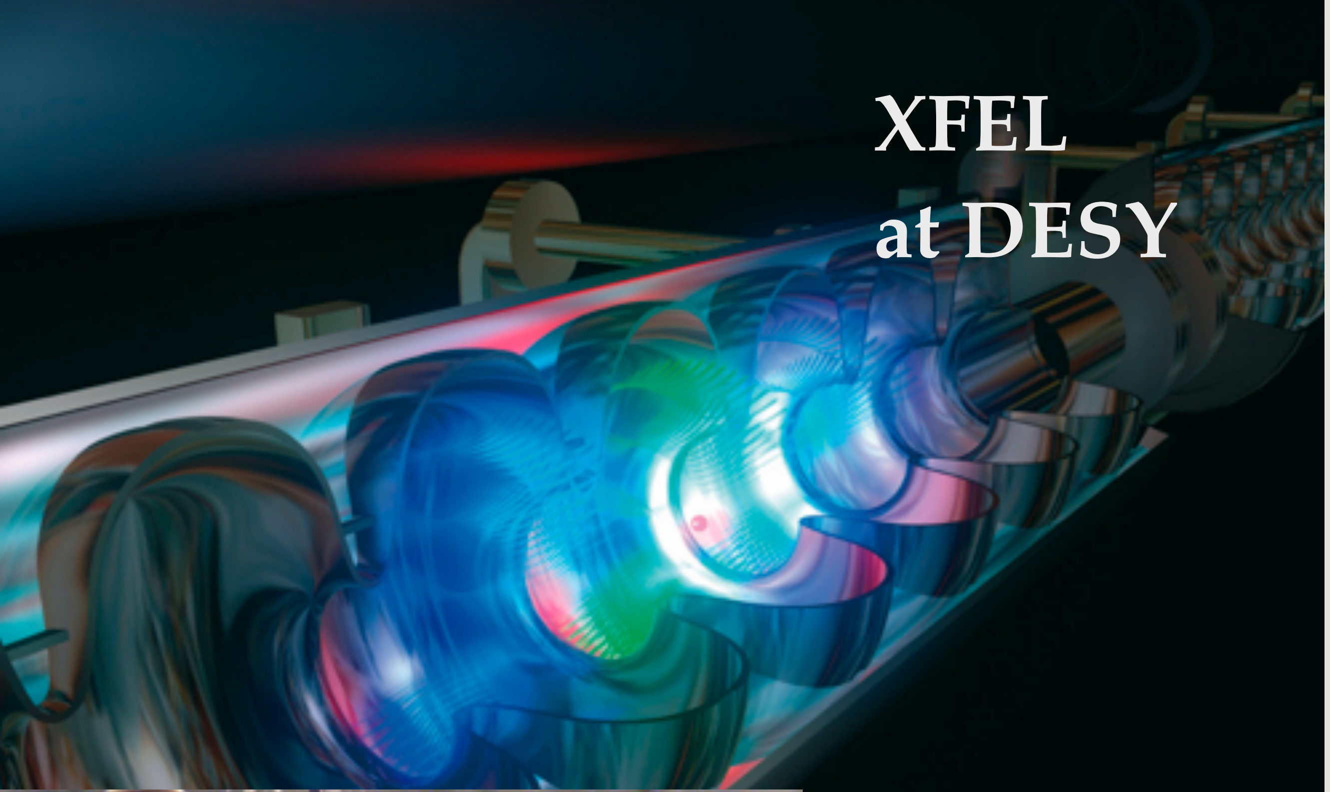


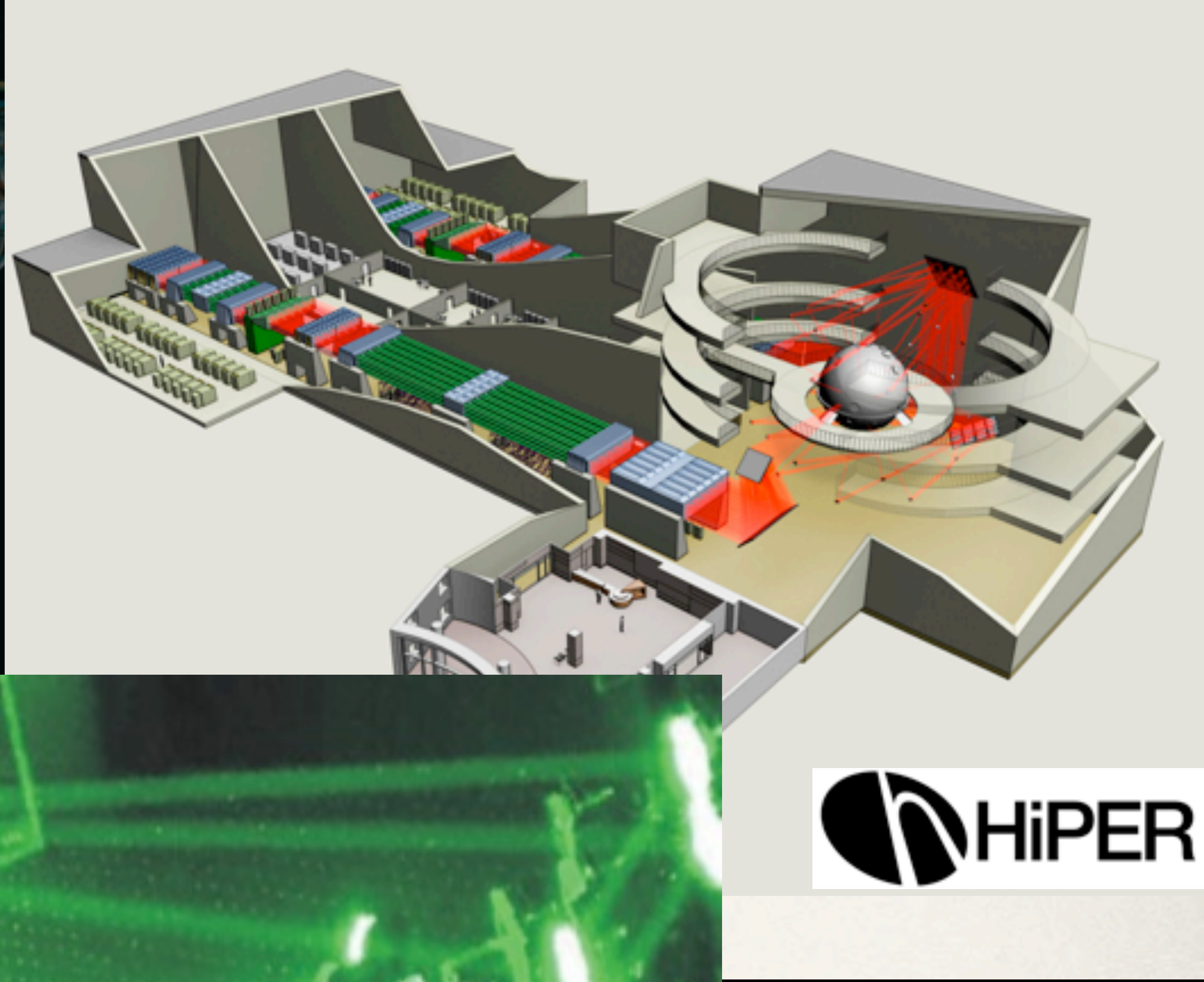
``Physicists are planning lasers powerful enough to rip apart the fabric of space and time''

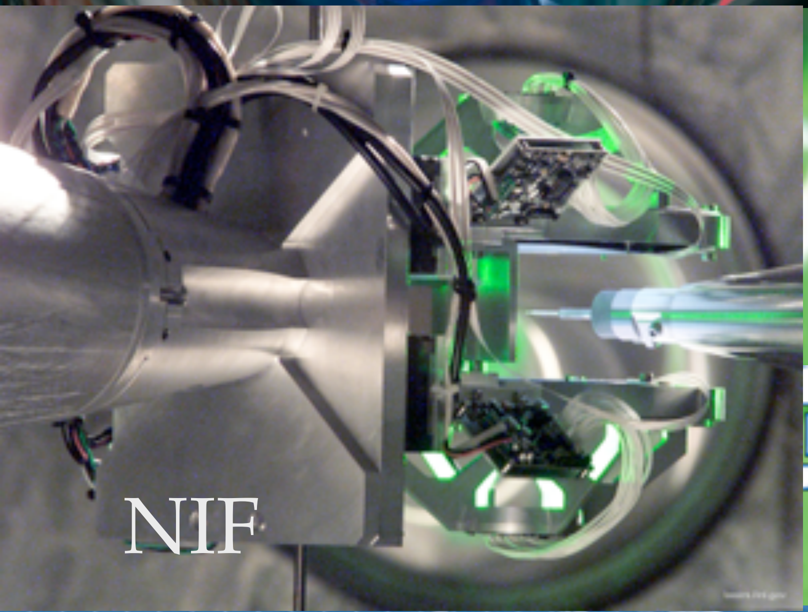
``We're going to change the index of refraction of the vacuum, and produce new particles.''

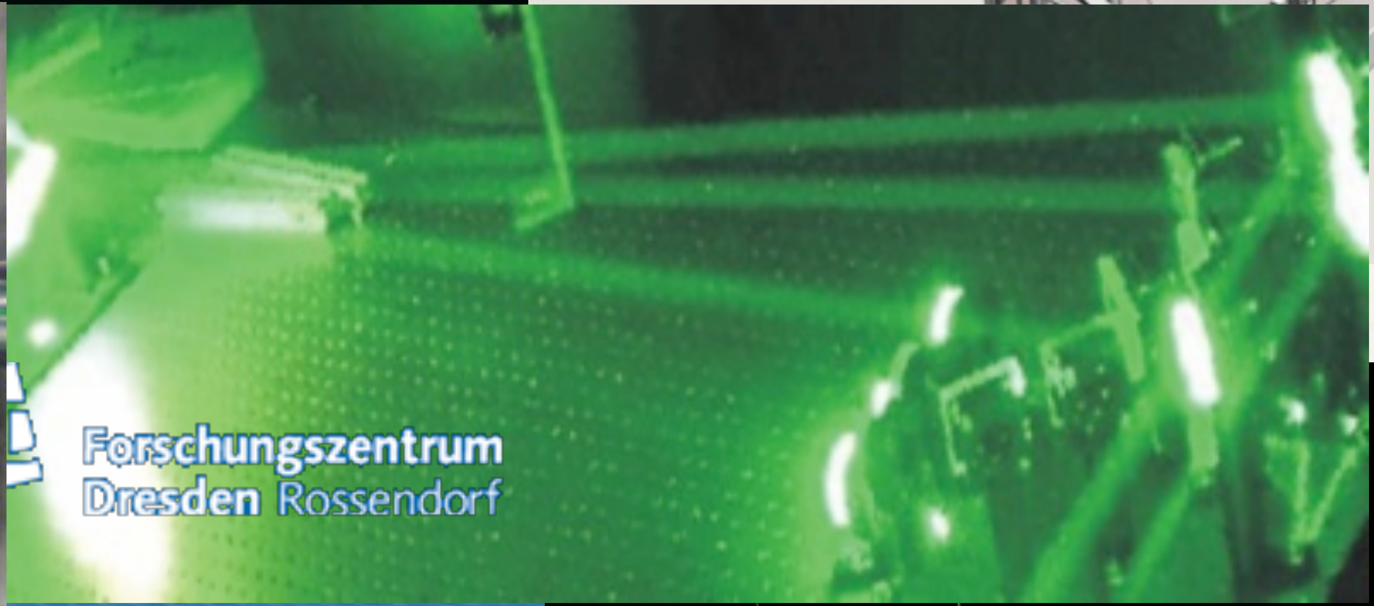
G. Mourou

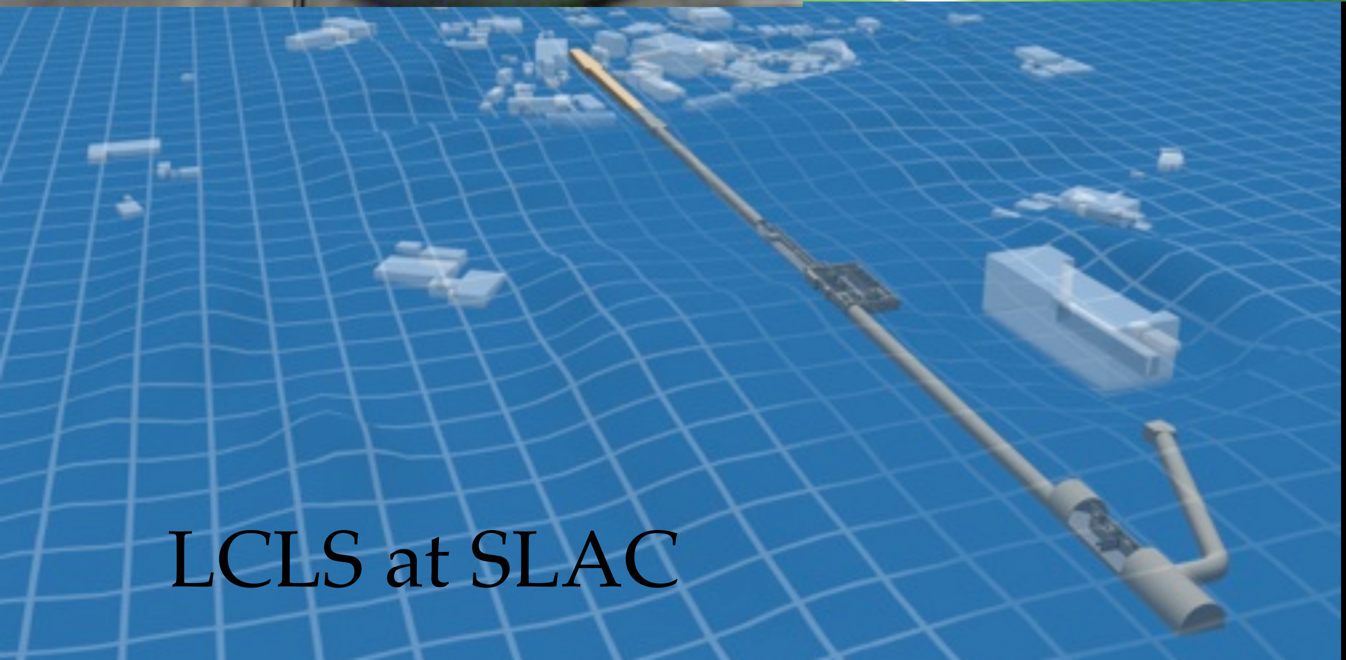
http://www.extreme-light-infrastructure.eu

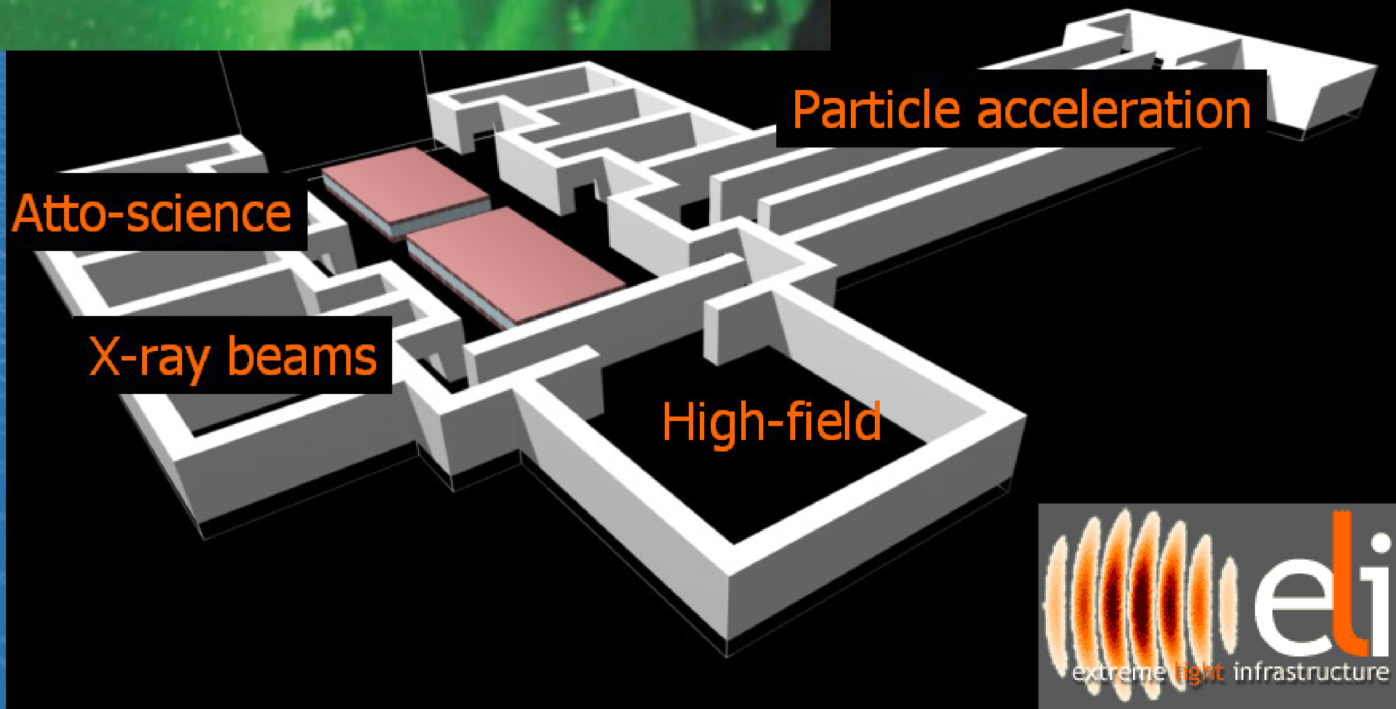

XFEL
at DESY

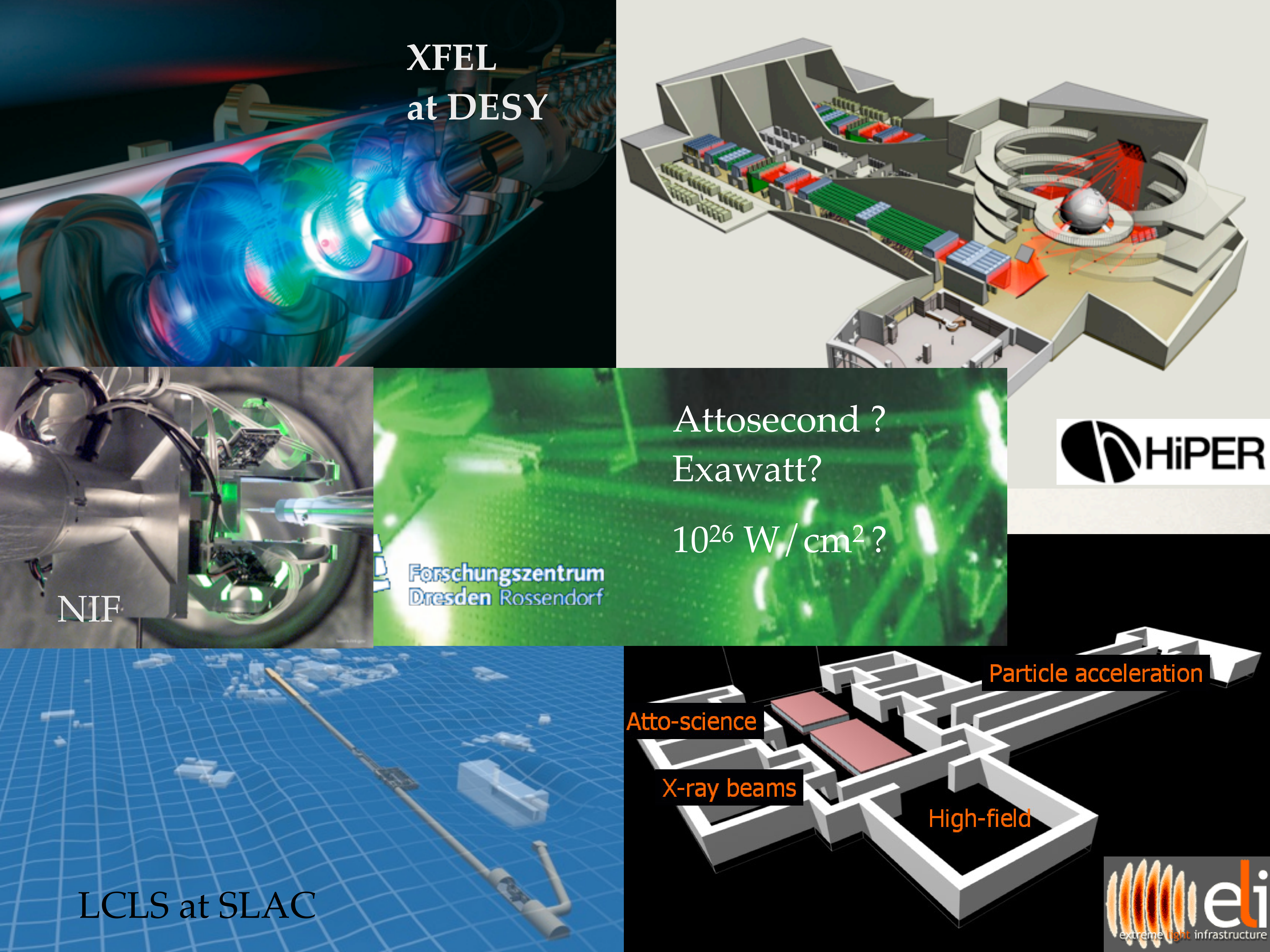

HiPER


NIF


Forschungszentrum
Dresden Rossendorf


LCLS at SLAC


Particle acceleration
Atto-science
X-ray beams
High-field
eli
extreme light infrastructure


XFEL
at DESY
HiPER
Attosecond ?
Exawatt?
10^26 W / cm^2 ?
Forschungszentrum
Dresden Rossendorf
NIF
Particle acceleration
Atto-science
X-ray beams
High-field
LCLS at SLAC
eli
extreme light infrastructure

Scientific legacy of Heisenberg & Euler's paper

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PHYSICAL REVIEW D VOLUME 2, NUMBER 10 15 NOVEMBER 1970

Nonlinear Effects in Quantum Electrodynamics. Photon Propagation and Photon Splitting in an External Field*

Z. Bialynicka-Birula† and I. Bialynicki-Birula‡
Department of Physics, University of Pittsburgh, Pittsburgh, Pennsylvania 15213
(Received 14 July 1970)

The effective nonlinear Lagrangian derived by Heisenberg and Euler is used to describe the propagation of photons in slowly varying but otherwise arbitrary electromagnetic fields. The group and the phase velocities for both propagation modes are calculated, and it is shown that the propagation is always causal. The photon splitting processes are also studied, and it is shown that they do not play any significant role even in very strong magnetic fields surrounding neutron stars.

PHOTON SPLITTING IN A STRONG MAGNETIC FIELD

S. L. Adler, J. N. Bahcall,* C. G. Callan, and M. N. Rosenbluth
The Institute for Advanced Study, Princeton, New Jersey 08540
(Received 6 August 1970)

We determine the absorption coefficient and polarization selection rules for photon splitting in a strong magnetic field, and describe the possible application of our results to pulsars.

PHYSICAL REVIEW D VOLUME 10, NUMBER 2 15 JULY 1974

Photon pair creation in intense magnetic fields*

Wu-yang Tsai
Department of Physics, University of California, Los Angeles, California 90024

Thomas Erber
Department of Physics, Illinois Institute of Technology, Chicago, Illinois 60616
(Received 4 April 1974)

The vacuum polarization of photons in intense, homogeneous magnetic fields is recalculated, using a proper-time method presented by Schwinger. This result is applied to compute exactly, in closed form, the photon absorption coefficient due to pair creation, $\kappa_{\parallel,\perp}$, corresponding to the polarization of the photon parallel or perpendicular to the plane

a new field of strong-field/high-intensity particle physics is forming

some laser-based fundamental physics experiments

PVLAS: Polarizzazione del Vuoto con LASer

ALPS
Any Light Particle Search
DESY

Fermilab GammeV

cnrs Université Paul Sabatier TOULOUSE III

Biréfringence Magnétique du Vide (BMV)

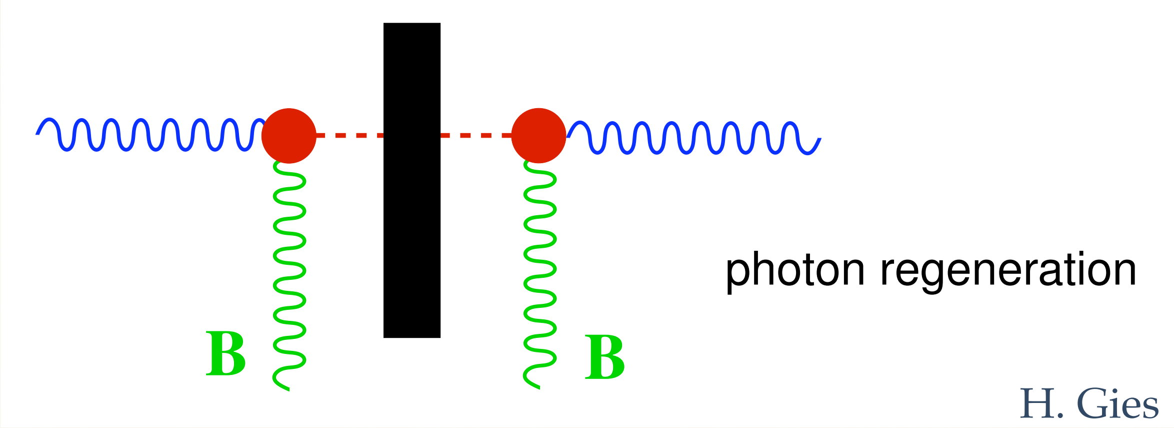
Jefferson Lab

LIPSS: Light Pseudoscalar and Scalar Search

CERN

OSQAR: Optical Search for QED vacuum magnetic birefringence, Axions and photon Regeneration

"light shining through walls" experiments



Ehret, ALPS collaboration, 2010

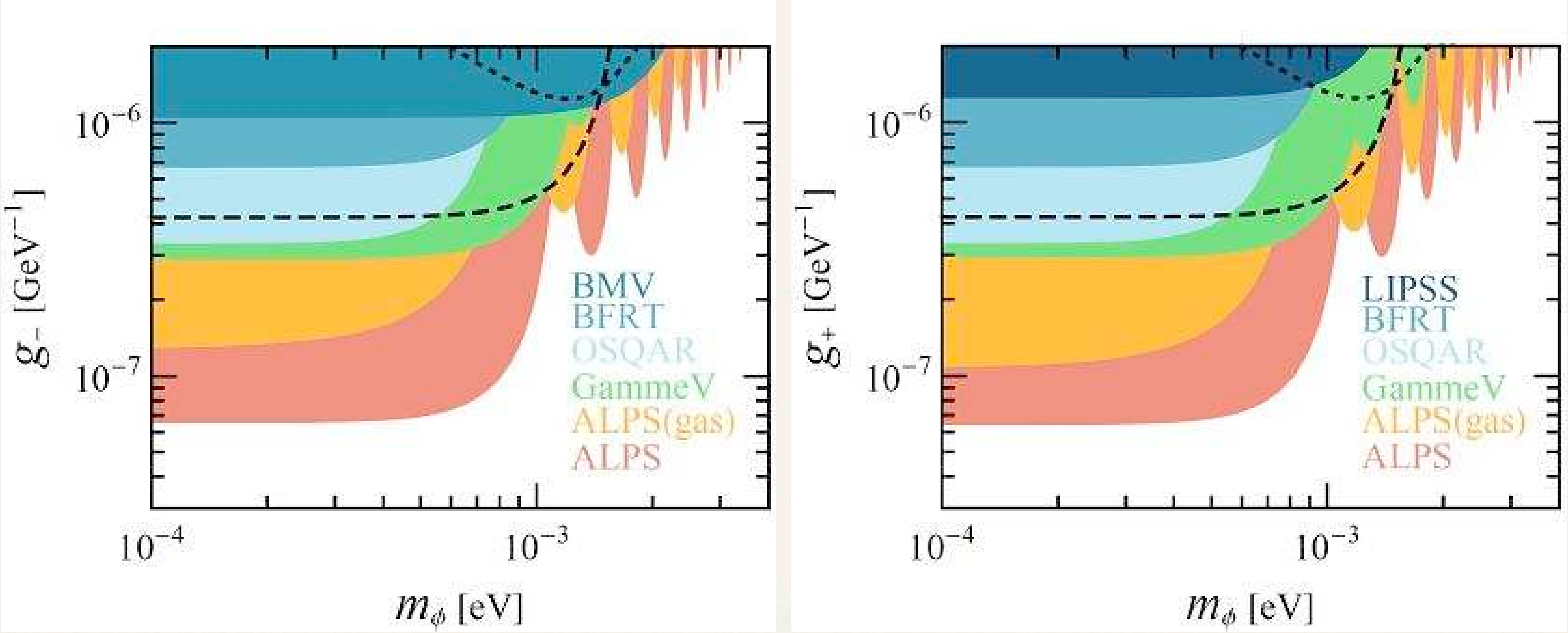


Figure 6: Exclusion limit (95% C.L.) for pseudoscalar (left) and scalar (right) axion-like particles obtained by the ALPS experiment from vacumm and gas runs together with the results from various other LSW experiments [10], see the text for details. Dashed and dotted lines show the bounds derived form the PVLAS measurement on ALP induced dichroism and birefringence [17].

Scientific legacy of Heisenberg & Euler's paper

- pair production from vacuum
- light-light scattering
- vacuum polarization physics
- effective field theory
- gravitational effective actions
- zeta functions

from QED to QCD ...

PHYSICAL REVIEW D VOLUME 11, NUMBER 8 15 APRIL 1975

Exact results for effective Lagrangians

M. R. Brown and M. J. Duff*
Department of Astrophysics, Oxford University, Oxford, England
(Received 4 September 1974)

A simple method is presented for the evaluation in quantum field theory of the effective Lagrangian induced by one-loop quantum effects. Exact solutions may be obtained in the quasilocal situation where the resulting Lagrangian is allowed to depend on the fields and their first derivatives (and, in some cases, their second derivatives as well). The method is a general one and may be applied to any given field theory. For example, Schwinger's result for the effective Maxwell Lagrangian with constant external field and the Coleman-Weinberg results for effective potentials each emerge as special cases of the general method. By isolating the divergent part of the induced Lagrangian in the general case, moreover, one may recover the 't Hooft-Veltman expression for the one-loop counterterms of an arbitrary field theory. At no stage need Feynman diagrams be evaluated.

Nuclear Physics B134 (1978) 539–545

VACUUM POLARIZATION INDUCED BY THE INTENSE GAUGE FIELD

S G MATINYAN and G K SAVVIDY
Yerevan Physics Institute, Armenia, USSR

Received 8 March 1977
(Revised 27 October 1977)

The results obtained from consideration of the effective Lagrangian density asymptotic behaviour in gauge theories by means of the renormalization-group method are discussed Such a consideration allows one to relate the asymptotic behaviour of the effective Lagrangian density in strong fields to the short-range behaviour of gauge theories

Fortschr. Phys. **32** (1984) 11, 585–622

Calculations in External Fields in Quantum Chromodynamics. Technical Review.

V. A. Novikov, M. A. Shifman, A. I. Vainshtein, V. I. Zakharov

Institute of Theoretical and Experimental Physics, Moscow, 117259, USSR

Abstract

We review the technique of calculation of operator expansion coefficients. The main emphasis is put on gluon operators which appear in expansion of n-point functions induced by colourless quark currents. Two convenient schemes are discussed in detail: the abstract operator method and the method based on the Fock-Schwinger gauge for the vacuum gluon field. We consider a large number of instructive examples important from the point of view of physical applications.

$$S_{\text{eff}} = S_{\text{cl}} + \Delta S,$$

$$\Delta S = -i \ln D_{\text{reg}} = -i \,\mathrm{Tr} \ln \left\| \frac{(\not{P} - m)_A}{(\not{P} - m)_{A=0}} \frac{(\not{P} - M_R)_{A=0}}{(\not{P} - M_R)_A} \right\|$$

$$-\frac{1}{32\pi^2} \int d^4x \,\mathrm{Tr}_c \left\{ \frac{2}{3} g^2 G_{\mu\nu}^2 \ln \frac{M_R{}^2}{m^2} - \frac{2}{45} i g^3 G_{\mu\nu} G_{\nu\gamma} G_{\gamma\mu} \frac{1}{m^2} \right.$$
$$\left. + \frac{g^4}{18} \left[(G_{\mu\nu} G_{\mu\nu})^2 - \frac{7}{10} \{G_{\mu\alpha}, G_{\alpha\nu}\}_+{}^2 - \frac{29}{70} [G_{\mu\alpha}, G_{\alpha\nu}]_-{}^2 + \frac{8}{35} [G_{\mu\nu}, G_{\alpha\beta}]_-{}^2 \right] \frac{1}{m^4} \right\}$$

Nuclear Physics B, **333**, 471 (1990)

Covariant perturbation theory (II). Second order in the curvature. General algorithms

A.O. Barvinsky, G.A. Vilkovisky

Nuclear Safety Institute, Bolshaya Tulskaya 52, Moscow 113191, USSR
Lebedev Physical Institute, Leninsky Prospect 53, Moscow 117924, USSR
Received 7 July 1989; Available online 18 October 2002.

Abstract

Covariant perturbation theory proposed in the previous paper is worked up to the second order in field strengths (curvatures). The trace of the heat kernel and the one-loop effective action for the generic second-order operator are obtained with this accuracy. The calculational scheme for higher orders is presented. The large time behaviour of the trace of the heat kernel is obtained to all orders in the curvature.

supersymmetry, helicity, superstrings, ...

SELF-DUALITY, HELICITY, AND SUPERSYMMETRY: THE SCATTERING OF LIGHT BY LIGHT

M.J. DUFF [1] and C.J. ISHAM
Physics Department, Imperial College, London SW7 2AZ, UK

Received 18 June 1979

The scattering of light by light is used to provide a concrete example of the connection between self-duality, helicity and supersymmetry first suggested in supergravity.

Let us first consider fermions. As shown by Euler and Heisenberg [10] and emphasized by Schwinger [9], the process $\gamma\gamma \to \gamma\gamma$ via a closed fermion loop may be represented, in the limit of low frequencies, by an effective four-photon lagrangian,

$$\mathcal{L}_{1/2} = (\alpha^2/90m^4)\,[\tfrac{11}{2}F_+^2F_-^2 - \tfrac{3}{4}(F_+^4 + F_-^4)]\,, \qquad (3)$$

$$\mathcal{L}_{\text{super}} = \frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s^3}\, e^{-m^2 s}(es)^2 \times \left[\mathcal{G}\left(\frac{1-\operatorname{Re}\cos(es\mathcal{H}_-^{1/2})}{\operatorname{Im}\cos(es\mathcal{H}_+^{1/2})}\right) - \mathcal{F}\right]. \qquad (13)$$

Note that the quadratic divergence in the first term has cancelled ‡2 and also that $\mathcal{L}_{\text{super}}$ vanishes when F_+ or F_- vanishes. If we expand the lagrangian (13) and keep only terms quartic in the field strength, we then recover the four-photon lagrangian (7).

If we repeat the above argument for a single charged scalar loop (with the same m) one finds [9]

$$\mathcal{L}_0 = (\alpha^2/90m^4)\,[F_+^2F_-^2 + \tfrac{3}{8}(F_+^4 + F_-^4)]\,, \qquad (6)$$

and once again both types of amplitude are present.

If we now consider the supersymmetric combination, however, then from (3) and (6)

$$\mathcal{L}_{\text{super}} = \mathcal{L}_{1/2} + 2\mathcal{L}_0 = (\alpha^2/12m^4)F_+^2F_-^2\,. \qquad (7)$$

The terms involving $(F_+^4 + F_-^4)$ have cancelled and we recover, as anticipated, the helicity-conservation rule. Thus the simple example of scattering of light by light in supersymmetric QED provides a concrete demonstration, in a spin-1 gauge theory, of the connection between self-duality, helicity, and supersymmetry first suggested in supergravity.

Nuclear Physics B
Volume 261, 1985

Quantum string theory effective action

E. S. Fradkin and A. A. Tseytlin

Department of Theoretical Physics, P.N. Levedev Physical Institute, Leninsky pr. 53, Moscow 117924, USSR

Abstract

We present a covariant background field method for quantum string dynamics. It is based on the effective action Γ for fields corresponding to different string modes. A formalism is developed for the calculation of Γ in the $\alpha' \rightarrow 0$ limit. It is shown that in the case of closed Bose strings Γ contains the standard kinetic terms for the scalar, external metric and the antisymmetric tensor. Our approach makes possible a consistent formulation and solution of a ground state problem (including the problem of space-time compactification) in the string theory. We suggest a solution to the old "tachyon problem" based on the generation of non-trivial vacuum values for the scalar field, metric and antisymmetric tensor. It is shown that a preferred compactification in the closed Bose string theory is to three (anti-de Sitter) space-time dimensions.

Physics Letters B
Volume 163, 1985

Non-linear electrodynamics from quantized strings

E. S. Fradkin and A. A. Tseytlin

P.N. Lebedev Physical Institute, Leninsky pr. 53, Moscow 117924, USSR

Abstract

We compute the effective action for an abelian vector field coupled to the virtual open Bose string. The problem is exactly solved (in the "tree" and "one-loop" approximation for the string theory) for the case of a constant field strength and the number of space-time dimensions D=26. The resulting tree-level effective lagrangian is shown to coincide with the Born-Infeld lagrangian, $[\det(\delta_{\mu\nu} + 2\Pi\alpha' F_{\mu\nu})]^{1/2}$.

Scientific legacy of Heisenberg & Euler's paper

- pair production from vacuum
- light-light scattering
- vacuum polarization physics
- effective field theory
- gravitational effective actions
- zeta functions

PHYSICAL REVIEW VOLUME 162, NUMBER 5 25 OCTOBER 1967

Quantum Theory of Gravity. II. The Manifestly Covariant Theory*

BRYCE S. DEWITT
Institute for Advanced Study, Princeton, New Jersey
and
Department of Physics, University of North Carolina, Chapel Hill, North Carolina†

(Received 25 July 1966; revised manuscript received 9 January 1967)

THE
PHYSICAL REVIEW
A journal of experimental and theoretical physics established by E. L. Nichols in 1893

SECOND SERIES, VOL. **183**, NO. 5 25 JULY 1969

Quantized Fields and Particle Creation in Expanding Universes. I

LEONARD PARKER
Department of Physics, University of Wisconsin–Milwaukee, Milwaukee, Wisconsin 53201

(Received 11 March 1969)

PHYSICAL REVIEW D VOLUME 12, NUMBER 4 15 AUGUST 1975

General-relativistic quantum field theory: An exactly soluble model

P. Candelas and D. J. Raine*
Department of Astrophysics, University of Oxford, South Parks Road, Oxford, England
(Received 3 March 1975)

The massive scalar and Dirac fields quantized on a de Sitter background geometry prove to be exactly soluble models in general-relativistic field theory. The Feynman Green's function is computed for both the scalar and Dirac fields. A dimensional regularization procedure applied in coordinate space facilitates the calculation of their respective effective Lagrangians, which describe the vacuum corrections due to closed matter loops. The model is found to be renormalizable. There is no creation of real particle pairs.

PHYSICAL REVIEW D VOLUME 13, NUMBER 12 15 JUNE 1976

Effective Lagrangian and energy-momentum tensor in de Sitter space

J. S. Dowker and Raymond Critchley
Department of Theoretical Physics, The University, Manchester, 13, England
(Received 29 October 1975)

The effective Lagrangian and vacuum energy-momentum tensor $\langle T^{\mu\nu}\rangle$ due to a scalar field in a de Sitter-space background are calculated using the dimensional-regularization method. For generality the scalar field equation is chosen in the form $(\Box^2 + \xi R + m^2)\varphi = 0$. If $\xi = 1/6$ and $m = 0$, the renormalized $\langle T^{\mu\nu}\rangle$ equals $g^{\mu\nu}(960\pi^2 a^4)^{-1}$, where a is the radius of de Sitter space. More formally, a general zeta-function method is developed. It yields the renormalized effective Lagrangian as the derivative of the zeta function on the curved space. This method is shown to be virtually identical to a method of dimensional regularization applicable to any Riemann space.

PHYSICAL REVIEW D VOLUME 23, NUMBER 12 15 JUNE 1981

Path-integral evaluation of Feynman propagator in curved spacetime

Jacob D. Bekenstein* and Leonard Parker
Department of Physics, University of Wisconsin at Milwaukee, Milwaukee, Wisconsin 53201
(Received 24 November 1980)

We develop an efficient approximation procedure for evaluating the scalar Feynman propagator in arbitrary spacetimes. In the familiar manner we represent it by an integral over the transition amplitude for a Schrödinger-type equation (proper-time method). The amplitude is then represented by a Feynman path integral which is dominated by the contribution of a certain extremal path. The contributions of adjacent paths are then simply expressed by working in Fermi normal coordinates based on the extremal path. In this manner the path integral becomes an ordinary multiple integral over "Fourier coefficients" which represent the various paths. For a conformal field, or for spacetimes with constant scalar curvature, we evaluate the integral in the Gaussian approximation in terms of the curvature along the (geodesic) extremal path. We show the result to be related to the Schwinger-DeWitt expansion for the amplitude, but valid for well-separated end points. In the Einstein universe our expression gives the exact amplitude and propagator. In the de Sitter spacetime it gives a good approximation for the amplitude even for well-separated points. We also evaluate the post-Gaussian corrections to the amplitude, though we do not implement them in a concrete spacetime. For nonconformal fields in spacetimes with varying scalar curvature, we evaluate the amplitude in the Gaussian approximation in terms of the values of the curvature along the extremal (nongeodesic) path. It is very different in form from the one mentioned earlier, which suggests the existence of novel effects arising from variation in the scalar curvature.

No better starting point for an approximation scheme offers itself than the Schwinger-DeWitt proper-time formalism.[1-3,11] In this method, one replaces our problem by that of solving the Schrödinger equation[12]

$$i\frac{\partial}{\partial s}\langle x, s\,|\,x', 0\rangle = (-\nabla^{\mu}\nabla_{\mu} + \xi R)\langle x, s\,|\,x', 0\rangle \quad (1.2)$$

subject to the boundary condition

$$\lim_{s\to 0}\langle x, s\,|\,x' 0\rangle = |g(x)|^{-1/2}\delta(x, x')\,. \quad (1.3)$$

The kernel $\langle x, s\,|\,x', 0\rangle$ is formally the amplitude for a particle coupled to the curvature to propagate from spacetime point x' to x in the course of a *fictitious* proper-time inverval s. The Green's function is recovered by

$$G_F(x, x') = i\int_0^{\infty}\langle x, s\,|\,x', 0\rangle e^{-im^2 s}ds\,, \quad (1.4)$$

Scientific legacy of Heisenberg & Euler's paper

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- effective field theory
- gravitational effective actions
- zeta functions

Commun. math. Phys. 55, 133—148 (1977)

Communications in Mathematical Physics

Zeta Function Regularization of Path Integrals in Curved Spacetime

S. W. Hawking

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 9EW, England

Abstract. This paper describes a technique for regularizing quadratic path integrals on a curved background spacetime. One forms a generalized zeta function from the eigenvalues of the differential operator that appears in the action integral. The zeta function is a meromorphic function and its gradient at the origin is defined to be the determinant of the operator. This technique agrees with dimensional regularization where one generalises to n dimensions by adding extra flat dimensions. The generalized zeta function can be expressed as a Mellin transform of the kernel of the heat equation which describes diffusion over the four dimensional spacetime manifold in a fith dimension of parameter time. Using the asymptotic expansion for the heat kernel, one can deduce the behaviour of the path integral under scale transformations of the background metric. This suggests that there may be a natural cut off in the integral over all black hole background metrics. By functionally differentiating the path integral one obtains an energy momentum tensor which is finite even on the horizon of a black hole. This energy momentum tensor has an anomalous trace.

One forms a generalized zeta function from the eigenvalues of the operator A:

$$\zeta(s)=\sum_n \lambda_n^{-s} . \tag{3.1}$$

In four dimensions this will converge for $\mathrm{Re}(s)>2$. It can be analytically extended to a merophorphic function of s with poles only at $s=2$ and $s=1$ [18]. In particular it is regular at $s=0$. The gradient of zeta at $s=0$ is formally equal to $-\sum_n \log\lambda_n$. One can therefore *define* $\det A$ to be $\exp(-d\zeta/ds|_{s=0})$ [19]. Thus the partition function

$$\log Z[\tilde{\phi}]=\tfrac{1}{2}\zeta'(0)+\tfrac{1}{2}\log(\tfrac{1}{4}\pi\mu^2)\zeta(0) . \tag{3.2}$$

Commun. math. Phys. 55, 133—148 (1977)

Communications in Mathematical Physics

Zeta Function Regularization of Path Integrals in Curved Spacetime

S. W. Hawking

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 9EW, England

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One-loop effective potentials in quantum electrodynamics

W Dittrich

Institut für Theoretische Physik der Universität Tübingen, Auf der Morgenstelle 14, D-7400, Tübingen 1, West Germany

Received 2 February 1976

$$\mathscr{L}^{(1)}_{\frac{1}{2}}[H] = -\frac{2}{64\pi^2}\Big\{(2m^4_{\frac{1}{2}} - 4m^2_{\frac{1}{2}}(eH) + \tfrac{4}{3}(eH)^2)\Big[\ln\Big(\frac{m^2_{\frac{1}{2}}}{eH}\Big) + 1\Big] + 4m^2_{\frac{1}{2}}(eH) - 3m^4_{\frac{1}{2}} + 2(4eH)^2\zeta'(-1; m^2_{\frac{1}{2}}/2eH)\Big\}$$

One forms a generalized zeta function from the eigenvalues of the operator A:

$$\zeta(s) = \sum_n \lambda_n^{-s} \,. \tag{3.1}$$

In four dimensions this will converge for $\mathrm{Re}(s) > 2$. It can be analytically extended to a merophorphic function of s with poles only at $s=2$ and $s=1$ [18]. In particular it is regular at $s=0$. The gradient of zeta at $s=0$ is formally equal to $-\sum_n \log\lambda_n$. One can therefore *define* $\det A$ to be $\exp(-d\zeta/ds|_{s=0})$ [19]. Thus the partition function

$$\log Z[\tilde{\phi}] = \tfrac{1}{2}\zeta'(0) + \tfrac{1}{2}\log(\tfrac{1}{4}\pi\mu^2)\zeta(0) \,. \tag{3.2}$$

ZETA FUNCTIONS ON THE SPHERE*

BY

S. MINAKSHISUNDARAM, *Andhra University.*

[Received 6 May 1949.]

Consider the eigenvalues and eigenfunctions of the following problem

$$\triangle u+\lambda u=0$$

$$u=0 \text{ or } \frac{\partial u}{\partial n}=0 \text{ on } B, \qquad (1)$$

where B is the boundary of a bounded Euclidean domain D of k dimensions and $\triangle$ is the Laplace operator,

In other words the harmonics are the eigenfunctions of

$$\wedge u+\lambda u=0$$

with the eigenvalues $n(n+k-1)$. The number of independent solutions of (4) is l, where

$$l=\frac{(k+n-2)!}{n!\,(k-1)!}(k+2n-1). \qquad (5)$$

$$\sum_{1}^{\infty}\frac{C_n^k(1)}{n^s(n+2\nu)^s}=\left(\frac{2}{\nu}\right)^{s-1/2}\frac{\Gamma(s+\frac{1}{2})}{\Gamma(2s)}\int_0^\infty\left\{\frac{\cosh t/2}{(\sinh t/2)^{-k}}-e^{-\nu t}\right\}\times t^{s-\frac{1}{2}}I_{s-\frac{1}{2}}(\nu t)\,dt. \quad (16)$$

Advances in Mathematics **156**, 107–132 (2000)
doi:10.1006/aima.2000.1946, available online at http://www.idealibrary.com on IDEAL®

On Barnes' Multiple Zeta and Gamma Functions

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Centre for Mathematics and Computer Science, P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

Communicated by Takahiro Kawai

Received February 7, 2000; accepted July 17, 2000

$$\zeta_N(s, w \mid a_1, ..., a_N) = \sum_{m_1, ..., m_N = 0}^{\infty} (w + m_1 a_1 + \cdots + m_N a_N)^{-s}, \ \mathrm{Re}\, w > 0, \ \mathrm{Re}\, s > N,$$

$$\zeta_N(s, w) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{dt}{t} t^s e^{-wt} \prod_{j=1}^{N} (1 - e^{-a_j t})^{-1}, \ \mathrm{Re}\, s > N, \ \mathrm{Re}\, w > 0.$$

To proceed, we introduce the multiple gamma function

$$\Gamma_N(w) \equiv \exp(\Psi_N(w)) = \exp(\partial_s \zeta_N(s, w)|_{s=0}).$$

W. Heisenberg & H. Euler, *Consequences of Dirac's theory of the positron*, Zeitschr. Phys., **98**, 714 (1936)

this paper was many years ahead of its time

significant scientific legacy continues today

- pair production from vacuum
- light-light scattering
- vacuum polarization physics
- effective field theory
- gravitational effective actions
- zeta functions
- ...

Euler-Heisenberg Session @ QFEXT11, Benasque

Thursday, September 22

09:00: Euler-Heisenberg and Beyond, G. Dunne

09:45: Electromagnetic superconductivity of vacuum induced by strong magnetic field, M. N. Chernodub

15.30: A. The Euler-Heisenberg Lagrangian beyond one-loop, C. Schubert

16.00: A. Radiation damping effects in high intensity laser fields, C. Harvey

16.30: A. QED processes in intense laser fields, A. Ilderton

17:30: A. Optical probes of the quantum vacuum - the photon polarization tensor in external fields, F. Karbstein

18.00: A. Generalizations of the Heisenberg-Euler energy to strong electric fields, S. Gavrilov

18.30: A. Creation of Neutral Fermions by Magnetic Barriers, T. Adorno

Friday, September 23

11:00: Strong-Field QED and High-Power Lasers, T. Heinzl

15:30: A. Interference Effects in Vacuum Pair Production in Time Dependent Laser Pulses, C. Dumlu

16:00: C. Effective actions of magnetic flux tubes, D. Mazur