Canonical quantization of macroscopic electromagnetism and the Casimir Effect

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The beginnings of quantum electromagnetism

Bethe, 1947:

Lamb shift in hydrogen:

$$\delta E_{2s} - \delta E_{2p} = 2\pi\hbar \times 1040\,\mathrm{MHz}$$

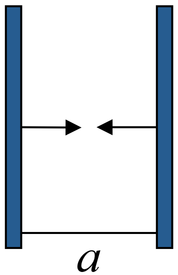
Schwinger, 1948:

Magnetic moment of the electron:

$$\mu = \frac{e\hbar}{2m}\left(1 + \frac{e^2}{8\pi^2\varepsilon_0\hbar c} + \dots\right)$$

Casimir, 1948:

Attractive force between neutral perfect mirrors:

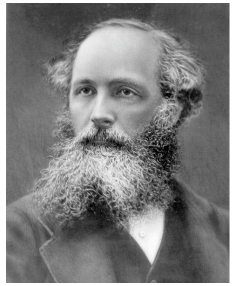


$$f = -\frac{\hbar c \pi^2}{240 a^4}$$

QED of light and particles

Canonical quantization: $[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$

Canonical formulation of Maxwell's equations:



$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \frac{\rho}{\varepsilon_0}, \quad \boldsymbol{\nabla}\times\boldsymbol{B} = \frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t} + \mu_0\boldsymbol{j}, \quad \boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}, \quad \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$$

Canonical fields: scalar and vector potentials: $\boldsymbol{E} = -\nabla\phi - \partial_t\boldsymbol{A}, \qquad \boldsymbol{B} = \nabla\times\boldsymbol{A}$

Action: $S[\phi,\boldsymbol{A}] = \int d^4x \left(\frac{\varepsilon_0}{2}\boldsymbol{E}\cdot\boldsymbol{E} - \frac{1}{2\mu_0}\boldsymbol{B}\cdot\boldsymbol{B} - \rho\,\phi + \boldsymbol{j}\cdot\boldsymbol{A}\right)$

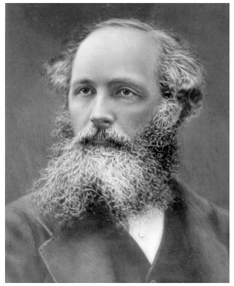
Canoncial momenta: $\Pi_\phi = \frac{\delta S}{\delta(\partial_t\phi)}, \qquad \boldsymbol{\Pi}_{\mathrm{A}} = \frac{\delta S}{\delta(\partial_t\boldsymbol{A})}$

Canonically quantize to obtain field operators:

$$\hat{\boldsymbol{E}} = \frac{1}{(2\pi)^{3/2}}\sum_\lambda \int d^3\boldsymbol{k}\sqrt{\frac{\hbar\omega}{2\varepsilon_0}}\boldsymbol{e}_{\boldsymbol{k}\lambda}\left(i\hat{a}_{\boldsymbol{k}\lambda}e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{r}} + \mathrm{h.c}\right) \qquad \left[\hat{a}_{\boldsymbol{k}\lambda}, \hat{a}^\dagger_{\boldsymbol{k}'\lambda'}\right] = \delta_{\lambda\lambda'}\delta(\boldsymbol{k}-\boldsymbol{k}')$$

Macroscopic electromagnetism

Another version of classical electromagnetism –
macroscopic Maxwell equations for light in macroscopic media –
media described by dielectric functions (permittivities and permeabilities):



$$\boldsymbol{\nabla}\cdot\boldsymbol{D}=\rho,\quad \boldsymbol{\nabla}\times\boldsymbol{H}=\frac{\partial\boldsymbol{D}}{\partial t}+\boldsymbol{j},\quad \boldsymbol{\nabla}\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t},\quad \boldsymbol{\nabla}\cdot\boldsymbol{B}=0$$

$$\boldsymbol{D}(\boldsymbol{r},t)=\frac{\varepsilon_0}{2\pi}\int_0^\infty d\omega[\varepsilon(\boldsymbol{r},\omega)\boldsymbol{E}(\boldsymbol{r},\omega)e^{-i\omega t}+\text{c.c.}]$$

$$\boldsymbol{H}(\boldsymbol{r},t)=\frac{\kappa_0}{2\pi}\int_0^\infty d\omega[\kappa(\boldsymbol{r},\omega)\boldsymbol{B}(\boldsymbol{r},\omega)e^{-i\omega t}+\text{c.c.}],\quad \kappa=1/\mu$$

Kramers-Kronig relation: $\varepsilon_{\text{R}}(\mathbf{r},\omega')-1=\frac{2}{\pi}\text{P}\int_0^\infty d\omega\frac{\omega\varepsilon_{\text{I}}(\mathbf{r},\omega)}{\omega^2-\omega'^2},$ and for $\kappa(\mathbf{r},\omega)$

This is arguably the most experimentally significant field theory in physics.

Macroscopic QED?
[Obvious way to generalize Casimir's result to realistic materials]
Find Lagrangian and Hamiltonian for macroscopic electromagnetism – canonically quantize.

Canonical quantization "widely agreed" to be impossible [Huttner and Barnett, PRA, 46 (1992) 4306].

Problem is dispersion and absorption (Kramers-Kronig).

Microscopic models and phenomenological theory

The lack of a canonical formulation of the macroscopic Maxwell equations led to two approaches to quantization.

Microscopic models

Introduce a simple microscopic model of a medium – typically a field of harmonic oscillators with some resonant frequency.
Absorption is dealt with using a reservoir (heat bath) which is a *continuum* of harmonic oscillators at *all* frequencies.
Microscopic matter degrees of freedom are specified; system can be canonically quantized; afterwards, an effective permittivity is extracted.
Many examples in the literature.

Phenomenological approaches

In the purely phenomenological approach, no rigorous quantization attempted.
Some reasonable quantum prescriptions are imposed, which can be justified in the case of the simple microscopic models.
Again, many papers over the decades.
[Sometimes called "macroscopic QED"]

Macroscopic QED

Canonical quantization of macroscopic electromagnetism *is* possible
[TP, *New J. Phys.* **12** (2010) 123008].

Action for macroscopic electromagnetism includes reservoir fields, because of dissipation:

$$S[\phi, \boldsymbol{A}, \boldsymbol{X}_\omega, \boldsymbol{Y}_\omega] = S_{\mathrm{em}}[\phi, \boldsymbol{A}] + S_{\mathrm{X}}[\boldsymbol{X}_\omega] + S_{\mathrm{Y}}[\boldsymbol{Y}_\omega] + S_{\mathrm{int}}[\phi, \boldsymbol{A}, \boldsymbol{X}_\omega, \boldsymbol{Y}_\omega]$$

Electromagnetic fields: $$S_{\mathrm{em}}[\phi, \boldsymbol{A}] = \frac{\kappa_0}{2}\int d^4x \left(\frac{1}{c^2}\boldsymbol{E}\cdot\boldsymbol{E} - \boldsymbol{B}\cdot\boldsymbol{B}\right)$$

Reservoir fields: $$S_{\mathrm{X}}[\boldsymbol{X}_\omega] = \frac{1}{2}\int d^4x \int_0^\infty d\omega \left(\partial_t\boldsymbol{X}_\omega\cdot\partial_t\boldsymbol{X}_\omega - \omega^2\boldsymbol{X}_\omega\cdot\boldsymbol{X}_\omega\right)$$

Coupling terms: $$S_{\mathrm{int}}[\phi, \boldsymbol{A}, \boldsymbol{X}_\omega, \boldsymbol{Y}_\omega] = \int d^4x \int_0^\infty d\omega \left[\alpha(\boldsymbol{r},\omega)\boldsymbol{X}_\omega\cdot\boldsymbol{E} + \beta(\boldsymbol{r},\omega)\boldsymbol{Y}_\omega\cdot\boldsymbol{B}\right]$$

Coupling functions: $$\alpha(\boldsymbol{r},\omega) = \left[\frac{2\varepsilon_0}{\pi}\omega\varepsilon_{\mathrm{I}}(\boldsymbol{r},\omega)\right]^{1/2}, \quad \beta(\boldsymbol{r},\omega) = \left[-\frac{2\kappa_0}{\pi}\omega\kappa_{\mathrm{I}}(\boldsymbol{r},\omega)\right]^{1/2}$$

No microscopic substructure underlying dielectric functions is assumed; valid for arbitrary linear inhomogeneous magneto-dielectrics.

Field equations of this action are the macroscopic Maxwell equations:

$$\boldsymbol{\nabla}\cdot\boldsymbol{D} = \rho, \quad \boldsymbol{\nabla}\times\boldsymbol{H} = \frac{\partial\boldsymbol{D}}{\partial t} + \boldsymbol{j}, \quad \boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}, \quad \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$$

Action depends on fields and their first derivatives: **canonical quantization can proceed**

Canonical macroscopic electromagnetism

Field equations of the action are

$$\varepsilon_0 \nabla \cdot \boldsymbol{E} + \int_0^\infty d\omega\, \nabla \cdot [\alpha(\boldsymbol{r},\omega)\boldsymbol{X}_\omega] = 0,$$

$$-\kappa_0 \nabla \times \boldsymbol{B} + \varepsilon_0 \partial_t \boldsymbol{E} + \int_0^\infty d\omega\, \{\alpha(\boldsymbol{r},\omega)\partial_t \boldsymbol{X}_\omega + \nabla \times [\beta(\boldsymbol{r},\omega)\mathbf{Y}_\omega]\} = 0,$$

$$-\partial_t^2 \boldsymbol{X}_\omega - \omega^2 \boldsymbol{X}_\omega + \alpha(\boldsymbol{r},\omega)\mathbf{E} = 0,$$

$$-\partial_t^2 \mathbf{Y}_\omega - \omega^2 \mathbf{Y}_\omega + \beta(\boldsymbol{r},\omega)\boldsymbol{B} = 0$$

$$\alpha(\boldsymbol{r},\omega) = \left[\frac{2\varepsilon_0}{\pi}\omega\varepsilon_{\mathrm{I}}(\boldsymbol{r},\omega)\right]^{1/2}, \quad \beta(\boldsymbol{r},\omega) = \left[-\frac{2\kappa_0}{\pi}\omega\kappa_{\mathrm{I}}(\boldsymbol{r},\omega)\right]^{1/2}$$

Solve the reservoir equations, with the retarded solution, and use the Kramers-Kronig relation

$$\varepsilon_{\mathrm{R}}(\mathbf{r},\omega') - 1 = \frac{2}{\pi}\mathrm{P}\int_0^\infty d\omega \frac{\omega\varepsilon_{\mathrm{I}}(\mathbf{r},\omega)}{\omega^2 - \omega'^2}, \qquad \text{and for} \quad \kappa(\mathbf{r},\omega)$$

the equations for the electromagnetic fields are the macroscopic Maxwell equations

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho, \quad \boldsymbol{\nabla} \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{j}, \quad \boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

Standard canonical quantization proceeds without difficulty.

Macroscopic QED

Electric field operator is

$$\hat{\boldsymbol{E}}(\boldsymbol{r},t) = \frac{\mu_0}{2\pi}\int_0^\infty d\omega \int d^3\boldsymbol{r'} \left[i\omega \boldsymbol{G}(\boldsymbol{r},\boldsymbol{r'},\omega)\cdot \hat{\boldsymbol{j}}(\boldsymbol{r'},\omega)\exp(-i\omega t) + \mathrm{h.c.}\right]$$

Green bi-tensor is the solution of

$$\nabla \times [\kappa(\boldsymbol{r},\omega)\nabla \times \boldsymbol{G}(\boldsymbol{r},\boldsymbol{r'},\omega)] - \frac{\omega^2}{c^2}\varepsilon(\boldsymbol{r},\omega)\boldsymbol{G}(\boldsymbol{r},\boldsymbol{r'},\omega) = \mathbb{1}\delta(\boldsymbol{r}-\boldsymbol{r'})$$

Free current operator is

$$\hat{\boldsymbol{j}}(\boldsymbol{r},\omega) = -2\pi i\omega\left[\frac{\hbar\varepsilon_0}{\pi}\varepsilon_{\mathrm{I}}(\boldsymbol{r},\omega)\right]^{1/2}\hat{\boldsymbol{C}}_{\mathrm{e}}(\boldsymbol{r},\omega) + 2\pi\nabla\times\left\{\left[-\frac{\hbar\kappa_0}{\pi}\kappa_{\mathrm{I}}(\boldsymbol{r},\omega)\right]^{1/2}\hat{\boldsymbol{C}}_{\mathrm{m}}(\boldsymbol{r},\omega)\right\}$$

Diagonalizing operators of the Hamiltonian are bosonic creation and annihilation operators:

$$\left[\hat{C}_{\mathrm{e}i}(\boldsymbol{r},\omega),\hat{C}^\dagger_{\mathrm{e}j}(\boldsymbol{r'},\omega')\right] = \delta_{ij}\delta_{\lambda\lambda'}\delta(\omega-\omega')\delta(\boldsymbol{r}-\boldsymbol{r'}), \quad \text{etc.}$$

General theory of the Casimir effect

Standard theory - Lifshitz theory - was not based on the principles of QED, rather on those of thermodynamics (fluctuation-dissipation theorem).

Lifshitz theory has no Hamiltonian, there are no quantized fields – yet it is used to calculate forces caused by quantum fields (zero-point and thermal).
This is the cause of long-standing doubts about the status of Lifshitz theory as a quantum theory.

Macroscopic QED is the obvious basis for the general theory of the Casimir effect:

- Casimir effect is caused by quantum electromagnetism in macroscopic media.
- Casimir effect is broadband so dispersion and dissipation cannot be ignored (arbitrary ε and μ, obeying Kramers-Kronig relations).

General theory of the Casimir effect

Canonical macroscopic QED, restricted to thermal equilibrium, provides a rigorous quantum foundation for Casimir effect. [TP, *New J. Phys.* **13** (2011) 063026]

Only a simple restriction to thermal equilibrium is required; place the bosonic eigenmodes in a mixed thermal state:

$$\left\langle \hat{C}^{\dagger}_{\mathrm{e}i}(\boldsymbol{r},\omega), \hat{C}_{\mathrm{e}j}(\boldsymbol{r}',\omega')\right\rangle = \left[\exp\left(\frac{\hbar\omega}{k_{\mathrm{B}}T}\right) - 1\right]^{-1} \delta_{ij}\delta_{\lambda\lambda'}\delta(\omega-\omega')\delta(\boldsymbol{r}-\boldsymbol{r}'), \quad \text{etc.}$$

Energy density and stress tensor of macroscopic QED are given by Noether's theorem (because we have a Lagrangian):

$$\hat{\rho} = \frac{\kappa_0}{2}\left[\frac{1}{c^2}\hat{\boldsymbol{E}}^2 + \hat{\boldsymbol{B}}^2\right] + \int_0^\infty d\omega \left[\frac{1}{2}(\partial_t\hat{\boldsymbol{X}}_\omega)^2 + \frac{1}{2}(\partial_t\hat{\boldsymbol{Y}}_\omega)^2 + \frac{1}{2}\omega^2(\hat{\boldsymbol{X}}_\omega^2 + \hat{\boldsymbol{Y}}_\omega^2) - \frac{1}{2}\beta\left(\hat{\boldsymbol{Y}}_\omega\cdot\hat{\boldsymbol{B}} + \hat{\boldsymbol{B}}\cdot\hat{\boldsymbol{Y}}_\omega\right)\right]$$

$$\begin{aligned}\hat{\sigma}_{ij} =& \frac{1}{2}\delta_{ij}(\varepsilon_0\hat{\boldsymbol{E}}^2 + \kappa_0\hat{\boldsymbol{B}}^2) - \varepsilon_0\hat{E}_i\hat{E}_j - \kappa_0\hat{B}_iB_j \\ &+ \int_0^\infty d\omega \left\{\delta_{ij}\left[\frac{1}{2}(\partial_t\hat{\boldsymbol{X}}_\omega)^2 + \frac{1}{2}(\partial_t\hat{\boldsymbol{Y}}_\omega)^2 - \frac{1}{2}\omega^2(\hat{\boldsymbol{X}}^2_\omega + \hat{\boldsymbol{Y}}^2_\omega) + \frac{1}{2}\alpha\left(\hat{\boldsymbol{X}}_\omega\cdot\hat{\boldsymbol{E}} + \hat{\boldsymbol{E}}\cdot\hat{\boldsymbol{X}}_\omega\right)\right]\right. \\ &\left. - \frac{1}{2}\alpha\left(\hat{E}_i\hat{X}_{\omega j} + \hat{X}_{\omega j}\hat{E}_i\right) + \frac{1}{2}\beta\left(\hat{Y}_{\omega i}\hat{B}_j + \hat{B}_j\hat{Y}_{\omega i}\right)\right\}\end{aligned}$$

Casimir (i.e. zero-point and thermal) energy density and stress tensor are the electromagnetic part of stress-energy in thermal state.

Casimir energy density and stress tensor

Electromagnetic energy density of zero-point and thermal fields inside medium:

$$\langle\hat{\rho}\rangle = k_B T {\sum_{n=0}}' \left\{ \frac{1}{c^2} \frac{d[\omega\varepsilon(\boldsymbol{r},\omega)]}{d\omega}\bigg|_{\omega=i\xi_n} {\Delta^E}_i{}^i(\boldsymbol{r},i\xi_n) + \left[\frac{1}{\mu^2(\boldsymbol{r},\omega)} \frac{d[\omega\mu(\boldsymbol{r},\omega)]}{d\omega}\right]_{\omega=i\xi_n} {\Delta^B}_i{}^i(\boldsymbol{r},i\xi_n) \right\}$$

Sum over imaginary frequencies $i\xi_n := i\frac{2\pi k_B T n}{\hbar}$

Correlation function <EE> in frequency domain

Correlation function <BB> in frequency domain

Conflicting assertions about Casimir energy density inside media have been made – correct form emerges automatically from canonical macroscopic QED.

Electromagnetic stress tensor of zero-point and thermal fields inside medium:

$$\langle\hat{\sigma}_{ij}\rangle = 2k_B T {\sum_{n=0}}' \left\{ \frac{1}{c^2}\varepsilon(\mathbf{r},i\xi_n)\left[\frac{1}{2}\delta_{ij}{\Delta^E}_k{}^k(\mathbf{r},i\xi_n) - {\Delta^E}_{ij}(\mathbf{r},i\xi_n)\right]\right.$$
$$\left. +\kappa(\mathbf{r},i\xi_n)\left[\frac{1}{2}\delta_{ij}{\Delta^B}_k{}^k(\mathbf{r},i\xi_n) - {\Delta^B}_{ij}(\mathbf{r},i\xi_n)\right]\right\}.$$

This is the Lifshitz theory result. [Dzyaloshinskii, Lifshitz & Pitaevskii *Adv. Phys.* **10** (1961) 165]

Restrictions on constitutive relations from QED

For anisotropic media the permittivity and permeability tensors are *necessarily* symmetric in the canonical formulation, and hence in macroscopic QED:

$$\mathrm{Im}(\boldsymbol{\varepsilon}) = \frac{\pi}{2\varepsilon_0\omega}\boldsymbol{\alpha}(\omega)\boldsymbol{\alpha}^T(\omega) = \mathrm{Im}(\boldsymbol{\varepsilon}^T), \qquad \mathrm{Im}(\boldsymbol{\kappa}) = -\frac{\pi}{2\kappa_0\omega}\boldsymbol{\beta}(\omega)\boldsymbol{\beta}^T(\omega) = \mathrm{Im}(\boldsymbol{\kappa}^T)$$

The restriction to symmetric dielectric functions is usually derived from thermodynamics. Here it is seen to be a requirement for a canonical formulation and hence for a quantum theory.

Canonical macroscopic QED can be extended to magnetoelectric coupling [S. Horsley, arXiv:1106.2178].
Again, there are *automatic* restrictions, on the size of the magnetoelectric susceptibility:

$$\tilde{\boldsymbol{D}}(\omega) = \varepsilon_0\boldsymbol{\varepsilon}\cdot\tilde{\boldsymbol{E}}(\omega) + \boldsymbol{\chi}^{\mathrm{me}}\cdot\tilde{\boldsymbol{B}}(\omega), \qquad \tilde{\boldsymbol{H}}(\omega) = \kappa_0\boldsymbol{\kappa}\cdot\tilde{\boldsymbol{B}}(\omega) - \boldsymbol{\chi}^{\mathrm{me}\,T}\cdot\tilde{\boldsymbol{E}}(\omega)$$

$$\left[\mathrm{Im}(\chi^{\mathrm{me}}_{ij}(\omega))\right]^2 \leq -\varepsilon_0\kappa_0\mathrm{Im}(\varepsilon_{ii}(\omega))\mathrm{Im}(\kappa_{jj}(\omega)) \qquad \text{(no summation on repeated indices)}$$

In the special case of chiral media, this restriction has been derived from thermodynamics. Here, a more general restriction is an automatic requirement for a canonical formulation and hence for a quantum theory.

Summary

- Macroscopic electromagnetism has been canonically quantized.
- Macroscopic QED provides a rigorous quantum theory of light in general dispersive, absorptive media, including the Casimir effect.
- There is no need for phenomenological approaches to quantizing the macroscopic Maxwell equations.
- Macroscopic QED can be generalized to include magneto-electric coupling and spatial dispersion.
- Macroscopic QED, in requiring a canonical formulation of the classical theory, gives restrictions on allowable constitutive relations, some (but not all) of which have previously been understood only from thermodynamic considerations.