QFEXT2011, Benasque, September 22, 2011

# The Euler-Heisenberg Lagrangian beyond one loop

(with I. Huet and M. Rausch de Traubenberg)

1. QED in a constant external field in the wordline formalism.
2. The one-loop Euler-Heisenberg Lagrangian: weak field expansion, amplitudes, imaginary part.
3. The two-loop Euler-Heisenberg Lagrangian: mass renormalization, Lebedev-Ritus exponentiation.
4. An all-loop conjecture from worldline instantons.
5. The three-loop Euler-Heisenberg Lagrangian for 2D QED.
6. Conclusions.

## 1. QED in a constant external field in the wordline formalism.

Feynman 1950, 1951; E.S. Fradkin 1966; A.M. Polyakov 1987;
Z. Bern & D.A. Kosower 1992; M.J. Strassler 1992; M.G. Schmidt & C.S. 1993;
R. Shaisultanov 1996; M. Reuter, M.G. Schmidt & C.S. 1997; ...

Scalar QED:

*(Quenched) effective action* $\Gamma(A)$

$$\Gamma(A) \;=\; \int d^4x\, \mathcal{L}(A) = \int_0^\infty \frac{dT}{T}\, \mathrm{e}^{-m^2T} \int_{x(T)=x(0)} \mathcal{D}x(\tau)\, e^{-S[x(\tau)]}$$

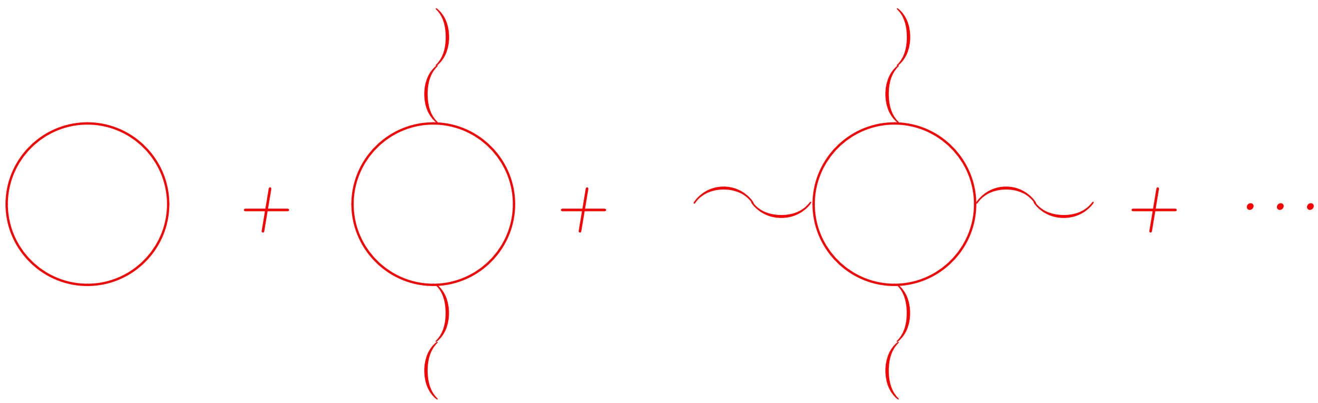
$T \;=\;$ *proper time of the loop scalar*

$S[x(\tau)] \;=\;$ *worldline action*

$$S \;=\; S_0 + S_{\rm ext} + S_{\rm int}$$

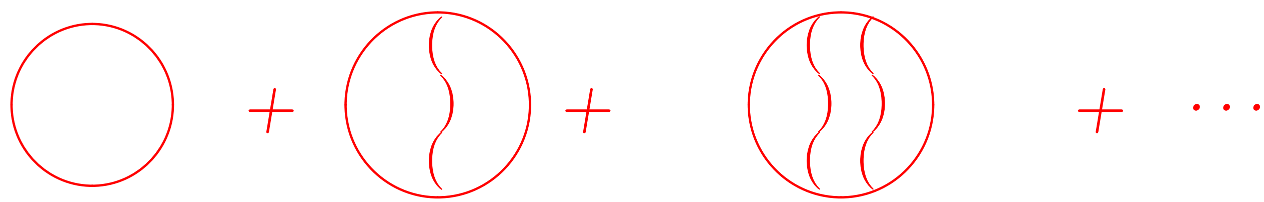
$$S_0 \;=\; \int_0^T d\tau \frac{\dot{x}^2}{4} \qquad \text{(free propagation)}$$

$$S_{\text{ext}} \;=\; ie\int_0^T \dot{x}^\mu A_\mu(x(\tau)) \qquad \text{(external photons)}$$



$$S_{\rm int} \;=\; -\frac{e^2}{8\pi^2}\int_0^T d\tau_1 \int_0^T d\tau_2 \frac{\dot{x}(\tau_1)\cdot\dot{x}(\tau_2)}{(x(\tau_1)-x(\tau_2))^2}$$

(internal photons)



*Generalize: Multiple scalar loops, open scalar lines, . . .*

$\rightarrow$ *First – quantized representation of* $\Gamma(A)$

$\rightarrow$ *First – quantized representation of the S-matrix*

## *Spinor QED*

*Spin 0 → Spin* $\frac{1}{2}$

*Represent electron spin by a Grassmann path integral*
*E.S. Fradkin, NPB 76 (1966) 588*

$$S[x,A] \to \int \mathcal{D}\psi(\tau)\, \exp\Big[-\int_0^T d\tau \Big(\frac{1}{2}\psi\cdot\dot\psi - ie\psi^\mu F_{\mu\nu}\psi^\nu\Big)\Big]$$

$$\begin{aligned} \psi(\tau_1)\psi(\tau_2) &= -\psi(\tau_2)\psi(\tau_1) \\ \psi(T) &= -\psi(0) \end{aligned}$$

*Calculation methods:*

- *The analytical ("string-inspired") approach (A. Polyakov 1987, M. J . Strassler 1992,...).*
- *The semiclassical ("worldline instanton") approach (I.K. Affleck, O. Alvarez and N.S. Manton 1982,...).*
- *Worldline Monte Carlo (H. Gies and K. Langfeld 2001 .....).*

## The string-inspired approach

*Strategy:*

1. *Manipulate the path integral into gaussian form*
2. *Wick contract using worldline correlators*

$$
\begin{aligned}
\langle x^\mu(\tau_1) x^\nu(\tau_2)\rangle &= -G_B(\tau_1,\tau_2)\,\delta^{\mu\nu} \\
G_B(\tau_1,\tau_2) &= |\tau_1-\tau_2| - \frac{(\tau_1-\tau_2)^2}{T}
\end{aligned}
$$

$$
\begin{aligned}
\langle \psi^\mu(\tau_1) \psi^\nu(\tau_2)\rangle &= G_F(\tau_1,\tau_2)\,\delta^{\mu\nu} \\
G_F(\tau_1,\tau_2) &= \mathrm{sign}(\tau_1-\tau_2)
\end{aligned}
$$

*Bern-Kosower master formula*

$$\Gamma[\{k_i,\varepsilon_i\}] \;=\; (-ie)^N(2\pi)^D\delta(\sum k_i)\int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}}e^{-m^2T}\prod_{i=1}^{N}\int_0^T d\tau_i$$
$$\times \exp\{\sum_{i,j=1}^{N}[\frac{1}{2}G_{Bij}k_i\cdot k_j + i\dot{G}_{Bij}k_i\cdot\varepsilon_j + \frac{1}{2}\ddot{G}_{Bij}\varepsilon_i\cdot\varepsilon_j]\}\;|_{\mathrm{lin}(\varepsilon_1,\dots,\varepsilon_N)}$$

# *QED in a constant external field*

*Easy implementation of a constant background field* $F_{\mu\nu}$*:*
*In Fock-Schwinger gauge,*

$$A^\mu(x) = -\frac{1}{2}F^{\mu\nu}x_\nu$$

*so that the worldline Lagrangian changes only by quadratic terms:*

$$L \to L + \frac{1}{2}iex^\mu F_{\mu\nu}\dot{x}^\nu - ie\psi^\mu F_{\mu\nu}\psi^\nu$$

*Thus this change can be absorbed into the Green's functions and path integral determinants:*

- *Change worldline Green's functions*

$$G_B(\tau_1,\tau_2) \to \mathcal{G}_B(\tau_1,\tau_2) = \frac{1}{2(eF)^2}\Bigl(\frac{eF}{\sin(eFT)}\mathrm{e}^{-ieFT\dot{G}_{B12}} + ieF\dot{G}_{B12} - \frac{1}{T}\Bigr)$$

$$G_F(\tau_1,\tau_2) \to \mathcal{G}_F(\tau_1,\tau_2) = G_{F12}\,\frac{\mathrm{e}^{-ieFT\dot{G}_{B12}}}{\cos(eFT)}$$

- *Change free path integral determinants*

$$(4\pi T)^{-\frac{D}{2}} \;\to\; (4\pi T)^{-\frac{D}{2}}\,\mathrm{det}^{-\frac{1}{2}}\Bigl[\frac{\sin eFT}{eFT}\Bigr] \qquad \text{(Scalar QED)}$$

$$(4\pi T)^{-\frac{D}{2}} \;\to\; (4\pi T)^{-\frac{D}{2}}\,\mathrm{det}^{-\frac{1}{2}}\Bigl[\frac{\tan eFT}{eFT}\Bigr] \qquad \text{(Spinor QED)}$$

*R. Shaisultanov, PLB 378, 354 (1996)*
*M. Reuter, M.G. Schmidt, C.S., Ann. Phys. 259, 313 (1997)*

*Bern-Kosower type master formula for the scalar QED $N$ - photon scattering amplitude in a constant field:*

$$\Gamma_{\rm scal}[\{k_i,\varepsilon_i\}] = (-ie)^N (2\pi)^D \delta(\sum k_i) \int_0^\infty \frac{dT}{T} (4\pi T)^{-\frac{D}{2}} e^{-m^2 T} {\rm det}^{-\frac{1}{2}} [\frac{\sin(eFT)}{eFT}]$$
$$\times \prod_{i=1}^N \int_0^T d\tau_i \exp\{\sum_{i,j=1}^N [\frac{1}{2} k_i \cdot \mathcal{G}_{Bij} \cdot k_j - i\varepsilon_i \cdot \dot{\mathcal{G}}_{Bij} \cdot k_j + \frac{1}{2}\varepsilon_i \cdot \ddot{\mathcal{G}}_{Bij} \cdot \varepsilon_j]\} \mid_{{\rm lin}(\varepsilon_1,\ldots,\varepsilon_{\rm N})}$$

*This formula is valid off-shell* $\rightarrow$ *can use it to construct multiloop Euler-Heisenberg Lagrangians by sewing off pairs of photons with propagators.*

## 2. The one-loop Euler-Heisenberg Lagrangian: weak field expansion, amplitudes, imaginary part

*The one-loop EHL just comes from the determinant factors* $\det^{-\frac{1}{2}}\left[\frac{\sin eFT}{eFT}\right], \det^{-\frac{1}{2}}\left[\frac{\tan eFT}{eFT}\right]$, *up to renormalization:*

*Scalar QED:*

$$\mathcal{L}^{(1)}_{\rm scal}(F) = \frac{1}{16\pi^2}\int_0^\infty \frac{dT}{T^3}\, e^{-m^2T}\Big[\frac{(eaT)(ebT)}{\sinh(eaT)\sin(ebT)} + \frac{e^2}{6}(a^2-b^2)T^2 - 1\Big]$$

*Spinor QED:*

$$\mathcal{L}^{(1)}_{\rm spin}(F) = -\frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T^3}\, e^{-m^2T}\Big[\frac{(eaT)(ebT)}{\tanh(eaT)\tan(ebT)} - \frac{e^2}{3}(a^2-b^2)T^2 - 1\Big]$$

*Here* $a, b$ *are the two invariants of the Maxwell field, related to* $\mathbf{E}$, $\mathbf{B}$ *by* $a^2 - b^2 = B^2 - E^2, \quad ab = \mathbf{E}\cdot\mathbf{B}$.

*The EHL has the information on*

- *The $N$ photon amplitudes in the low energy limit (where all photon energies are small compared to the electron mass, $\omega_i \ll m$ ). The amplitudes can be constructed explicitly from the weak field expansion coefficients $c_{kl}$, defined by*

$$\mathcal{L}(F) = \sum_{k,l} c_{kl}\, a^{2k} b^{2l}$$

- *Schwinger pair creation, represented by the imaginary part of the EHL:*

$$\begin{aligned}
\mathrm{Im}\mathcal{L}^{(1)}_{\mathrm{spin}}(E) &= \frac{m^4}{8\pi^3}\beta^2 \sum_{k=1}^{\infty} \frac{1}{k^2} \exp\left[-\frac{\pi k}{\beta}\right] \\
\mathrm{Im}\mathcal{L}^{(1)}_{\mathrm{scal}}(E) &= -\frac{m^4}{16\pi^3}\beta^2 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \exp\left[-\frac{\pi k}{\beta}\right]
\end{aligned}$$

*($\beta = eE/m^2$).*

*The electric field creates electron-positron pairs (vacuum tunneling).*

*Weak field limit $\beta \ll 1 \Rightarrow$ only $k = 1$ relevant.*

*At any loop order, the photon amplitudes and Schwinger pair creation rates are connected by a Borel dispersion relation :*

*Weak field expansion at $l$ loops:*

$$\mathcal{L}^{(l)}(E) = \sum_{n=2}^{\infty} c^{(l)}(n) (\frac{eE}{m^2})^{2n}$$

$$c^{(l)}(n) \overset{n\to\infty}{\sim} c_{\infty}^{(l)} \Gamma[2n-2]$$

*Leading asymptotic growth rate same at each loop order*

*(S.L. Lebedev, V.I. R 1984; G.V. Dunne & CS)*

*Borel dispersion relation:*

$$\mathrm{Im}\mathcal{L}^{(l)}(E) \sim c_{\infty}^{(l)} \, \mathrm{e}^{-\frac{\pi m^2}{eE}}$$

*for* $\beta \to 0$.

## 3. The two-loop Euler-Heisenberg Lagrangian: mass renormalization, Lebedev-Ritus exponentiation

$\mathcal{L}^{(2)}_{\rm scal,spin}(F)$ *known only in terms of intractable two parameter integrals* *(V. I. Ritus 1975, S.L. Lebedev & V.I. Ritus 1984, M. Reuter, M.G. Schmidt & C.S. 1997)*

*However, the first few weak field coefficients can be calculated, and there is a Schwinger-type formula for* ${\rm Im}\mathcal{L}^{(2)}_{\rm spin}(E)$*:*

$$
{\rm Im}\mathcal{L}^{(2)}_{\rm spin}(E) \;=\; \frac{m^4}{8\pi^3}\beta^2 \sum_{k=1}^{\infty} \alpha\pi K_k(\beta)\, \exp\left[-\frac{\pi k}{\beta}\right]
$$

$(\alpha = \frac{e^2}{4\pi})$

$$
K_k(\beta) \;=\; -\frac{c_k}{\sqrt{\beta}} + 1 + {\rm O}(\sqrt{\beta})
$$

$$
c_1 = 0, \qquad\qquad c_k = \frac{1}{2\sqrt{k}} \sum_{l=1}^{k-1} \frac{1}{\sqrt{l(k-l)}}, \quad k \geq 2
$$

*Weak field limit:*

$$\mathrm{Im}\mathcal{L}^{(1)}_{\mathrm{spin}}(E) + \mathrm{Im}\mathcal{L}^{(2)}_{\mathrm{spin}}(E) \overset{\beta\to 0}{\sim} \frac{m^4\beta^2}{8\pi^3}(1+\alpha\pi)\,\mathrm{e}^{-\frac{\pi}{\beta}}$$

*S.L. Lebedev & V.I. Ritus 1984: Assuming that higher orders will lead to exponentiation*

$$\mathrm{Im}\mathcal{L}^{(1)}_{\mathrm{spin}}(E) + \mathrm{Im}\mathcal{L}^{(2)}_{\mathrm{spin}}(E) + \mathrm{Im}\mathcal{L}^{(3)}_{\mathrm{spin}}(E) + \dots \overset{\beta\to 0}{\sim} \frac{m^4\beta^2}{8\pi^3}\,\mathrm{e}^{\alpha\pi}\,\mathrm{e}^{-\frac{\pi}{\beta}}$$

*then the result can be interpreted in the tunneling picture as the corrections to the Schwinger pair creation rate due to the pair being created with a negative Coulomb interaction energy* .

For Scalar QED, this formula was already known.....

## 4. An all-loop conjecture from worldline instantons

AAM conjecture (I.K. Affleck, O. Alvarez, N.S. Manton 1982):

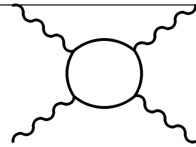
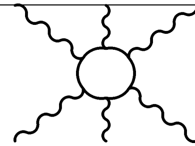
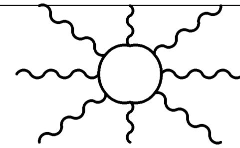
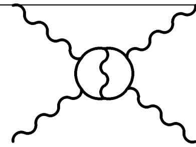
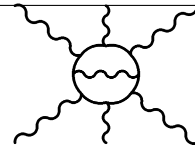
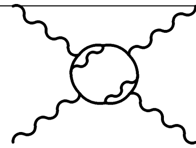
For Scalar QED and in the weak field limit,

$$\sum_{l=1}^{\infty} \mathrm{Im}\mathcal{L}^{(l)}_{\mathrm{scal}}(E) \overset{\beta\to 0}{\sim} -\frac{m^4\beta^2}{16\pi^3}\exp[-\frac{\pi}{\beta}+\alpha\pi]$$
$$= \mathrm{Im}\mathcal{L}^{(1)}_{\mathrm{scal}}(E)\, \mathrm{e}^{\alpha\pi}$$

Remarkable:

- True all-loop result, receives contributions from an infinite set of graphs of arbitrary loop order (although non-quenched diagrams get suppressed in this limit).
- Includes mass renormalization! (!?)
- Extremely simple derivation (semi-classical worldline instanton approximation of the worldline path integral representation of $\mathcal{L}_{\mathrm{scal}}(E)$).

## In terms of Feynman diagrams:

| | Number of external legs | | | |
|---|---|---|---|---|
| Number of loops | 4 | 6 | 8 | ⋯ |
| 1 |  |  |  | ⋯ |
| 2 |  |  | ⋯ | ⋯ |
| 3 |  | ⋯ | ⋱ | ⋮ |
| ⋮ | ⋮ | ⋱ | ⋱ | ⋮ |

But a direct calculation of these diagrams is hopeless...

Strange: All-order loop summation has produced the finite factor $e^{\alpha\pi}$ !
Such things are not supposed to happen in QED.

C.V. Dunne & C.S., 2004:
AAM conjecture + modest assumptions $\rightarrow$ Convergence of the perturbation series for the QED $N$ photon in the quenched (one electron loop) approximation.

(Using Borel analysis and spinor helicity).

- This generalizes a 1977 conjecture by Cvitanovic on $g-2$.
- If true, it would indicate extensive cancellations between Feynman diagrams, presumably due to gauge invariance.

## Three predictions/consistency checks for the three-loop EHL:

1. $\lim_{n\to\infty} \frac{c^{(3)}(n)}{c^{(1)}(n)} = \frac{1}{2}\alpha^2$
2. Only the quenched part contributes to this limit.
3. The convergence of $\frac{c^{(3)}(n)}{c^{(1)}(n)}$ should not be slower than the one of $\frac{c^{(2)}(n)}{c^{(1)}(n)}$.

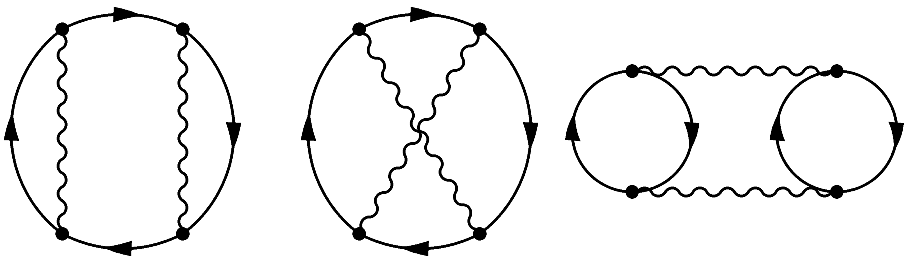
NOT EASY TO VERIFY:

- First attempt: Calculation of the 3-loop EHL in $D = 4$ – too hard! No essential dependence on dimension $\rightarrow$ Try to show the analogous statements for the massive Schwinger model.
- Second attempt: Feynman diagram calculation of the 3-loop 2D EHL failed because of spurious IR divergences ($\rightarrow$ QFEXT1009).
- Third attempt: Calculation of the 3-loop 2D EHL in the worldline formalism yielded a manifestly IR (and UV) finite parameter integral.

## 5. The three-loop Euler-Heisenberg Lagrangian for 2D QED

(I. Huet, M. Rausch de Traubenberg and C.S., work in progress)

Three Feynman diagrams at three-loop:



A B C

Solid line = fermion propagator in the constant field,

$$F = \begin{pmatrix} 0 & f \\ -f & 0 \end{pmatrix}$$

A,B quenched, C non-quenched.

Diagram C is easy, and gives

$$
\begin{aligned}
\mathcal{L}^{3C}(f) &= \frac{e^3}{16\pi^3 f}\int_0^\infty dz dz' d\hat{z} dz'' \frac{\sinh z \sinh z' \sinh \hat{z} \sinh z''}{[\sinh(z+z')\sinh(\hat{z}+z'')]^2} \\
&\times \frac{\mathrm{e}^{-2\kappa(z+z'+\hat{z}+z'')}}{\sinh z \sinh z' \sinh(\hat{z}+z'') + \sinh \hat{z} \sinh z'' \sinh(z+z')}
\end{aligned}
$$

($\kappa = m^2/2ef$).

From this we got the first 12 weak field expansion coefficients, which was sufficient to verify that they are indeed asymptotically (exponentially) suppressed.

The results for diagrams A + B are lengthy:

$$\mathcal{L}^{3(A+B)}_{\rm spin}(f) \;=\; -\frac{e^4}{(4\pi)^3}\int_0^\infty \frac{dT}{T^2}\mathrm{e}^{-m^2T}\frac{Z}{\tanh Z}\prod_{i=1}^4\int_0^T d\tau_i$$
$$\times(2I_{1234}+I_{1324}+4I_{123}+2I_{12}+4I_{13}+I_{12,34}+2I_{13,24})$$

where, for example,

$$I_{ijkl} \;=\; \frac{\mathrm{tr}(\{ijkl\}_S)}{\Delta}$$

with

$$\{i_1 i_2 \ldots i_n\}_S \;:=\; \dot G_{Bi_1i_2}\dot G_{Bi_2i_3}\cdots \dot G_{Bi_ni_1} - \mathcal{G}_{Fi_1i_2}\mathcal{G}_{Fi_2i_3}\cdots\mathcal{G}_{Fi_ni_1}$$

($\Delta$ is a determinant also involving the worldline Green's functions).

- This is already the sum of A and B.
- Manifestly UV and IR finite term-by-term.

IN PROGRESS: Calculation of the expansion coefficients for the quenched part...

FINAL RESULTS WILL BE SHOWN IN QFEXT2013 !