QFEXT11

Nonstandard recursion relations for Fresnel coefficients: Casimir force across a layered medium

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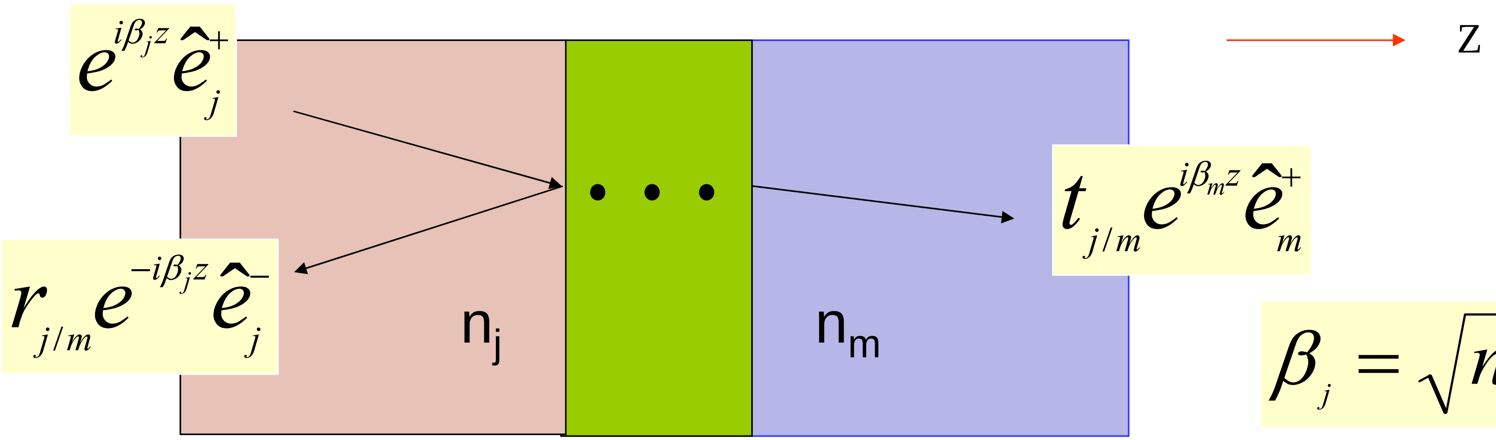
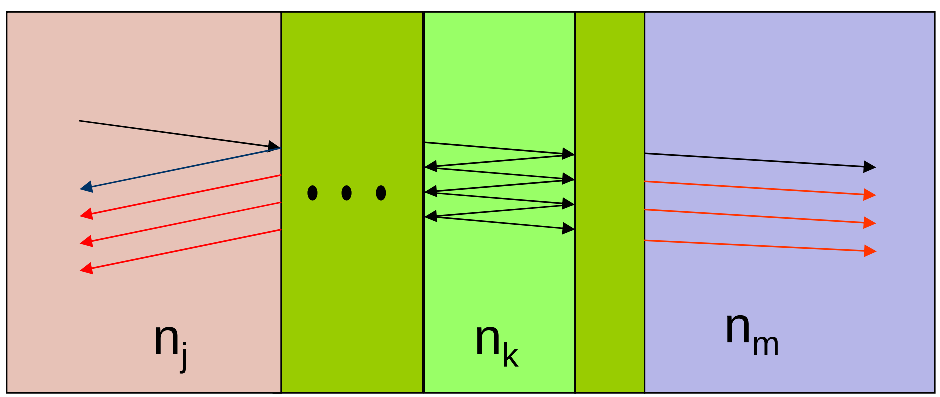
Abstract. Instead of using usual recursion relations involving successive layers, Fresnel coefficients of a complex layered system can equivalently be calculated using recursion relations involving successive stacks of layers. Using these nonstandard recursion relations, in this work we extend the Lifshitz formula to a configuration with an inhomogeneous, n-layered, medium separating two planar objects. The theory correctly reproduces very recently obtained results for the Casimir force/energy in some simple systems of this kind. As a by product, we obtain the formula for the force on an (unspecified) stack of layers between two planar objects which generalizes our previous result for the force on a slab in a planar cavity.

Generalized Fresnel coefficients: Definition



$$\beta_j = \sqrt{n_j^2(\omega)\omega^2/c^2 - k^2}$$

Note that $r_{j/m}$, $t_{j/m}$, $r_{m/j}$ and $t_{m/j}$ are Fresnel coefficients of a planar object (a stack of layers) between two semi-infinite local layers j and m (that is, of the system j…m denoted shortly as j/m).



If there is an intermediate local layer k in the stack, from the above definition it follows that

$$t_{j/m} = t_{j/k} e^{i\beta_k d_k}\left(1 + r_{k/m} r_{k/j} e^{2i\beta_k d_k} + \ldots\right) t_{k/m},$$

$$r_{j/m} = r_{j/k} + t_{j/k} r_{k/m} e^{2i\beta_k d_k}\left(1 + r_{k/j} r_{k/m} e^{2i\beta_k d_k} + \ldots\right) t_{k/j}.$$

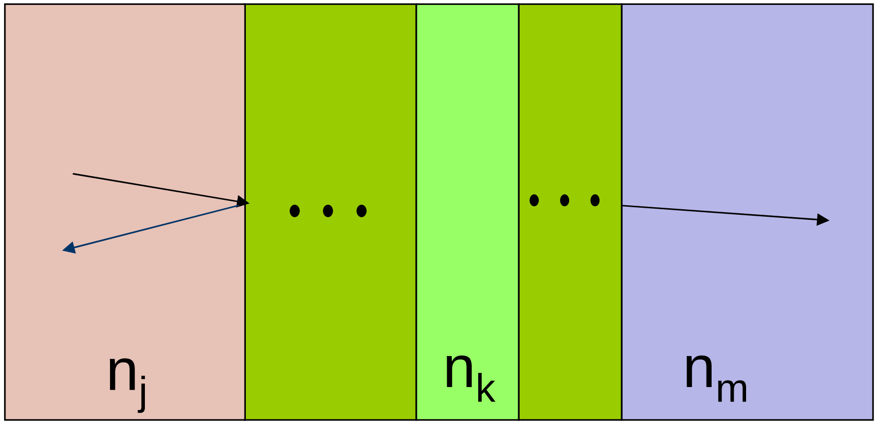
Nonstandard recurrence relations

This leads to the following recursion relations [1]:

$$t_{j/m} = \frac{t_{j/k} t_{k/m} e^{i\beta_k d_k}}{1 - r_{k/j} r_{k/m} e^{2i\beta_k d_k}} \equiv t_{j/k/m},$$

$$r_{j/m} = \frac{r_{j/k} + a_{j/k} r_{k/m} e^{2i\beta_k d_k}}{1 - r_{k/j} r_{k/m} e^{2i\beta_k d_k}} \equiv r_{j/k/m}, \qquad \textbf{(1)}$$

$$a_{j/k} = t_{j/k} t_{k/j} - r_{j/k} r_{k/j} = a_{k/j}.$$



Quantity $a_{j/m}$ itself obeys recurrence relation

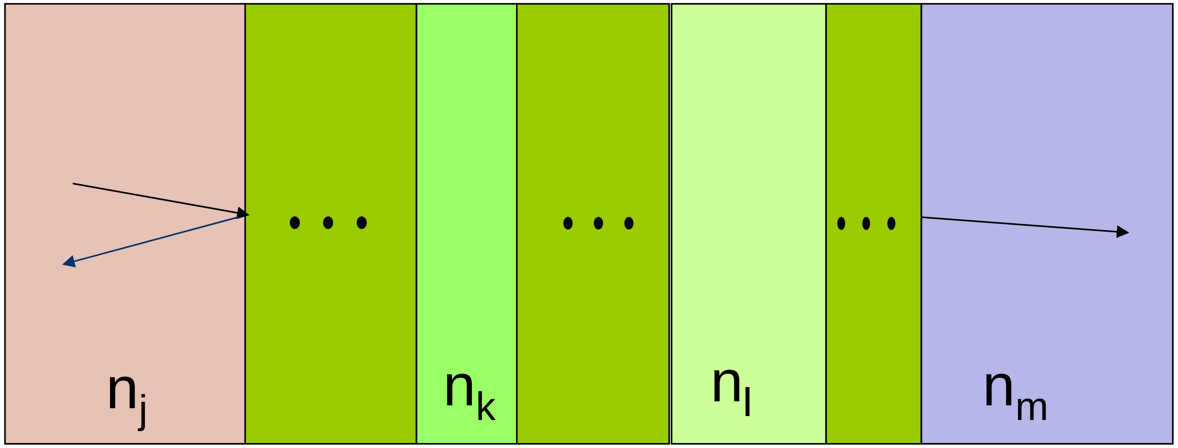
$$a_{j/m} = \frac{a_{j/k} a_{m/k} e^{2i\beta_k d_k} - r_{j/k} r_{m/k}}{1 - r_{k/j} r_{k/m} e^{2i\beta_k d_k}} \equiv a_{j/k/m},$$

which can be regarded as the generalized Stokes relation [1].

Equivalence

Clearly, Fresnel coefficients do not depend on the choice of the intermediate layer. Thus, as can be easily verified using (1) [1], if there is another local layer l in the stack in addition to the layer k one has

$$t_{j/k/m} = t_{j/l/m} \text{ and } r_{j/k/m} = r_{j/l/m},$$



Accordingly, since for successive local layers j and k the Stokes relation

$$a_{j/k} \equiv a_{jk} = t_{jk} t_{kj} - r_{jk} r_{kj} = 1$$

holds, for a fully local system the above recurrence relations are equivalent to the standard ones:

$$t_{j/m} \equiv t_{jk/m} = \frac{t_{jk} t_{k/m} e^{i\beta_k d_k}}{1 - r_{kj} r_{k/m} e^{2i\beta_k d_k}}, \quad r_{j/m} \equiv r_{jk/m} = \frac{r_{jk} + r_{k/m} e^{2i\beta_k d_k}}{1 - r_{kj} r_{k/m} e^{2i\beta_k d_k}}.$$

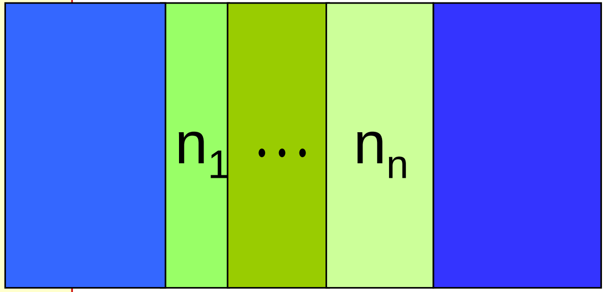
Casimir force across a layered medium

We consider the Casimir effect between two planar objects (plates) separated by an n-layered magnetodielectric medium. According to the theory of the Casimir force in multilayers [2], the forces on the left (L) and right (R) plate are given by

$$F_L = \mathrm{T}_{zz}^{(1)} \quad \text{and} \quad F_R = -\mathrm{T}_{zz}^{(n)},$$

$$\mathrm{T}_{zz}^{(j)} = \frac{\hbar}{2\pi^2}\int_0^\infty d\xi \int_0^\infty dk k \kappa_j \sum_{pol} \frac{r_{j-} r_{j+} e^{-2\kappa_j d_j}}{1 - r_{j-} r_{j+} e^{-2\kappa_j d_j}},$$

$$\kappa_j = -i\beta_j(i\xi, k), \qquad r_{j\pm} \Rightarrow \text{right/left refl. coeff's}$$



Letting $r_{1-} = R_L$ and $r_{n+} = R_R$, and using (1) to write

$$r_{1+} = \frac{r_{1/n} + a_n R_R e^{-2\kappa_n d_n}}{1 - r_{n/1} R_R e^{-2\kappa_n d_n}}, \quad r_{n-} = \frac{r_{n/1} + a_n R_L e^{-2\kappa_1 d_1}}{1 - r_{1/n} R_L e^{-2\kappa_1 d_1}}, a_n = t_{1/n} t_{n/1} - r_{1/n} r_{n/1},$$

we obtain [1]

$$F_L = \frac{\hbar}{2\pi^2}\int_0^\infty d\xi \int_0^\infty dk k \kappa_1 \sum_{pol} \frac{1}{N_n}\left(r_{1/n} + a_n R_R e^{-2\kappa_n d_n}\right) R_L e^{-2\kappa_1 d_1},$$

$$F_R = \frac{-\hbar}{2\pi^2}\int_0^\infty d\xi \int_0^\infty dk k \kappa_n \sum_{pol} \frac{1}{N_n}\left(r_{n/1} + a_n R_L e^{-2\kappa_1 d_1}\right) R_R e^{-2\kappa_n d_n},$$

$$N_n = 1 - \left(r_{1/n} R_L e^{-2\kappa_1 d_1} + r_{n/1} R_R e^{-2\kappa_n d_n}\right) - a_n R_L e^{-2\kappa_1 d_1} R_R e^{-2\kappa_n d_n}. \qquad \textbf{(3)}$$

Note that the forces are not equal in magnitude unless the medium is symmetric across the gap.

Casimir energy

From [2] $F_L = \partial E / \partial d_1$ or $F_R = -\partial E / \partial d_n$, we have

$$E = \frac{\hbar}{(2\pi)^2}\int_0^\infty d\xi \int_0^\infty dk k \sum_{pol} \ln N_n. \qquad \textbf{(4)}$$

Using recursion relations (1), E for more complex systems can be written in a number of different ways. Clearly, to obtain the effective Casimir energy, we can drop from these results the terms not involving d_1 or d_n.

Force on the stack

$$F = \mathrm{T}_{zz}^{(n)} - \mathrm{T}_{zz}^{(1)} = -F_R - F_L =$$

$$\frac{\hbar}{2\pi^2}\int_0^\infty d\xi \int_0^\infty dkk \sum_{pol} \frac{1}{N_n}\left(\kappa_n r_{n/1} R_R e^{-2\kappa_n d_n} - \kappa_1 r_{1/n} R_L e^{-2\kappa_1 d_1}\right).$$

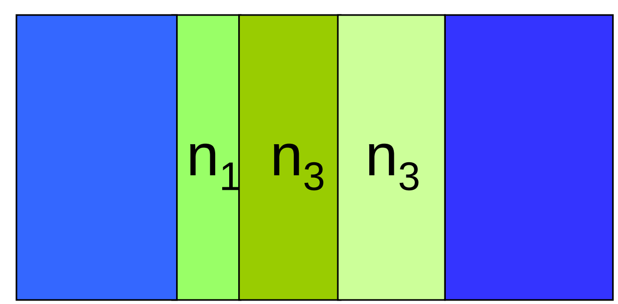
This generalizes our previous result for the force on a slab in a planar cavity [1,2].

System with a three-layer medium

In order to compare this result with those of several very recent papers dealing with the theory of the Casimir effect across a layered medium [3-5], we calculate the Casimir energy for a system with a three-layer intervening medium. From (1) and (2), we have

$$r_{1/3} = \frac{r_{12} + r_{23} e^{-2\kappa_2 d_2}}{D_2}, \quad r_{3/1} = \frac{r_{32} + r_{21} e^{-2\kappa_2 d_2}}{D_2}$$

$$a_3 = \frac{e^{-2\kappa_2 d_2} - r_{12} r_{32}}{D_2}, \quad D_2 = 1 - r_{21} r_{23} e^{-2\kappa_2 d_2},$$



which leads to $N_3 = \tilde{N}_3 / D_2$.

Thus, when dropping the (ineffective) term involving D_2,

$$E_{\text{eff}} = \frac{\hbar}{(2\pi)^2}\int_0^\infty d\xi \int_0^\infty dkk \sum_{pol} \ln \tilde{N}_3,$$

where

$$\tilde{N}_3 = \left(1 - r_{12} R_L e^{-2\kappa_1 d_1}\right)\left(1 - r_{32} R_R e^{-2\kappa_3 d_3}\right) - e^{-2\kappa_2 d_2}\left(R_L e^{-2\kappa_1 d_1} - r_{12}\right)\left(R_R e^{-2\kappa_3 d_3} - r_{32}\right).$$

With [1,2] $r_{ij}^{\mathrm{TM}} = \dfrac{\varepsilon_j \kappa_i - \varepsilon_i \kappa_j}{\varepsilon_j \kappa_i + \varepsilon_i \kappa_j}$, $r_{ij}^{\mathrm{TE}} = \dfrac{\mu_j \kappa_i - \mu_i \kappa_j}{\mu_j \kappa_i + \mu_i \kappa_j}$,

and assuming perfectly reflecting dielectric plates so that $R_{L(R)}^{\mathrm{TM}} = 1$ and $R_{L(R)}^{\mathrm{TE}} = -1$, this result agrees with that obtained in Ref. [3] whereas the results obtained in Refs. [4] and [5] correspond to plates with $R_{L(R)}^{\mathrm{TM\,(TE)}} = -1$ and $R_{L(R)}^{\mathrm{TM\,(TE)}} = 1$, respectively.

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