Sommerfeld's image method in the calculation of van der Waals force

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Motivation

- Perfectly conducting parallel plates -> Casimir force is always attractive.

- Technological problems: NEMS & MEMS.

- Atom-plane with a hole -> Levin et al. (2010)
  - Aim: Get the analytical result.

Setting the problem

- Atom in the presence of a surface.

- Non-retarded regime ($d \ll \lambda$):
  - Only the atom is quantized.

- Eberlein-Zietal method (2006)

Eberlein-Zietal Method

- Non-retarded regime -> EM field is not quantized.

- Force between an atom of dipole momentum operator **d** and an *arbitrary* perfectly conducting surface.

- Enables to change a QM problem in an electrostatic one.

Eberlein's Method

- Energy of interaction:

$$V = \frac{1}{2\varepsilon_0}(\mathbf{d}.\nabla)(\mathbf{d}.\nabla')G_H(\mathbf{r},\mathbf{r}')\Big|_{\mathbf{r}=\mathbf{r}_0;\mathbf{r}'=\mathbf{r}_0}.$$

where G_H is

$$\begin{cases} -\nabla^2 G_H(\mathbf{r},\mathbf{r}') = 0, \\ \left(\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} + G_H(\mathbf{r},\mathbf{r}')\right)_{\mathbf{r}\in \text{sup}.} = 0 \end{cases}$$

- If our problem admits an image, G_H will be the potential generated by the image.

Eberlein's Method

- Atom without permanent dipole and orthonormal basis

$$\langle d_i d_j \rangle = \delta_{ij} \langle d_i^2 \rangle$$

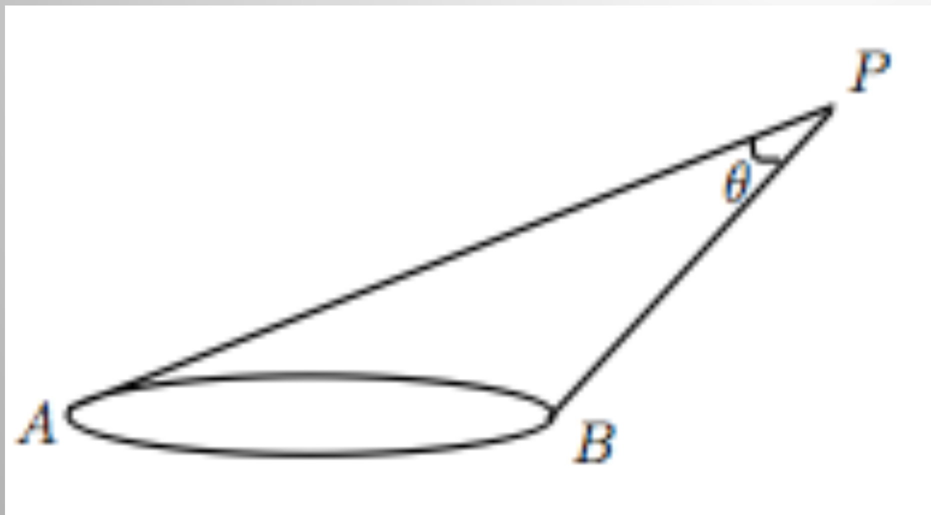
- First order:

$$\Delta E = \frac{1}{2\varepsilon_0} \sum_i \langle d_i^2 \rangle \partial_i{}' \partial_i G_H(\mathbf{r},\mathbf{r}')\Big|_{\mathbf{r}=\mathbf{r}_0;\mathbf{r}'=\mathbf{r}_0} .$$

C.Neumann's Peripolars

- Appropriate coordinate system for Levin's problem -> Peripolars:

- Symmetrical axis in the plane ΔAPB.



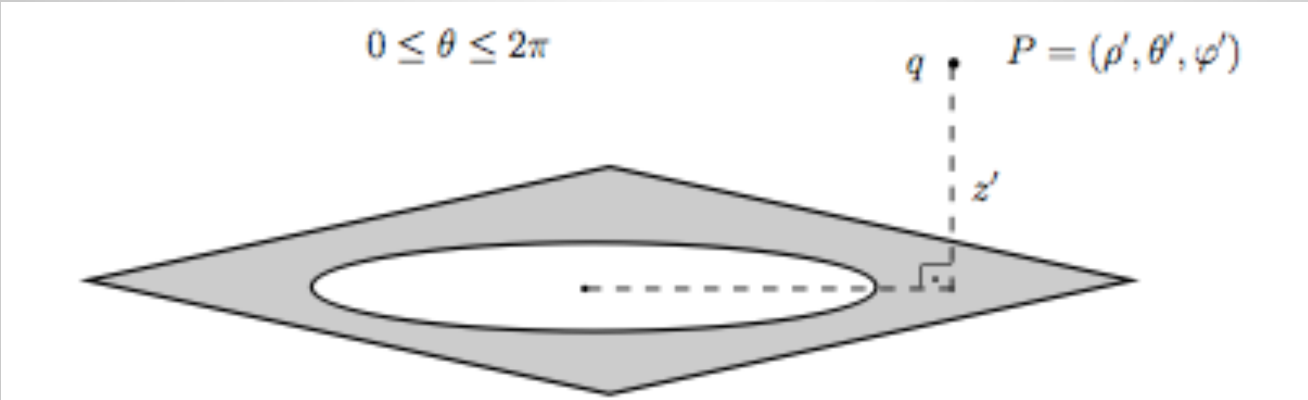
$$\theta = A\hat{P}B$$

$$\rho = \log \frac{PA}{PB}$$

$$\phi$$

The solution of Sommerfeld

- Charge-Plane with a hole (Davis-1971)



- Conducting surface: $\theta = 0,\ \theta = 2\pi$

The solution of Sommerfeld

- Discontinuity!

- Two fold space:

1º) REAL -> $0 \leq \theta < 2\pi$

2º) IMAGINARY -> $2\pi \leq \theta < 4\pi$

The solution of Sommerfeld

- Potential of a single charge at **r’**:

$$V = \frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{R}$$

- That’s wrong in the double space!

- It correspond to two charges:

$$\begin{cases} (\rho', \theta', \phi') \\ (\rho', \theta' + 2\pi, \phi') \end{cases}$$

The solution of Sommerfeld

- We must recognize in *V* the superposition of the potential of two charges!

- Cauchy's theorem:

$$\frac{1}{R(z)} = \oint \frac{R^{-1}(z')}{z' - z} dz'$$

- We choose the variable

$$z = e^{i\theta/2} \quad \text{and} \quad z' = e^{i\alpha/2}$$

The solution of Sommerfeld

- We may write

$$\frac{1}{R} = \oint \frac{R_\alpha^{-1}}{1 - e^{i(\theta - \alpha)/2}} d\alpha$$

- R_α^{-1} must be analytical in the contour.

$$\frac{1}{R_\alpha} = \frac{1}{a\sqrt{2}} \frac{(\cosh\rho - \cos\alpha)^{1/2}(\cosh\rho' - \cos\theta')^{1/2}}{\left\{\cosh\gamma - \cos(\alpha - \theta')\right\}^{1/2}},$$

The solution of Sommerfeld

- Where

$$\cosh\gamma = \cosh\rho\cosh\rho' - \sinh\rho\sinh\rho'\cos(\varphi - \varphi')$$

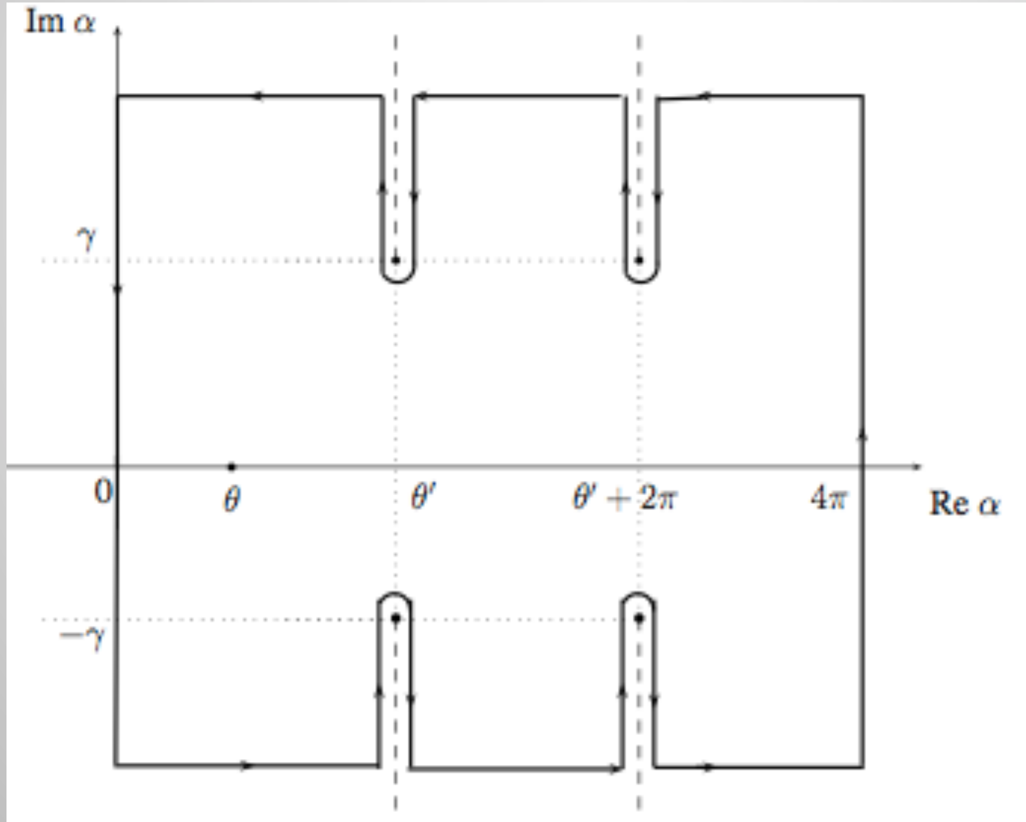
- Therefore:

$$R_{\alpha}^{-1} = 0 \Rightarrow \alpha = \theta' + 2m\pi \pm i\gamma$$

- Those are branch points!

The solution of Sommerfeld

- We choose the circuit:



The solution of Sommerfeld

- Hence,

$$\frac{1}{R} = \int_{A_0} \frac{R_\alpha^{-1}}{1 - e^{i(\theta-\alpha)/2}} d\alpha + \int_{A_1} \frac{R_\alpha^{-1}}{1 - e^{i(\theta-\alpha)/2}} d\alpha$$

- This must be the decomposition we seek!

The solution of Sommerfeld

- Sommerfeld has shown that the first term

1. Uniquely defined, finite and continuous except at $\mathbf{r} = (\rho', \theta', \phi')$
2. Obeys Laplacian equation, except at $\mathbf{r} = (\rho', \theta', \phi')$ and the conducting surface.
3. Vanishes at infinity
4. It's bivalent at ordinary space with a separate branch for each winding of Riemann space.

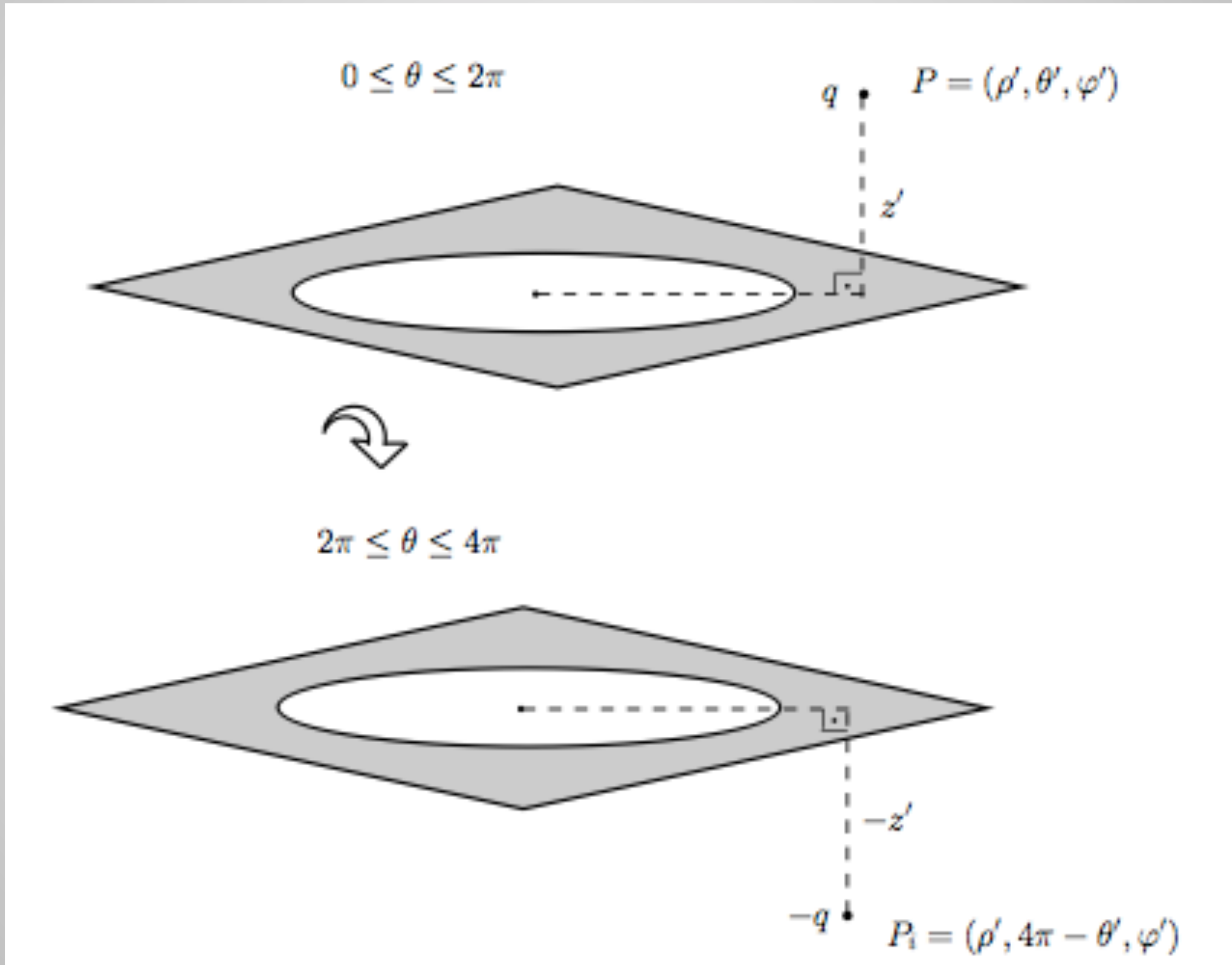
The solution of Sommerfeld

- Potential of one charge in the double space:

$$V_2(\rho,\theta,z) = \frac{q}{4\pi\varepsilon_0|\mathbf{r}-\mathbf{r}'|}\left[\frac{1}{2}+\frac{1}{\pi}\sin^{-1}\left\{\cos\frac{1}{2}(\theta-\theta')\sec h\frac{\gamma}{2}\right\}\right]$$

- Summing it with the potential of a charge at $(\rho',\theta'+2\pi,\phi')$ we obtain the newtonian potential.

Image



$0 \leq \theta \leq 2\pi$
q
$P = (\rho', \theta', \varphi')$
z'
$2\pi \leq \theta \leq 4\pi$
$-z'$
$-q$
$P_i = (\rho', 4\pi - \theta', \varphi')$

Image

- The homogeneous Green function is

$$V_{hole} = \frac{q}{4\pi\varepsilon_0 a\sqrt{2}} \times$$

$$\left\{ \frac{(\cosh\rho - \cos\theta)^{1/2}(\cosh\rho' - \cos\theta')^{1/2}}{\left\{\cosh\gamma - \cos(\theta - \theta')\right\}^{1/2}} \left[\frac{1}{2} + \frac{1}{\pi}\sin^{-1}\left\{ \cos\frac{1}{2}(\theta - \theta')\sec h\frac{\gamma}{2} \right\} \right] + \right.$$

$$\left. - \frac{(\cosh\rho - \cos\theta)^{1/2}(\cosh\rho' - \cos\theta')^{1/2}}{\left\{\cosh\gamma - \cos(\theta + \theta')\right\}^{1/2}} \left[\frac{1}{2} + \frac{1}{\pi}\sin^{-1}\left\{ \cos\frac{1}{2}(\theta + \theta')\sec h\frac{\gamma}{2} \right\} \right] \right\}$$

- It obeys BC!

The Homogeneous Green function

- We can use the method of the images by introducing images in the imaginary space!

- We must put it at $\mathbf{r}_i = (\rho', 4\pi - \theta', \phi')$.

- The solution is

$$G_H = \frac{\varepsilon_0 V_{hole}}{q} - \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}.$$

Dispersive interaction

- From Eberlein-Zietal method:

$$\Delta E = \frac{1}{2\varepsilon_0}\sum_i \langle d_i^2 \rangle \partial_i' \partial_i G_H(\mathbf{r},\mathbf{r}')\Big|_{\mathbf{r}=\mathbf{r}_0;\mathbf{r}'=\mathbf{r}_0}.$$

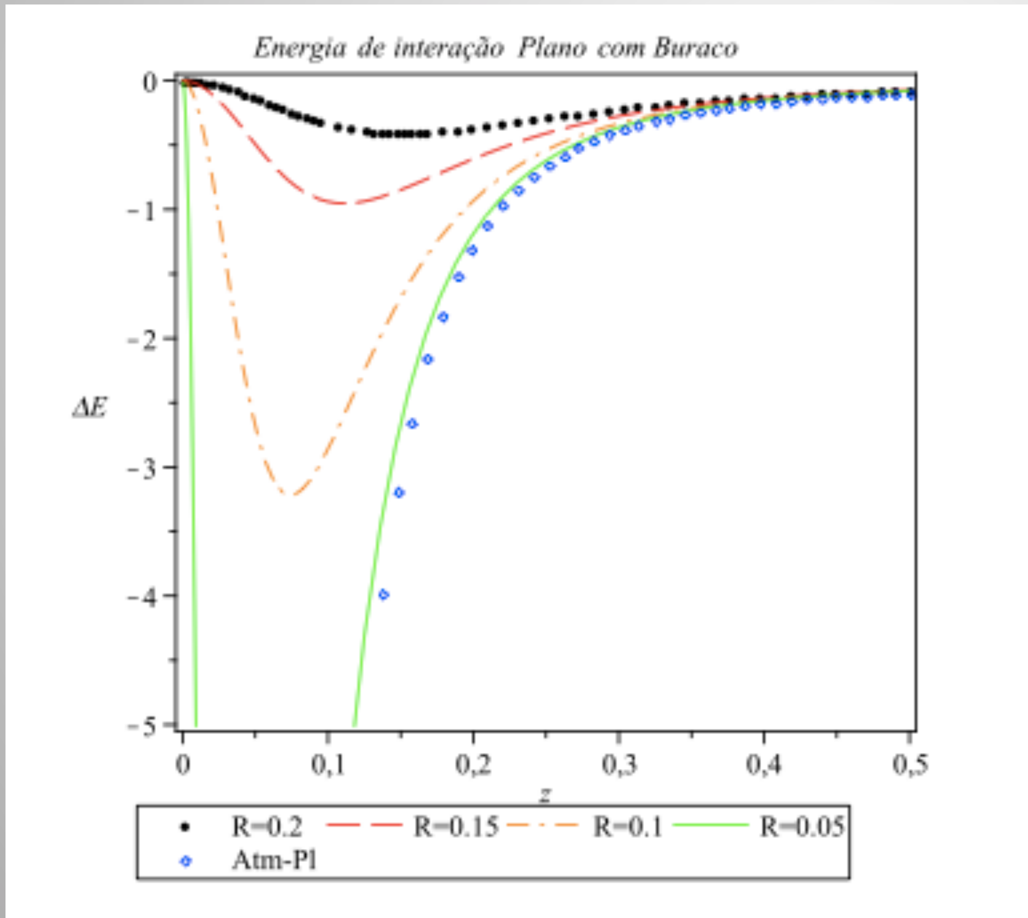
- Atom at the symmetry axis, polarizable in *z*:

$$E_{pb} = -\frac{1}{64\varepsilon_0\pi z_0^3}\left[1+\frac{2}{\pi}\,\mathrm{sen}^{-1}\left(\frac{z_0^2-a^2}{z_0^2+a^2}\right)-\frac{4az(3a^4+8a^2z_0^2-3z_0^4)}{3\pi(a^2+z_0^2)^3}\right] \text{ se } z \geq a$$

$$E_{pb} = -\frac{1}{64\varepsilon_0\pi z_0^3}\left[1-\frac{2}{\pi}\,\mathrm{sen}^{-1}\left(\frac{z_0^2-a^2}{z_0^2+a^2}\right)-\frac{4az(3a^4+8a^2z_0^2-3z_0^4)}{3\pi(a^2+z_0^2)^3}\right] \text{ se } z < a\,.$$

Dispersive interaction

- Graphically

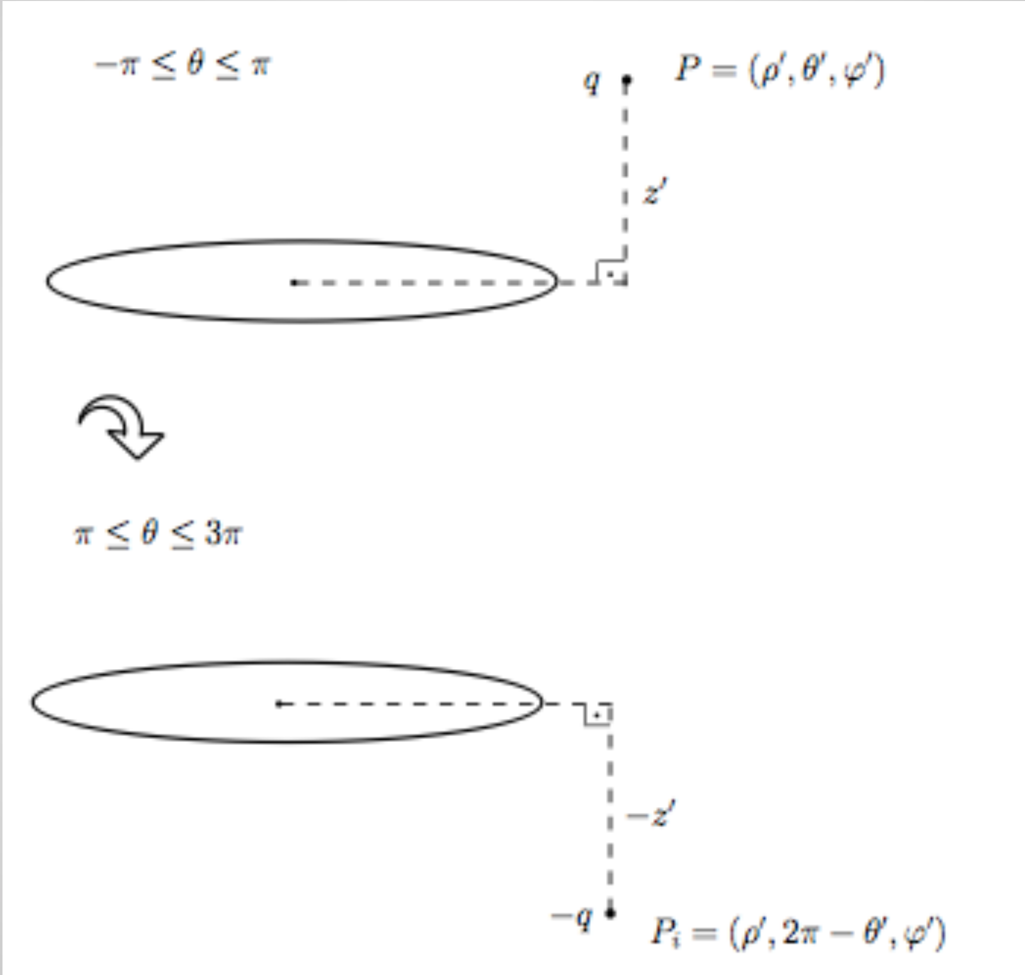


- There is repulsion!!

$$z_{eq} = 0,74235a$$

$$z_{eq}^{eb} = 0,7422a$$

Atom-Disc



$-\pi \leq \theta \leq \pi$
q
$P = (\rho', \theta', \varphi')$
z'
$\pi \leq \theta \leq 3\pi$
$-z'$
$-q$
$P_i = (\rho', 2\pi - \theta', \varphi')$

Atom-Disc

- Same procedure yields:

$$V_{disc} = \frac{q}{4\pi\varepsilon_0 a\sqrt{2}} \times$$

$$\left\{ \frac{(\cosh\rho - \cos\theta)^{1/2}(\cosh\rho' - \cos\theta')^{1/2}}{\left\{\cosh\gamma - \cos(\theta - \theta')\right\}^{1/2}} \left[\frac{1}{2} + \frac{1}{\pi}\sin^{-1}\left\{\cos\frac{1}{2}(\theta - \theta')\sec h\frac{\gamma}{2}\right\}\right] + \right.$$

$$\left. - \frac{(\cosh\rho - \cos\theta)^{1/2}(\cosh\rho' - \cos\theta')^{1/2}}{\left\{\cosh\gamma - \cos(\theta + \theta')\right\}^{1/2}} \left[\frac{1}{2} + \frac{1}{\pi}\sin^{-1}\left\{-\cos\frac{1}{2}(\theta + \theta')\sec h\frac{\gamma}{2}\right\}\right] \right\}$$

Non-additivity

- Force exerted on atom by the disc:

$$F_{disc} = -\partial_z E_{disc}$$

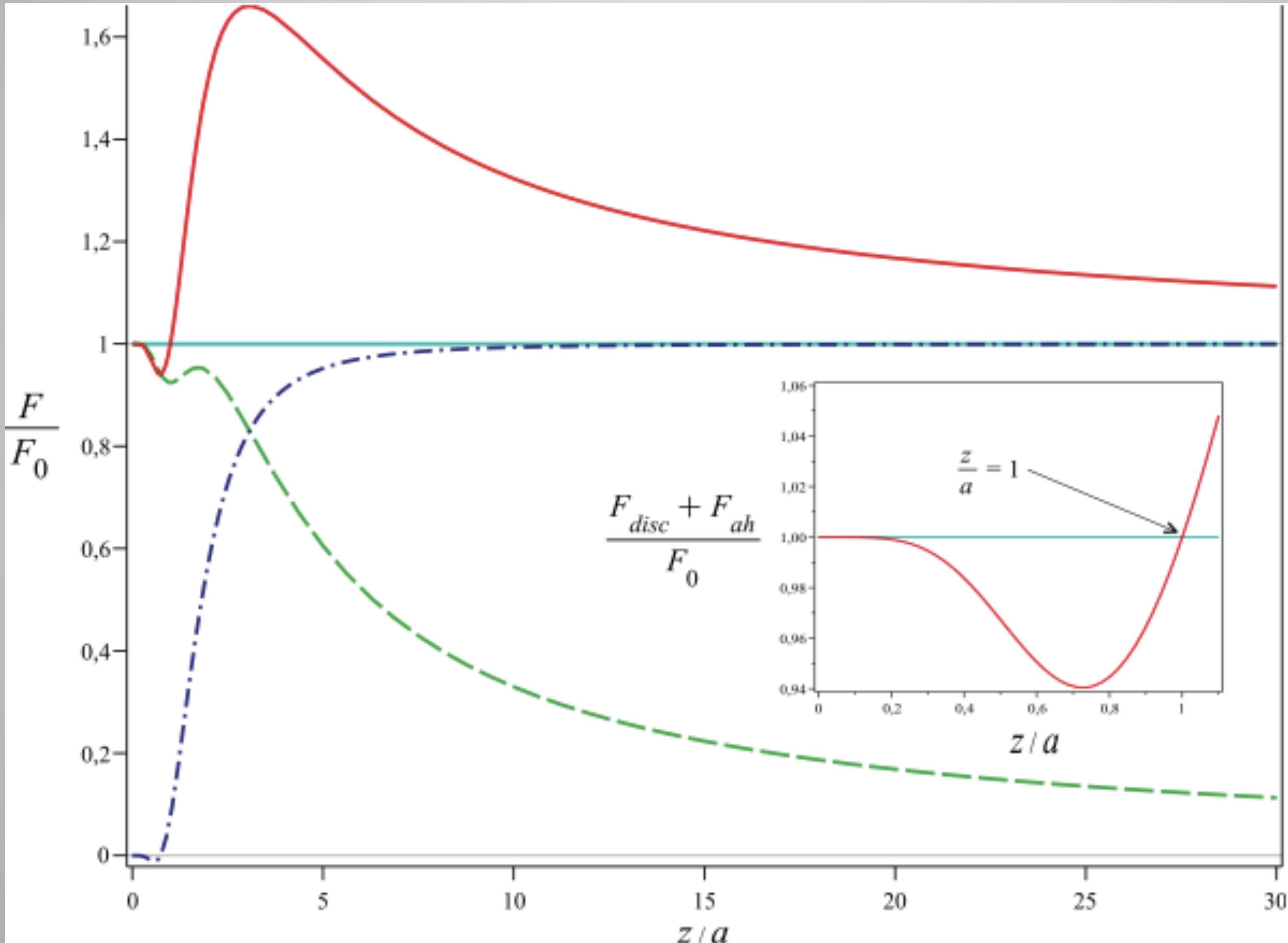
- Force exerted on atom by the plane with hole:

$$F_{ah} = -\partial_z E_{hole}$$

- Generally,

$$F := F_{ah} + F_{disc} \neq F_0$$

Non-additivity



Non-additivity

- For z=a the force is additive.
- Maybe the existence of a point to which the force is additive is a general properties for plane complementary surfaces.

Final Remarks

- Image method together with Eberlein-Zietal method is a powerful method to treat non-retarded dispersive interaction.

- We could treat analytically non-trivial geometries employing Sommerfeld's extension.

Final Remarks

- Analytical solutions allow careful studies of finite-size effects, non-additivity, …
- We intend to study Sommerfeld's extension to the Helmholtz equation.

References

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- [3] E.W. Hobson –http://www.archive.org/stream/memoirspresente00socigoog#page/n318/mode/2up (1900)
- [4] L.C. Davis, J.R. Reitz – Am.J.Phys. **39,** 1255.
- [5] A. Sommerfeld, Proc.London Math.Soc, **29**, 395.