# Effective Action and Phase Transitions in Thermal Yang-Mills Theory on Spheres

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September 18, 2011

# Introduction

- Yang-Mills at high energy
  $\Rightarrow$ asymptotic freedom, perturbation theory
- Yang-Mills at low energy
  $\Rightarrow$ strong coupling, non-perturbative regime
- Chromomagnetic vacuum (Savvidy)
- Instability of Savvidy vacuum due to a negative mode
- In curved space, the gluon operator is $-\Delta - 2\mathcal{F} + R$
- Stabilization of Savvidy vacuum by positive space curvature

# Yang-Mills Theory

- Yang-Mills action on $M = S^1 \times S^1 \times S^2$

$$S = -\frac{1}{2e^2} \int_M d\mathrm{vol}\, |\mathcal{F}|^2$$

- Covariantly constant background

$$\nabla_\rho R_{\mu\nu\alpha\beta} = 0, \qquad \nabla_\mu \mathcal{F}_{\alpha\beta} = 0$$

- Effective action

$$\Gamma_{(1)} = \frac{1}{2} \log \mathrm{Det}\, L_{\mathrm{gluon}} - \log \mathrm{Det}\, L_{\mathrm{ghost}}$$

- Relevant operators in minimal gauge

$$\begin{aligned} L_{\mathrm{gluon}} &= -\Delta_{T_1 \otimes Ad} + Q, \qquad Q = \mathbb{I}_{Ad} Ric - 2\mathcal{F} \\ L_{\mathrm{ghost}} &= -\Delta_{Ad} \end{aligned}$$

# Heat Trace

- Heat trace of Laplace-type operators $L = -\Delta + Q$

$$\mathrm{Tr}\,\exp(-tL) = \mathrm{vol}\,(M)\mathrm{tr}\,\exp(-tQ)U^{\mathrm{diag}}_{S^1\times S^1}(t)U^{\mathrm{diag}}_{S^2}(t)$$

- Heat kernel diagonal on $S^1 \times S^1$

$$U^{\mathrm{diag}}_{S^1\times S^1}(t) = \frac{1}{4\pi t}\Omega\left(\frac{t}{a_1^2}\right)\Omega\left(\frac{t}{a_2^2}\right)$$

- Jacobi theta function

$$\Omega(t) = \sqrt{\frac{t}{\pi}}\Omega\left(\frac{\pi^2}{t}\right) = \theta_3(0, e^{-\pi^2/t})$$

# Renormalization and Infrared Regularization

- Asymptotics of $\Theta_L(t) = (4\pi t)^2 \mathrm{Tr}\, \exp(-tL)$ as $t \to 0$

$$\Theta_L(t) \sim \sum_{k=0}^{\infty} B_k(L) t^k \sim e^{-t\lambda^2} \sum_{k=0}^{\infty} A_k(\lambda, L) t^k$$

- Renormalization

$$\Theta_L^{\mathrm{ren}}(t) = \Theta_L(t) - e^{-t\lambda^2} \left[ A_0(L) + A_1(\lambda, L) t + A_2(\lambda, L) t^2 \right] ,$$

- Infrared regularization $(z \to 0)$

$$\log \mathrm{Det}_{\,\mathrm{ren}}(L + z^2) = -(4\pi)^{-2} \int_0^{\infty} \frac{dt}{t^3} e^{-tz^2} \Theta_L^{\mathrm{ren}}(t)$$

# Effective Action

- Effective action

$$\Gamma_{(1)} = -\frac{1}{2}(4\pi)^{-2}\left\{\beta \log\frac{\mu^2}{\lambda^2} + \int_0^\infty \frac{dt}{t^3} e^{-tz^2}\Theta^{\mathrm{ren}}(t)\right\}$$

$$\Theta^{\mathrm{ren}}(t) = \Theta^{\mathrm{ren}}_{\mathrm{gluon}}(t) - 2\Theta^{\mathrm{ren}}_{\mathrm{ghost}}(t)$$

$$\beta = A_2(z, L_{\mathrm{gluon}}) - 2A_2(z, L_{\mathrm{ghost}})$$

# Geometry of $S^2$

- Metric on $S^2$

$$ds^2 = dr^2 + a^2 \sin^2\left(\frac{r}{a}\right) d\varphi^2$$

- Connection (monopole)

$$\mathcal{A} = -X\left[1 - \cos\left(\frac{r}{a}\right)\right] d\varphi$$

- Spectrum of $X$

$$\mathrm{Spec}(X) = \left\{\underbrace{0,\ldots,0}_{r}, i\frac{n_1}{2}, -i\frac{n_1}{2}, \ldots, i\frac{n_p}{2}, -i\frac{n_p}{2}\right\}$$

- For any compact simple group, some $n_i$ (monopole numbers) will have absolute value greater than or equal to 2.
- In particular, for $SU(N)$

$$n_{ij} = k_i - k_j\,, \qquad k \in \mathbb{Z}$$

# Laplacian

- Combined spin and gauge connection

$$G = \mathbb{I}_X \otimes T - X \otimes \mathbb{I}_T$$

- Twisted Lie derivatives

$$\mathcal{L}_A = \nabla_{\xi_A} - \frac{1}{2}\xi_{A[1;2]} G$$

form a representation of the isometry group $SO(3)$

- Cartan-Killing metric on $SO(3)$

$$(\gamma^{AB}) = \mathrm{diag}\left(1, 1, \frac{1}{a^2}\right)$$

- Laplacian

$$\Delta = \gamma^{AB}\mathcal{L}_A\mathcal{L}_B - \frac{1}{a^2}G^2$$

# Heat Trace on $S^2$

- Heat semigroup as integral over isometry group

$$\exp(t\Delta) = \frac{1}{4\pi t}\exp\left\{\left(\frac{1}{4} - G^2\right)\frac{t}{a^2}\right\}\int\limits_C \frac{d\omega}{\sqrt{4\pi t/a^2}}$$

$$\times \int\limits_{\mathbb{R}^2} dq\ \exp\left\{-\frac{|q|^2 + a^2\omega^2}{4t}\right\}\frac{\sin(\omega/2)}{\omega/2}\exp\left[q^k\mathcal{L}_k + \omega\mathcal{L}_3\right]$$

- Heat kernel diagonal

$$U^{\rm diag}(t) = \frac{1}{4\pi t}\exp\left[\frac{t}{a^2}\left(\frac{1}{4} - G^2\right)\right]\Psi\left(\frac{t}{a^2}; -2iG\right),$$

$$\Psi(t;n) = \int\limits_C \frac{d\omega}{\sqrt{4\pi t}}\ \exp\left\{-\frac{\omega^2}{4t} + in\omega/2\right\}\frac{\omega/2}{\sin(\omega/2)}$$

# Total Heat Trace

- Total heat trace

$$\Theta(t) = 16\pi^3 a^2 a_1 a_2 \Omega\left(\frac{t}{a_1^2}\right)\Omega\left(\frac{t}{a_2^2}\right) W\left(\frac{t}{a^2}\right)$$

$$\begin{aligned} W(t) &= \int_C \frac{d\omega}{\sqrt{4\pi t}} \exp\left\{-\frac{\omega^2}{4t}\right\} \frac{\omega/2}{\sin[\omega/2]} \\ &\times \mathrm{tr}_{Ad}\left\{\exp\left[t\left(\frac{1}{4} - X^2\right) - X\omega\right][\mathrm{tr}\exp(E\omega) - 2]\right\} \end{aligned}$$

where

$$E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

# Total Heat Trace (cont.)

$$W(t) = 2\sum_{j=1}^{p} \exp\left[\frac{t(1+n_j^2)}{4}\right]\{\Psi(t;2+|n_j|)+\Psi(t;2-|n_j|)\}$$
$$+2re^{t/4}\Psi(t;2)$$

# Effective Action

- Effective action

$$\Gamma = \frac{1}{xy}\left\{\frac{2\pi^3}{e^2}\sum_{i=1}^{p} n_i^2 - \frac{\pi}{2}\left(C_2 - z^2a^2C_1 + z^4a^4m\right)\log\frac{\mu^2}{\lambda^2}\right.$$
$$\left. -\frac{\pi}{2}\Phi\left(x, y; a\lambda, az\right)\Big]\right\}$$

where $x = a/a_1,\ y = a/a_2,\ m = \dim G = r + 2p$

$$\Phi(x, y; a\lambda, az) = \int_0^\infty \frac{dt}{t^3} e^{-ta^2z^2}\left\{\Omega\left(x^2t\right)\Omega\left(y^2t\right)W(t)\right.$$
$$\left. -e^{-ta^2\lambda^2}R(t; a\lambda)\right\}$$

$$R(t; a\lambda) = 2m + \left(C_1 + 2ma^2\lambda^2\right)t + \left(C_2 + C_1a^2\lambda^2 + ma^4\lambda^4\right)t^2$$

# Instability

- Asymptotically as $t \to 0$

$$W(t) \sim 2tce^{-t\lambda_{\min}}$$

where

$$\lambda_{\min} = \min_{1 \le j \le p} \left\{ 1 - |n_j| + \left| 1 - \frac{|n_j|}{2} \right| \right\}$$

- This model is always unstable because there is always an $|n_j| \geq 2$ with negative minimum eigenvalue

$$\lambda_{\min} = -\frac{1}{2} n_{\max} \qquad \text{where} \qquad n_{max} = \max_j |n_j|$$

- For $SU(N)$, $n_{\max} \geq 2$ for even $N$ and $n_{\max} \geq 3$ for odd $N$.

# Thermodynamics

- Free energy

$$F = T\Gamma = \frac{\Gamma}{2\pi a_1}$$

- Heat capacity

$$C_v = -T\frac{\partial^2}{\partial T^2}F = -a_1^2\left(a_1\frac{\partial^2}{\partial a_1^2}\frac{\Gamma}{a_1} + 2\frac{\partial}{\partial a_1}\frac{\Gamma}{a_1}\right) .$$

- Per unit volume:

$$\frac{C_v}{V} = \frac{1}{16\pi a_1 a^2}\partial_x^2\Phi(x, y, a\lambda, az) ,$$

- Letting $y \to 0$, $(a_2 \to \infty)$, $\lambda = 1/a$ and $z \to 0$

$$\partial_x^2\Phi(x, 0; 1, 0) = 2x^2\int_0^\infty \frac{dt}{t^2}\left\{\Omega'(t) + 2t\Omega''(t)\right\} W\left(\frac{t}{x^2}\right)$$

# Phase Transition

- High-temperature limit for $SU(2N)$ ($T \to \infty$)

$$\frac{C_v}{V} \sim 2\pi^2(4N^2 - 1)\nu_2 T^3 + \cdots$$

- Near critical temperature ($T \to T_c^+$)

$$\frac{C_v}{V} \sim \frac{N^2}{4\pi\sqrt{2}a^3}\left(\frac{T - T_c}{T_c}\right)^{-3/2}$$

- Critical temperature

$$T_c = \frac{1}{2\pi a}$$

- Second-order phase transition near $T_c$

# References

I. G. Avramidi and S. Collopy, *Effective Action And Phase Transitions In Yang-Mills Theory On Spheres*, `arXiv:1012.2414`, Commun. Math. Phys. (2011)