Repulsive Casimir Effects

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I. Quantum Vacuum Repulsion

- Like Casimir's original force between conducting plates in vacuum, Casimir forces are usually attractive.

- But repulsive Casimir forces can be achieved in special circumstances.

- These might prove useful in nanotechnology.

II. Multiple Scattering Technique

The multiple scattering approach starts from the well-known formula for the vacuum energy or Casimir energy (for simplicity here we first restrict attention to a massless scalar field)(τ is the "infinite" time that the configuration exists) [Schwinger, 1975]

$$E = \frac{i}{2\tau}\mathsf{Tr}\ln G \longrightarrow \frac{i}{2\tau}\mathsf{Tr}\ln GG_0^{-1},$$

where G (G_0) is the Green's function,

$$(-\partial^2 + V)G = 1, \quad +\mathsf{BC}, \quad -\partial^2 G_0 = 1.$$

T-matrix

Define the T-matrix (Lippmann-Schwinger)

$$T = S - 1 = V(1 + G_0 V)^{-1}.$$

If the potential has two disjoint parts, $V = V_1 + V_2$ it is easy to derive the interaction between the two bodies (potentials):

$$E_{12} = -\frac{i}{2\tau} \mathsf{Tr} \ln(1 - G_0 T_1 G_0 T_2)$$

$$= -\frac{i}{2\tau} \mathsf{Tr} \ln(1 - V_1 G_1 V_2 G_2),$$

where $G_i = (1 + G_0 V_i)^{-1} G_0, \quad i = 1, 2.$

III. Interaction between ε, μ bodies

Consider material bodies characterized by a permittivity $\varepsilon(\mathbf{r})$ and a permeability $\mu(\mathbf{r})$, so we have corresponding electric and magnetic potentials

$$V_e(\mathbf{r}) = \varepsilon(\mathbf{r}) - 1, \quad V_m(\mathbf{r}) = \mu(\mathbf{r}) - 1.$$

Then the trace-log is ($\boldsymbol{\Phi}_0 = -\frac{1}{\zeta}\boldsymbol{\nabla} \times \boldsymbol{\Gamma}_0$)

$$\mathsf{Tr} \ln \boldsymbol{\Gamma}\boldsymbol{\Gamma}_0^{-1} = -\mathsf{Tr} \ln(\mathbf{1} - \boldsymbol{\Gamma}_0 V_e) - \mathsf{Tr} \ln(\mathbf{1} - \boldsymbol{\Gamma}_0 V_m)$$
$$-\mathsf{Tr} \ln(\mathbf{1} + \boldsymbol{\Phi}_0\mathbf{T}_e\boldsymbol{\Phi}_0\mathbf{T}_m),$$
$$\mathbf{T}_{e,m} = V_{e,m}(\mathbf{1} - \boldsymbol{\Gamma}_0 V_{e,m})^{-1}.$$

Factorization

If we have *disjoint* electric bodies, the interaction term separates out:

$$\text{Tr} \ln (1 - \boldsymbol{\Gamma}_0(V_1 + V_2)) = -\text{Tr} \ln(1 - \boldsymbol{\Gamma}_0 \mathbf{T}_1)$$
$$-\text{Tr} \ln(1 - \boldsymbol{\Gamma}_0 \mathbf{T}_2) - \text{Tr} \ln(1 - \boldsymbol{\Gamma}_0 \mathbf{T}_1 \boldsymbol{\Gamma}_0 \mathbf{T}_2),$$

so only the latter term contributes to the interaction energy,

$$E_{\text{int}} = \frac{i}{2} \text{Tr} \ln(1 - \boldsymbol{\Gamma}_0 \mathbf{T}_1 \boldsymbol{\Gamma}_0 \mathbf{T}_2).$$

ε-μ **Lifshitz force**

The same is true if one body is electric and the other magnetic,

$$E_{\mathrm{int}} = -\frac{i}{2} \mathsf{Tr} \ln(1 + \mathbf{\Phi}_0 \mathbf{T}_1^e \mathbf{\Phi}_0 \mathbf{T}_2^m).$$

Using this, it is easy to show that the Lifshitz energy between a parallel dielectric and diamagnetic slabs is

$$E_{\varepsilon\mu} = \frac{1}{16\pi^3} \int d\zeta \int d^2 k \left[\ln\left(1 - r_1 r_2' e^{-2\kappa a}\right) \right.$$

$$\left. + \ln\left(1 - r_1 r_2' e^{-2\kappa a}\right) \right]$$

Repulsive Casimir force

where

$$r_i = \frac{\kappa - \kappa_i}{\kappa + \kappa_i}, \quad r_i' = \frac{\kappa - \kappa_i'}{\kappa + \kappa_i'},$$

with $\kappa^2 = k^2 + \zeta^2$, $\kappa_1^2 = k^2 + \varepsilon\zeta^2$, $\kappa_1' = \kappa_1/\varepsilon$, $\kappa_2^2 = k^2 + \mu\zeta^2$, $\kappa_2' = \kappa_2/\mu$.
This means in the perfect reflecting limit, $\varepsilon \to \infty$, $\mu \to \infty$,

$$E_{\text{Boyer}} = +\frac{7}{8}\frac{\pi^2}{720a^3},$$

we get Boyer's repulsive result.

Repulsion in Lifshitz situation

It is also well known in the Lifshitz-Dzyaloshinskii-Pitaevskii situation of parallel dielectric media, with the intermediate medium having an intermediate value of the permittivity:

$$\varepsilon_1 > \varepsilon_3 > \varepsilon_2$$

there is a Casimir repulsion between the upper and lower media. This was demonstrated in the Munday-Capasso experiment.
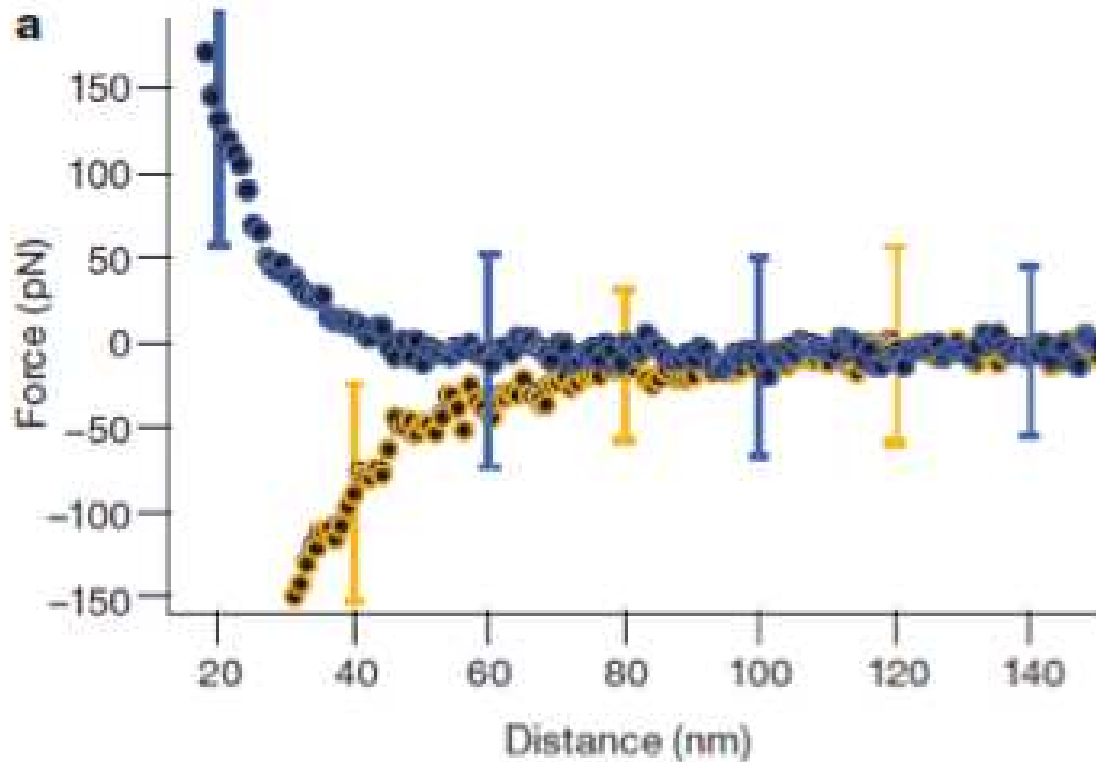
Yellow gold-gold, blue gold-silca, both separated by bromobenzene.

IV. Casimir effect on sphere, etc.

Earlier Boyer had shown that the Casimir self-energy of a spherical shell was positive, that is repulsive. Such calculations have been generalized.

Type	$E_{\mathrm{Sphere}}a$	$\mathcal{E}_{\mathrm{Cylinder}}a^2$	References
EM	0.04618	-0.01356	Boyer, DeRaad
D	0.002817	0.0006148	Bender,Gosdzinsl
$(\varepsilon - 1)^2$	$\frac{23}{1536\pi}$	0	Brevik, Cavero
ξ^2	$\frac{5}{32\pi}$	0	Klich, Milton
λ^2/a^2	$\frac{1}{32\pi}$	0	Milton, Cavero

Dimensional dependence

Bender and Milton, "Scalar Casimir effect for a D-dimensional sphere," Phys. Rev. D **50**, 6547 - 6555 (1994)

Here, we considered the Casimir effect due to fluctuations in a scalar field interior and exterior to a Dirichlet sphere.

Poles occurs in even dimensions!
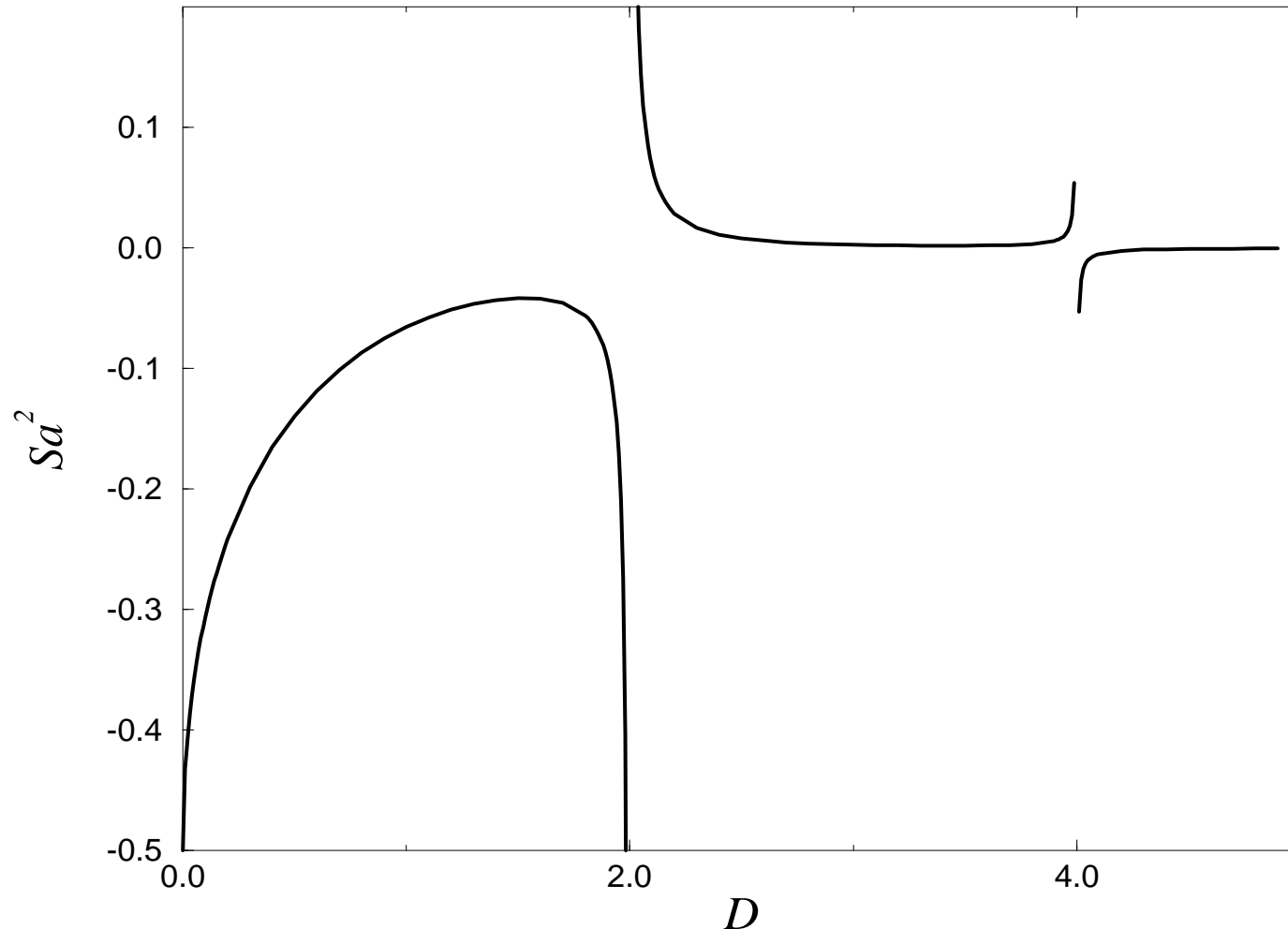
Scalar Casimir stress $\mathcal{S}$ for $0 < D < 5$ on a spherical shell.

V. Triangular cylinders

For an equilateral triangle of height h, the scalar eigenmodes corresponding to Dirichlet boundary conditions are known explicitly [Schwinger et al., *Classical Electrodynamics, Electromagnetic Radiation*]

$$\gamma_l^2 = \frac{2}{3} \left(\frac{\pi}{h} \right)^2 (l_1^2 + l_2^2 + l_3^2),$$

$$l_1 + l_2 + l_3 = 0, \quad l_i \neq 0.$$

[Elom Abalo, K.A.M, and Lev Kaplan, Phys. Rev. D **82**, 125007 (2010).]

Casimir energy

In d transverse dimensions, the Casimir energy is

$$\mathcal{E} = -\frac{\Gamma(-1/2 - d/2)}{2^{2+d}\pi^{(d+1)/2}} \sum_l (\gamma_l^2)^{(d+1)/2}$$

which can be analytically continued and summed by means of the Chowla-Selberg formula (which we used to find the temperature dependence for the diaphanous wedge), which is exceedingly rapidly convergent.

$$\mathcal{E} = +\frac{0.0177891}{h^2}.$$

Poisson sum formula

We can also evaluate the eigenvalue sum by use of the Poisson sum formula,

$$\sum_{l=-\infty}^{\infty} f(l) = 2\pi \sum_{k=-\infty}^{\infty} \tilde{f}(k),$$

in terms of the Fourier transform

$$\tilde{f}(k) = \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi} e^{2\pi i k \alpha} f(\alpha).$$

Point-splitting regularization

We use the Poisson sum formula together with point-split regularization, starting from

$$E = \frac{1}{2i} \int (d\mathbf{r}) \int \frac{d\omega}{2\pi} 2\omega^2 \mathcal{G}(\mathbf{r}, \mathbf{r}) e^{-i\omega\tau},$$

which for a cylindrical waveguide, gives

$$\mathcal{E} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\zeta}{2\pi} 2(-\zeta^2) \int \frac{dk}{2\pi} \sum_{m,n} \frac{1}{\zeta^2 + k^2 + \gamma_{mn}^2} e^{i\zeta\tau}$$

$$= \frac{1}{2} \lim_{\tau \to 0} \left(-\frac{d}{d\tau} \right) \int_{-\infty}^{\infty} \frac{dk}{2\pi} \sum_{m,n} e^{-\tau\sqrt{k^2+\gamma_{mn}^2}}.$$

Divergences

A virtue of the point-splitting method is that we can isolate the divergences in the energy:

$$\widehat{\mathcal{E}}_{\mathrm{Eq}}^{(D)} = \lim_{\tau \to 0} \left(\frac{3A}{2\pi^2 \tau^4} - \frac{P}{8\pi \tau^3} + \frac{1}{6\pi \tau^2} \right).$$

We note that the "volume" and "surface" divergent terms, which are respectively proportional to the area of the triangle $A = h^2/\sqrt{3}$ and the perimeter $P = 2\sqrt{3}h$, are as expected, and are presumably not of physical relevance.

Corner divergences

The last term, a constant in h, certainly does not contribute to the self-stress on the cylinder. Only this term reflects the corner divergences. For a general polygon, with interior angles α_i, the last term is

$$\frac{1}{48\pi} \sum_i \left(\frac{\pi}{\alpha_i} - \frac{\alpha_i}{\pi} \right) \frac{1}{\tau^2}.$$

These coefficients are proportional to the heat kernel coefficients –in particular there is no a_2 HK coefficient, which means that the CE can be identified unambiguously.

Closed-form result

Remarkably, for the integrable polygonal figures we are considering, the Casimir energy can be given in closed form. in terms of the polygamma function. Thus

$$\mathcal{E}_{\mathrm{Eq}}^{(D)} = -\frac{1}{96h^2} \left[\frac{\sqrt{3}}{9} \left[\psi'(1/3) - \psi'(2/3) \right] - \frac{8}{\pi} \zeta(3) \right]$$

$$= \frac{0.0177891}{h^2}.$$

It is *a priori* remarkable that such an explicit form can be achieved for a strong-coupling problem.

$\mathcal{E}_c$ for square waveguide

Same methods for evaluating the Casimir energy for a square waveguide (side a) (Lukosz/ Ambjørn and Wolfram):

$$\mathcal{E} = \frac{-1}{32\pi^2 a^2}\left[2\zeta(4) - \pi\zeta(3) + 8\pi^2 \sum_{l=1}^{\infty} l^{3/2}\sigma_3(l)K_{3/2}(2\pi l)\right]$$

$$= -\frac{1}{32\pi^2 a^2}\left[4\zeta(4) - 2\pi\zeta(3) + 4\sum_{k,l=1}^{\infty} \frac{1}{(k^2 + l^2)^2}\right]$$

$$= \frac{1}{16\pi a^2}\left[\zeta(3) - \frac{\pi}{3}G\right] = \frac{0.00483155}{a^2}.$$

Other triangles

By bifurcating the square, we can obtain the isosceles right triangle, and by bifurcating the equilateral triangle we can get the $30°$-$60°$-$90°$ triangle:

$$\mathcal{E}_{\mathrm{iso}} = \frac{1}{2}\mathcal{E}_{\mathrm{sq}} + \frac{\zeta(3)}{16\pi a^2} = \frac{0.0263299}{a^2},$$

$$\mathcal{E}_{369} = \frac{1}{2}\mathcal{E}_{\mathrm{et}} + \frac{\zeta(3)}{8\pi h^2} = \frac{0.0567229}{h^2},$$

To be compared to the result for a circle

$$\mathcal{E}_{\mathrm{circ}} = \frac{0.0006148}{a^2}.$$

H modes

We can also get results for Neumann boundary conditions (H or TE modes)

$$\mathcal{E}_{\text{sq}}^{N} = \mathcal{E}_{\text{sq}}^{D} - \frac{\zeta(3)}{8\pi a^2} = -\frac{0.0429968}{a^2},$$
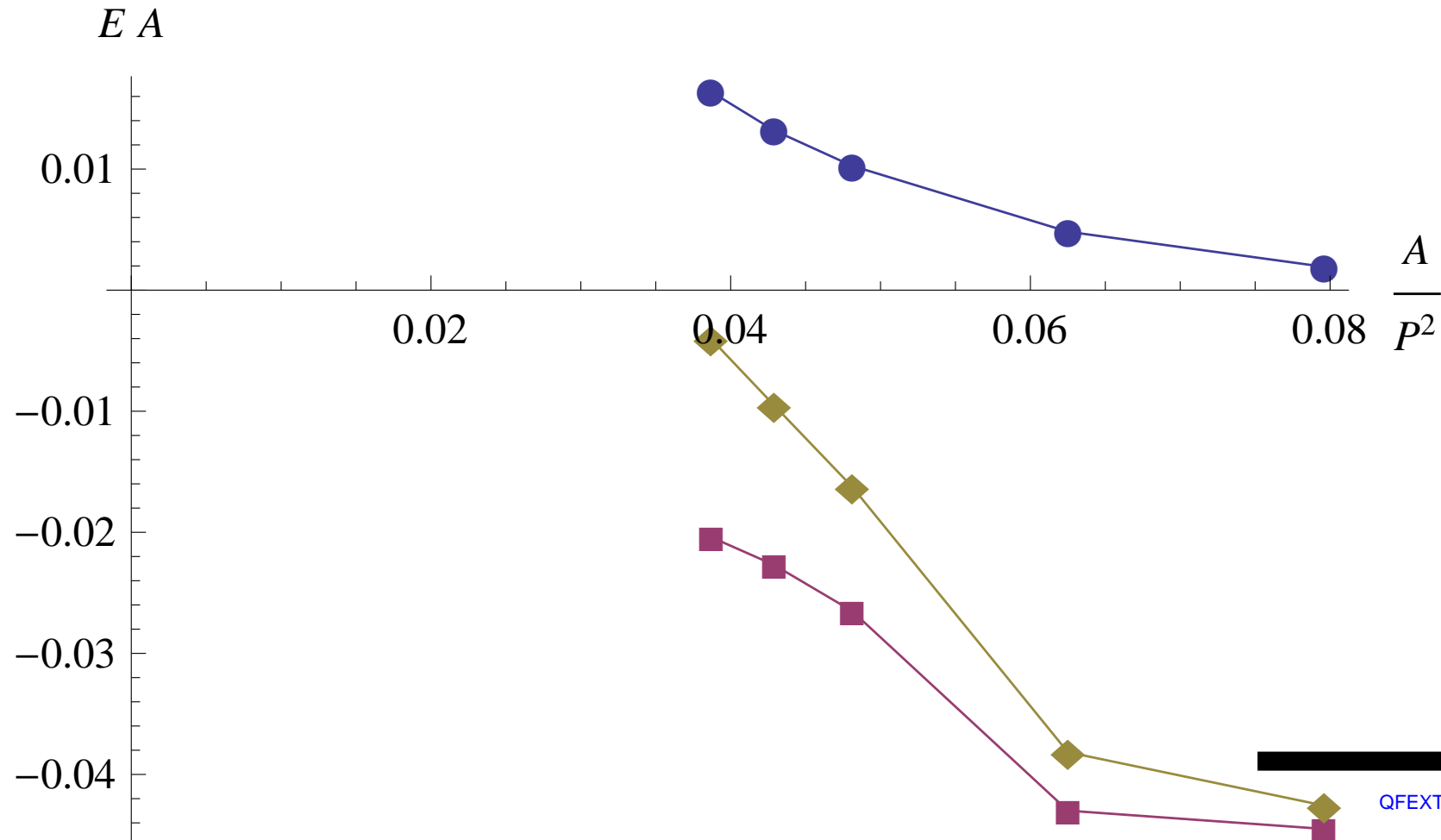
$$\mathcal{E}_{\text{et}}^{N} = \mathcal{E}_{\text{et}}^{D} - \frac{\zeta(3)}{6\pi h^2} = -\frac{0.045982}{h^2},$$

$$\mathcal{E}_{\text{iso}}^{N} = \frac{1}{2}\mathcal{E}_{\text{sq}}^{N} - \frac{\zeta(3)}{16\pi a^2} = -\frac{0.0454125}{a^2},$$

$$\mathcal{E}_{369}^{N} = \frac{1}{2}\mathcal{E}_{\text{et}}^{N} - \frac{\zeta(3)}{8\pi h^2} = -\frac{0.0708193}{h^2}.$$
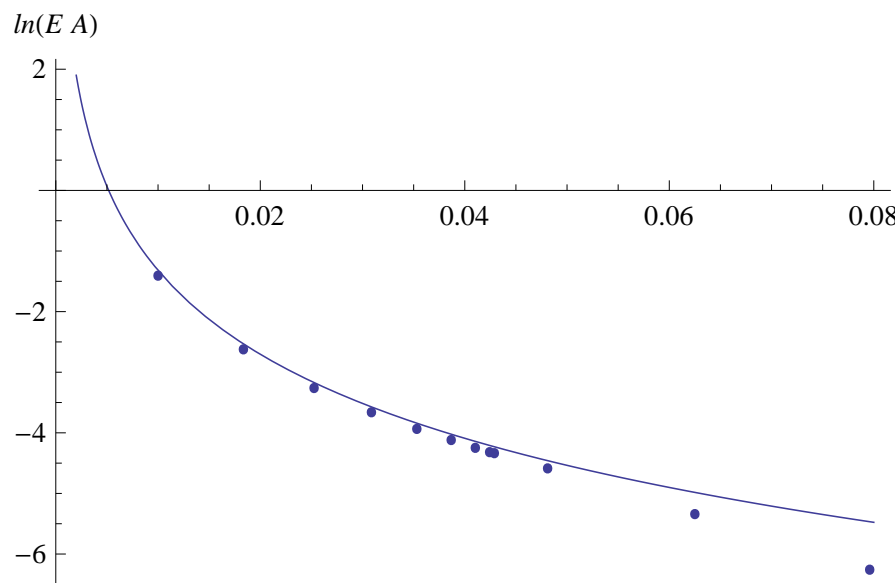
Systematic dependence of $\mathcal{E}_c^D$

$$\mathcal{E}(a) \quad \rightarrow \quad \mathcal{E}A(A/P^2), \quad A \quad = \quad \text{cross-sectional}$$

area, P = perimeter of waveguide.

Numerical work

Lev Kaplan has used a numerical method to extract eigenvalues for right triangles with arbitrary acute angles. Those results lie on our universal curve, and agree with the PFA (solid line) for small acute angles. (Dirichlet BC.)
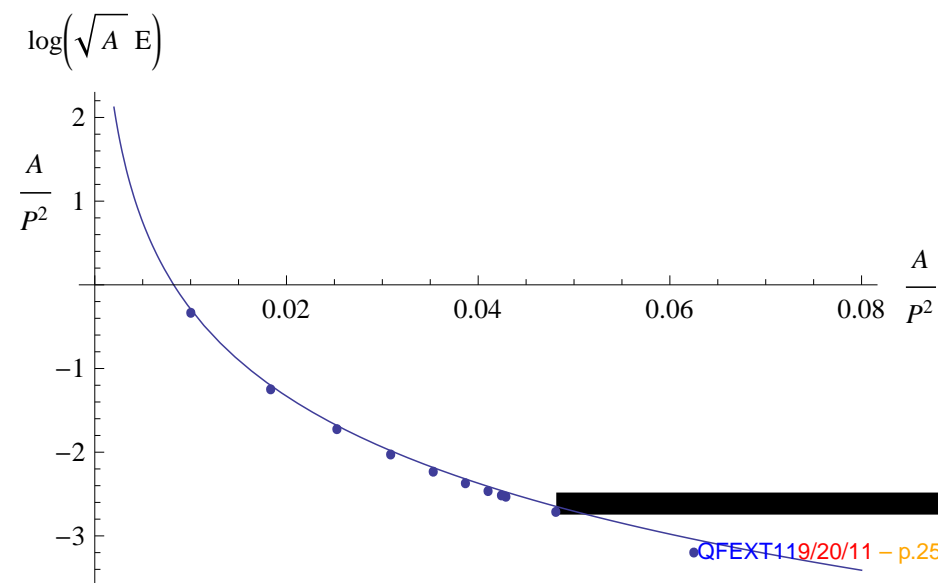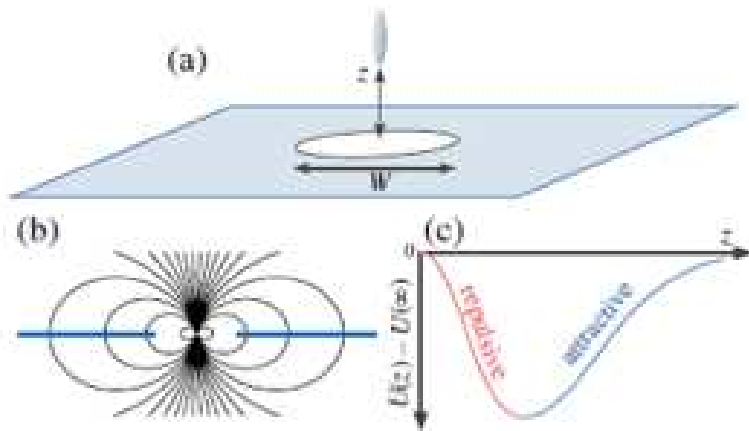
VI. Classical repulsion

FIG. 1 (color online). (a) Schematic geometry achieving Casimir repulsion: an elongated metal particle above a thin metal plate with a hole. The idealized version is the limit of an infinitesimal particle polarizable only in the z direction. (b) At $z = 0$, vacuum-dipole field lines are perpendicular to the plate by symmetry, and so dipole fluctuations are unaffected by the plate (for *any* ω, shown here for $\omega = 0$). (c) Schematic particle-plate interaction energy $U(z) - U(\infty)$: zero at $z = 0$ and at $z \to \infty$, and attractive for $z \gg W$, so there must be Casimir repulsion (negative slope) close to the plate.

Levin et al., PRL, **105**, 090403 (2010).

Classical dipole interaction

It is possible to achieve a repulsive force
between a configuration of fixed dipoles.
Consider the situation illustrated in the figure.

$$\begin{array}{c} \uparrow d_1 \\[1em] \vdots Z \\[1em] d_2 \quad\cdots\quad\downarrow\quad\cdots\quad d_3 \\ a \end{array}$$
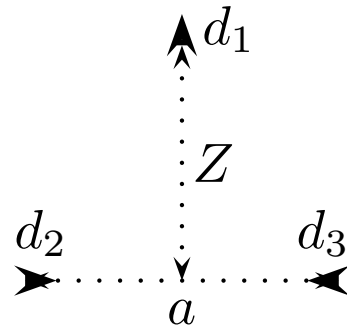
Configuration of three dipoles, two of which are
antiparallel, and one perpendicular to the other
two.

Force on perpendicular dipole

$$\mathbf{d}_2 = -\mathbf{d}_3 = d_2\hat{\mathbf{x}},$$

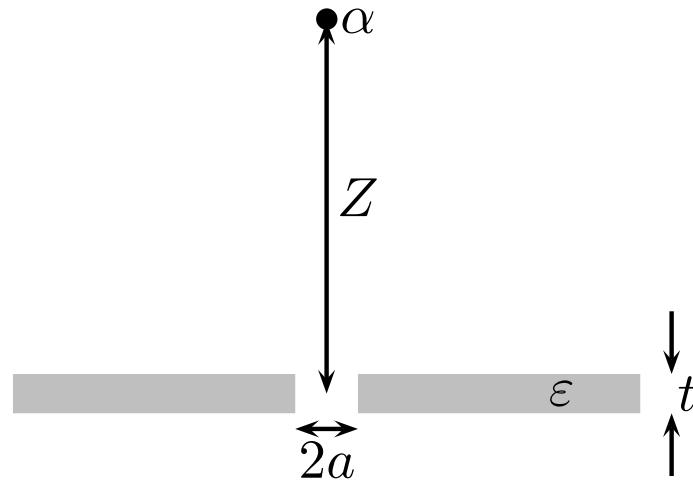equally distant from the z axis, and the dipole on the z axis is directed along that axis,

$$\mathbf{d}_1 = d_1\hat{\mathbf{z}},$$

the force on that dipole is along the z axis:

$$F_z = 3ad_1d_2\frac{a^2/4 - 4Z^2}{(Z^2 + a^2/4)^{7/2}},$$

which changes sign at $Z = a/4$, that is, close to $Z = 0$ the force is repulsive!

Interaction of atom with aperture

Three-dimensional geometry of a polarizable atom a distance Z above a dielectric slab with a circular aperture of radius a.

Green's function which vanishes on the entire line $z = 0$:

$$G(\mathbf{r}, \mathbf{r}') = -\ln[(x-x')^2+(z-z')^2]+\ln[(x-x')^2+(z+z')^2],$$

with BC:

$$G(x, 0; x', z') = 0,$$

the electrostatic potential at any point above the $z = 0$ plane:

$$\phi(\mathbf{r}) = \int_{z>0}(d\mathbf{r}')G(\mathbf{r}, \mathbf{r}')\rho(\mathbf{r}')+\frac{1}{4\pi}\int_{\text{ap}}dS'\frac{\partial}{\partial z'}G(\mathbf{r}, \mathbf{r}')\phi(\mathbf{r}')$$

Point dipole

where

$$\rho(\mathbf{r}) = -\mathbf{d} \cdot \boldsymbol{\nabla}\delta(\mathbf{r} - \mathbf{R}), \quad \mathbf{R} = (0, Z).$$

The surface integral extends only over the aperture because the potential vanishes on the conducting sheet. If we choose $\mathbf{d}$ to point along the z axis we easily find (a = width of aperture)

$$\phi(x, z > 0) = 2d \left[\frac{z - Z}{x^2 + (z - Z)^2} + \frac{z + Z}{x^2 + (z + Z)^2} \right]$$

$$+ \frac{1}{\pi} \int_{-a/2}^{a/2} dx' \frac{z}{(x - x')^2 + z^2} \phi(x', 0).$$

Fourier transforms

Now the free Green's function in two dimensions is

$$G_0(\mathbf{r}, \mathbf{r}') = 4\pi \int \frac{(d\mathbf{k})}{(2\pi)^2} \frac{e^{ik_x(x-x')}e^{ik_z(z-z')}}{k_x^2 + k_z^2}$$

$$= \int_{-\infty}^{\infty} dk_x \frac{1}{|k_x|} e^{ik_x(x-x')} e^{-|k_x||z-z'|}.$$

Then the surface integral above is

$$\int_{-\infty}^{\infty} \frac{dk_x}{2\pi} e^{ik_x x} e^{-|k_x|z} \tilde{\phi}(k_x),$$

in terms of the Fourier transform of the field

$$\tilde{\phi}(k_x) = \int_{-\infty}^{\infty} dx' e^{-ik_x x'} \phi(x', 0)$$

$$= 2 \int_{0}^{a/2} dx' \cos k_x x' \phi(x', 0),$$

since $\phi(x, 0)$ must be an even function for the geometry considered.

Fields above aperture

Thus we conclude

$$\phi(x, z > 0) = 2d \left[\frac{z - Z}{x^2 + (z - Z)^2} + \frac{z + Z}{x^2 + (z + Z)^2} \right]$$
$$+ \frac{1}{\pi} \int_0^\infty dk \cos kx \, e^{-kz} \tilde{\phi}(k).$$

$$E_z(x, z = 0+) = -\frac{\partial}{\partial z} \phi(x, z) \Big|_{z - 0+}$$
$$= -4d \frac{x^2 - Z^2}{(x^2 + Z^2)^2} + \frac{1}{\pi} \int_0^\infty dk \, k \cos kx \, \tilde{\phi}(k).$$

Below aperture

On the other side of the aperture, there is no charge density, so for $z < 0$ the potential is

$$\phi(x, z < 0) = \frac{1}{\pi} \int_0^\infty dk \cos kx \, e^{kz} \tilde{\phi}(k),$$

so the z-component of the electric field in the aperture is

$$E_z(x, z = 0-) = -\frac{\partial}{\partial z} \phi(x, z) \Big|_{z=0-}$$

$$= -\frac{1}{\pi} \int_0^\infty dk \, k \cos kx \, \tilde{\phi}(k).$$

Coupled integral equations

Because we require that the electric field be continuous in the aperture, and the potential vanish on the conductor, we obtain the two coupled integral equations for this problem,

$$4d\frac{x^2 - Z^2}{(x^2 + Z^2)^2} = \frac{2}{\pi}\int_0^\infty dk\, k\cos kx\, \tilde{\phi}(k), \quad 0 < |x| < a/2$$

$$0 = \int_0^\infty dk\cos kx\, \tilde{\phi}(k), \quad |x| > a/2.$$

Simple solution

$$\tilde{\phi}(k) = -\frac{4Zd\pi}{a} \int_0^1 dx\, x \frac{J_0(kax/2)}{(x^2 + 4Z^2/a^2)^{3/2}}.$$

From this, we can work out the energy of the system from

$$U = -\frac{1}{2}dE_z(0, Z) = \frac{1}{2}d\frac{\partial\phi}{\partial z}\bigg|_{z=Z, x=0},$$

1/2 comes from the fact that this must be the energy required to assemble the system. We must drop the self-energy of the dipole due to its own field.

Int. of atom with line/aperture

We are then left with

$$U_{\text{int}} = -\frac{d^2}{4Z^2} - \frac{d}{2\pi} \int_0^\infty dk\, k\, e^{-kZ} \tilde{\phi}(k)$$

$$= -\frac{d^2}{4Z^2} + Z^2 d^2 \left(\frac{2}{a}\right)^4 \int_0^1 \frac{1}{2} dx^2 \frac{1}{(x^2 + 4Z^2/a^2)^3}$$

$$= -\frac{4Z^2 d^2}{(a^2 + 4Z^2)^2}.$$

Twice that of Levin et al. Since this vanishes at $Z = 0$ and $Z = \infty$, the force must change from attractive to repulsive, which happens at $Z = a/2$.

3D aperture int. with dipole

It is quite straightforward to repeat the above calculation in three dimensions. Again we are considering a dipole, polarized on the symmetry axis, a distance Z above a circular aperture of radius a in a conducting plate.
The free three-dimensional Green's function in cylindrical coordinates has the representation

$$\frac{1}{\sqrt{\rho^2 + z^2}} = \int_0^\infty dk \, J_0(k\rho) e^{-k|z|},$$

Following previous procedure:

$$\phi(\mathbf{r}_\perp, z > 0) = d \left[\frac{z - Z}{[r_\perp^2 + (z - Z)^2]^{3/2}} + \frac{z + Z}{[r_\perp^2 + (z + Z)^2]^{3/2}} \right]$$

$$+ \int_0^\infty dk \, k \, e^{-kz} J_0(kr_\perp) \Phi(k),$$

where the Bessel transform of the potential in the aperture is

$$\Phi(k) = \int_0^\infty d\rho \, \rho \, J_0(k\rho) \phi(\rho, 0).$$

Integral equations

Thus the integral equations resulting from the continuity of the z-component of the electric field in the aperture and the vanishing of the potential on the conductor are

$$d\frac{r_\perp^2 - 2Z^2}{[r_\perp^2 + Z^2]^{5/2}} = \int_0^\infty dk\, k^2 J_0(kr_\perp)\Phi(k), \quad r_\perp < a,$$

$$0 = \int_0^\infty dk\, k J_0(kr_\perp)\Phi(k), \quad r_\perp > a.$$

Solution

The solution to these equations is given in Titchmarsh's book, and after a bit of manipulation we obtain

$$\Phi(k) = -\left(\frac{2ka}{\pi}\right)^{1/2} \frac{2dZ}{ka^2} \int_0^1 dx\, x^{3/2} \frac{J_{1/2}(xka)}{(x^2 + Z^2/a^2)^2}.$$

Then the energy may be easily evaluated using

$$\int_0^\infty dk\, k^{3/2} e^{-kZ} J_{1/2}(kax) = 2\sqrt{\frac{2xa}{\pi}} \frac{Z}{(x^2 a^2 + Z^2)^2}.$$

Closed-form energy

The energy can again be expressed in closed form:

$$U = -\frac{d^2}{8Z^3} + \frac{d^2}{4\pi Z^3}\left[\arctan\frac{a}{Z}\right.$$
$$\left. + \frac{Z}{a}\frac{1 + 8/3(Z/a)^2 - (Z/a)^4}{(1 + Z^2/a^2)^3}\right].$$

This is always negative, but vanishes at infinity and at zero. Numerically, we find that the force changes sign at $Z = 0.742358a$.

VII.Quantum vacuum energy

The reason why the energy vanishes when the dipole is centered in the aperture is clear: Then the electric field lines are perpendicular to the conducting sheet on the surface, and the sheet could be removed without changing the field configuration.
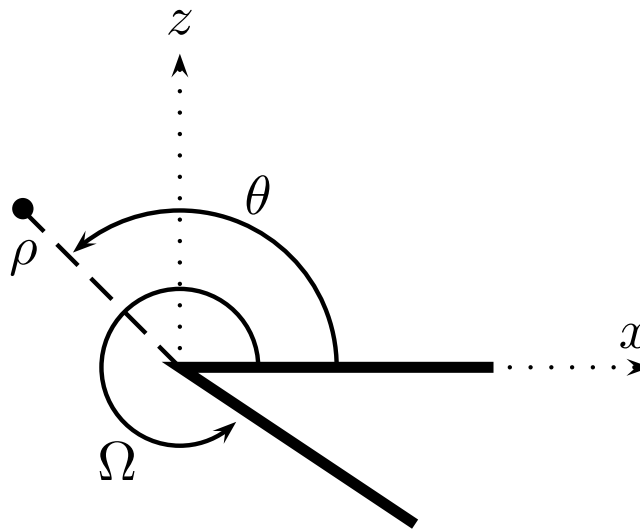
Our goal is to analytically find the quantum (Casimir) analog of this classical respulsion.

Casimir force: cylinder/aperture
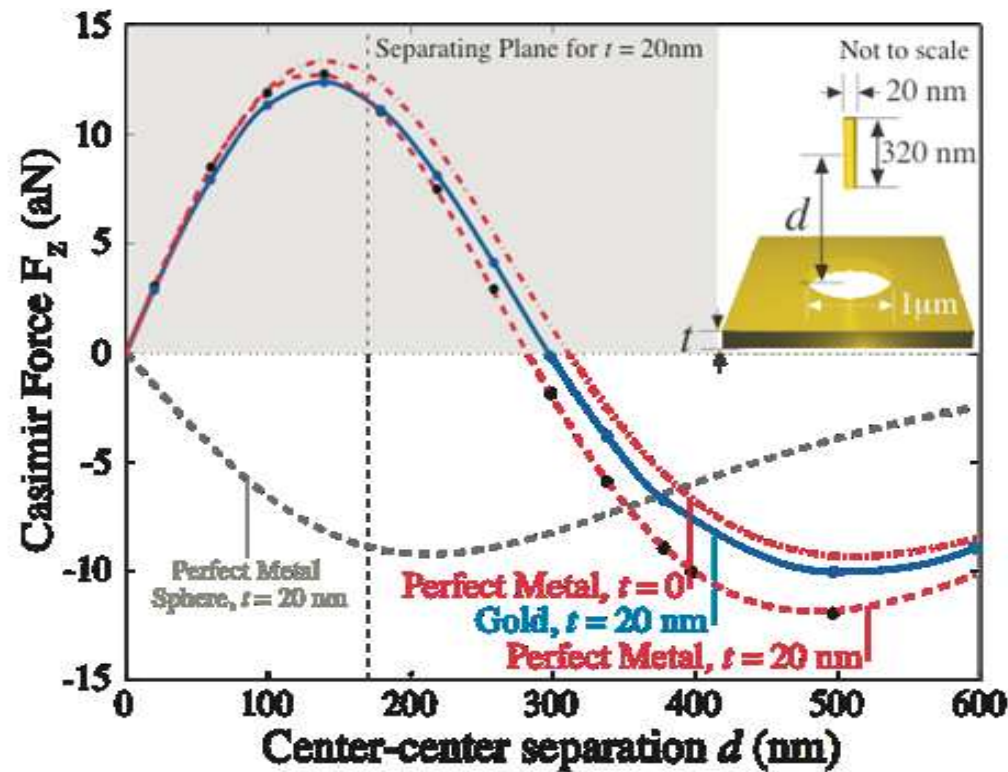
Levin et al., PRL **105**, 090403 (2010).

Polarizable atom, located at polar coordinates ρ, θ, within a conducting wedge with dihedral angle $\Omega = \pi/p$.

Green's dyadic

$$\boldsymbol{\Gamma}(\mathbf{r}, \mathbf{r}') = 2p \sum_{m=0}^{\infty}{}' \int \frac{dk}{2\pi} \Bigg[-\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{M}}'^{*}(\nabla_{\perp}^{2} - k^{2})$$

$$\times \frac{1}{\omega^{2}} F_{mp}(\rho, \rho') \frac{\cos mp\theta \cos mp\theta'}{\pi} e^{ik(y-y')}$$

$$+ \boldsymbol{\mathcal{N}}\boldsymbol{\mathcal{N}}'^{*} \frac{1}{\omega} G_{mp}(\rho, \rho') \frac{\sin mp\theta \sin mp\theta'}{\pi} e^{ik(y-y')}$$

$$\boldsymbol{\mathcal{M}} = \hat{\rho}\frac{\partial}{\rho\partial\theta} - \hat{\theta}\frac{\partial}{\partial\rho},$$

$$\boldsymbol{\mathcal{N}} = ik\left(\hat{\rho}\frac{\partial}{\partial\rho} + \hat{\theta}\frac{\partial}{\rho\partial\theta}\right) - \hat{\mathbf{y}}\nabla_{\perp}^{2}.$$

Free reduced Green's functions

In this situation, the boundaries are entirely in planes of constant θ, so the radial Green's functions are equal to the free Green's function

$$\frac{1}{\omega^2}F_{mp}(\rho,\rho') = \frac{1}{\omega}G_{mp}(\rho,\rho') = -\frac{i\pi}{2\lambda^2}J_{mp}(\lambda\rho_<)H_{mp}^{(1)}(\lambda\rho_>)$$

with $\lambda^2 = \omega^2 - k^2$. We will immediately make the Euclidean rotation, $\omega \to i\zeta$, where $\lambda \to i\kappa$, $\kappa^2 = \zeta^2 + k^2$, so the free Green's functions become $-\kappa^{-2}I_{mp}(\kappa\rho_<)K_{mp}(\kappa\rho_>)$.

Completely anisotropic atom

We start by considering the most favorable case for CP repulsion, where only $\alpha_{zz} \neq 0$. In the static limit, then the only component of the Green's dyadic that contributes is $\int \frac{d\zeta}{2\pi} \Gamma_{zz}$

$$
= \frac{2p}{4\pi^3} \int dk \, d\zeta \left\{ \left[\zeta^2 \sin^2 \theta \sin^2 mp\theta - k^2 \cos^2 \theta \cos^2 mp\theta \right. \right.
$$

$$
\times \frac{m^2 p^2}{\kappa^2 \rho_< \rho_>} I_{mp}(\kappa \rho_<) K_{mp}(\kappa \rho_>)
$$

$$
- \left[k^2 \sin^2 \theta \sin^2 mp\theta - \zeta^2 \cos^2 \theta \cos^2 mp\theta \right]
$$

$$
\times I'_{mp}(\kappa \rho_<) K'_{mp}(\kappa \rho_>) \bigg\}.
$$

Divergent term

Now the integral over the Bessel functions is given by

$$\int_0^\infty d\kappa\,\kappa\,I_\nu(\kappa\rho_<)K_\nu(\kappa\rho_>) = \frac{z^\nu}{\rho_>^2(1-\xi^2)},$$

where $\xi = \rho_</\rho_>$. After that the m sum is easily carried out by summing a geometrical series. Care must also be taken with the $m = 0$ term in the cosine series. The result of a straightforward calculation leads to

$$\int \frac{d\zeta}{2\pi}\Gamma_{zz} = -\frac{\cos 2\theta}{\pi^2\rho^4}\frac{1}{(\xi-1)^4} + \text{finite}.$$

Subtracted Casimir energy

The divergent term is that of the vacuum without the wedge, so we must subtract this term off, leaving for the static Casimir energy

$$U_{\mathrm{CP}}^{zz} = -\frac{\alpha_{zz}(0)}{8\pi} \frac{1}{\rho^4 \sin^4 p\theta} \left[p^4 - \frac{2}{3} p^2 (p^2 - 1) \sin^2 p\theta \right.$$
$$\left. + \frac{(p^2 - 1)(p^2 + 11)}{45} \sin^4 p\theta \cos 2\theta \right].$$

This result may also be easily derived from the closed form given by Lukosz.

Expected Casimir-Polder result

A small check of this result is that as $\theta \to 0$ (or $\theta \to \Omega$) we recover the expected Casimir-Polder result for an atom above an infinite plane:

$$U^{zz}_{\mathrm{CP}} \to -\frac{\alpha_{zz}(0)}{8\pi Z^4},$$

in terms of the distance of the atom above the plane, $Z = \rho\theta$. This limit is also obtained when $p \to 1$, for when $\Omega = \pi$ we are describing a perfectly conducting infinite plane.

Isotropic atom

A very similar calculation gives the result for an isotropic atom, $\boldsymbol{\alpha} = \alpha\mathbf{1}$, which was first given by Brevik, Lygren, and Marachevsky:

$$
U_{\text{CP}} = -\frac{3\alpha(0)}{8\pi\rho^4 \sin^4 p\theta}\left[p^4 - \frac{2}{3}p^2(p^2 - 1)\sin^2 p\theta \right.
$$
$$
\left. -\frac{1}{3}\frac{1}{45}(p^2 - 1)(p^2 + 11)\sin^4 p\theta \right].
$$

Note that this is not three times U_{CP}^{zz} in above, because the $\cos 2\theta$ factor in the last term in the latter is replaced by $-1/3$ here.

B. Repulsion by half-plane

Let us consider the special case $p = 1/2$, that is $\Omega = 2\pi$, the case of a semi-infinite conducting
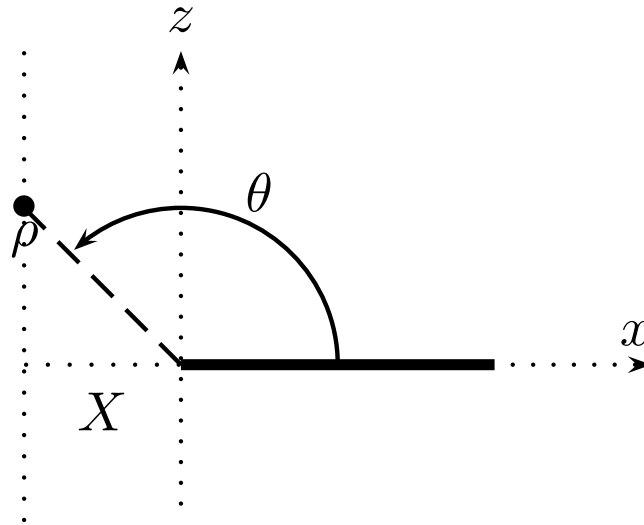
plane.   Consider a particle free to move along a line parallel to the z axis, a distance X to the left of the semi-infinite plane.

Infinite aperture

The half-plane $x < 0$ constitutes an aperture of infinite width. With X fixed, we can describe the trajectory by $u = X/\rho = -\cos\theta$, which variable ranges from zero to one. The polar angle is given by

$$\sin^2\frac{\theta}{2} = \frac{1+u}{2}.$$

The energy for an isotropic atom is given by

$$U_{\mathrm{CP}} = -\frac{\alpha(0)}{32\pi}\frac{1}{X^4}V(u),$$

$$V(u) = 3u^4\left[\frac{1}{(1+u)^2} + \frac{1}{u+1} + \frac{1}{4}\right].$$

Anisotropic atom

The energy for the completely anisotropic atom is

$$V_{zz} = \frac{1}{3}V(u) + \frac{u^4}{2}(1 - 3u^2).$$

If we consider instead a cylindrically symmetric polarizable atom in which

$$\boldsymbol{\alpha} = \alpha_{zz}\hat{\mathbf{z}}\hat{\mathbf{z}} + \gamma\alpha_{zz}(\hat{\mathbf{x}}\hat{\mathbf{x}} + \hat{\mathbf{y}}\hat{\mathbf{y}}) = \alpha_{zz}(1-\gamma)\hat{\mathbf{z}}\hat{\mathbf{z}} + \gamma\alpha_{zz}\mathbf{1},$$

where γ is the ratio of the transverse polarizability to the longitudinal polarizability of the atom, the effective potential is

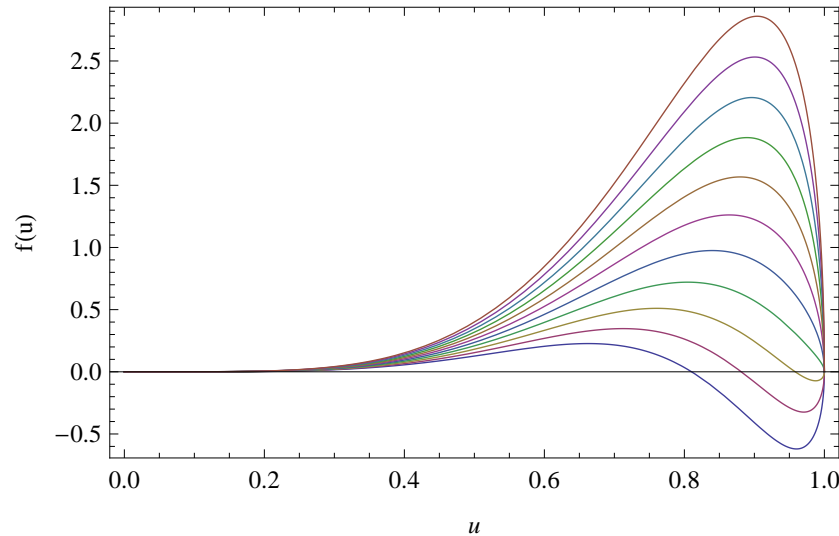$$(1 - \gamma)V_{zz} + \gamma V,$$

Force on anisotropic atom

and the z-component of the force on the atom is

$$F_z^{\gamma} = -\frac{\alpha_{zz}(0)}{32\pi}\frac{1}{X^5}u^2\sqrt{1-u^2}\frac{d}{du}\left[\frac{1}{2}u^4(1-\gamma)(1-3u^2)\right.$$

$$\left. + \frac{1}{3}(1+2\gamma)V(u)\right],$$

Note that the energy only vanishes at $u = 1$ (the plane of the conductor) when $\gamma = 0$. Thus, the argument given in Levin et al. applies only for the completely anisotropic case.

F_z on anisotropic atom

$F_z = -\alpha_{zz}/(32\pi X^5)f(u)$ $f > 0$ is attractive, $f < 0$ repulsive. γ goes from 0 to 1 by steps of 0.1, from bottom to top. For $\gamma < 1/4$ a repulsive regime always occurs when the atom is sufficiently close to the plane of the conductor.

Finer resolution
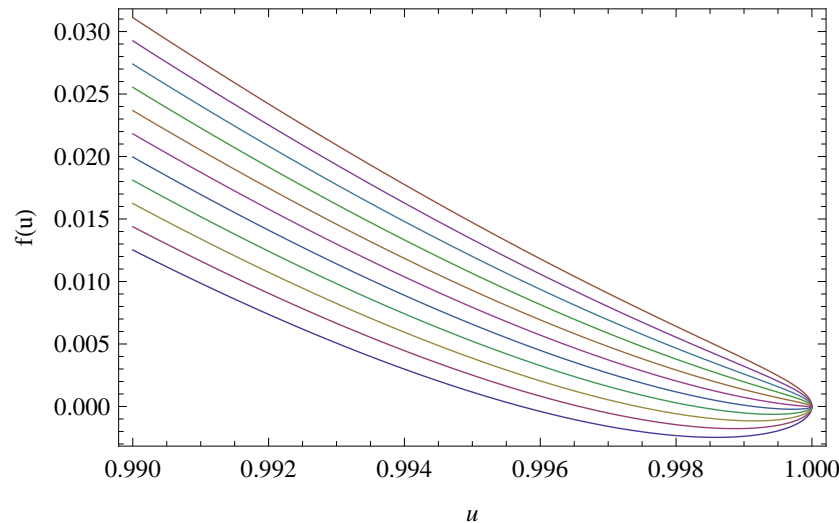
The region close to the plane, $1 \geq u \geq 0.99$, with γ near the critical value of 1/4. Here from bottom to top are shown the results for values of γ from 0.245 to 0.255 by steps of 0.001.

Same effect for vdW

It is interesting to observe that the same critical value of γ occurs for the nonretarded (electrostatic) regime of a circular aperture, as follows from a simple computation based on the result of Ebelein and Zietal.
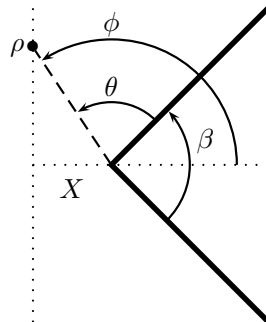
$$
U = -\frac{1}{16\pi^2} \int_{-\infty}^{\infty} d\zeta \, \alpha_{zz}(\zeta)
$$

$$
\times \frac{1}{Z^3} \left\{ (1+\gamma) \left(\frac{\pi}{2} + \arctan \frac{Z^2 - a^2}{2aZ} \right) \right.
$$

$$
+ \frac{2aZ}{(Z^2 + a^2)^3} \left[(1+\gamma)(Z^4 - a^4) - \frac{8}{3}(1-\gamma)a^2Z^2 \right.
$$

C. Repulsion by a wedge

It is very easy to generalize the above result for a wedge, $p > 1/2$. That is, we want to consider a strongly anisotropic atom, with only α_{zz} significant, to the left of a wedge of opening angle

$$\beta = 2\pi - \Omega,$$

as shown in the figure.

Changed coordinates

We want the z axis to be perpendicular to the symmetry axis of the wedge so the relation between the polar angle of the atom and the angle to the symmetry line is

$$\phi = \theta + \beta/2,$$

where, as before, θ is the angle relative to the top surface of the wedge. The C-P energy is changed only by the replacement $\cos 2\theta$ by $\cos 2\phi$, with no change in $\sin p\theta$. How does repulsion depends on the wedge angle β?

Dependence on wedge angle

Write for an atom on the line $x = -X$

$$U_{\mathrm{CP}}^{zz} = -\frac{\alpha_{zz}(0)}{8\pi X^4} V(\phi),$$

where

$$V(\phi) = \cos^4 \phi \left[\frac{p^4}{\sin^4 \frac{\pi}{2} \frac{\phi - \beta/2}{\pi - \beta/2}} - \frac{2}{3} \frac{p^2(p^2 - 1)}{\sin^2 \frac{\pi}{2} \frac{\phi - \beta/2}{\pi - \beta/2}} \right.$$
$$\left. + \frac{1}{45}(p^2 - 1)(p^2 + 11) \cos 2\phi \right].$$

Vanishing of energy?

At the point of closest approach,

$$V(\pi) = \frac{1}{45}(4p^2 - 1)(4p^2 + 11),$$

so the potential vanishes at that point only for the half-plane case, $p = 1/2$.

F_z

$$F_z = -\frac{\alpha_{zz}}{8\pi}\frac{1}{X^5}\cos^2\phi\frac{\partial V(\phi)}{\partial\phi}.$$

The figure shows the force as a function of ϕ for fixed X. It will be seen that the force has a repulsive region for angles close enough to the apex of the wedge, provided that the wedge angle is not too large. The critical wedge angle is actually rather large, $\beta_c = 1.87795$, or about $108°$. For larger angles, the z-component of the force exhibits only attraction.
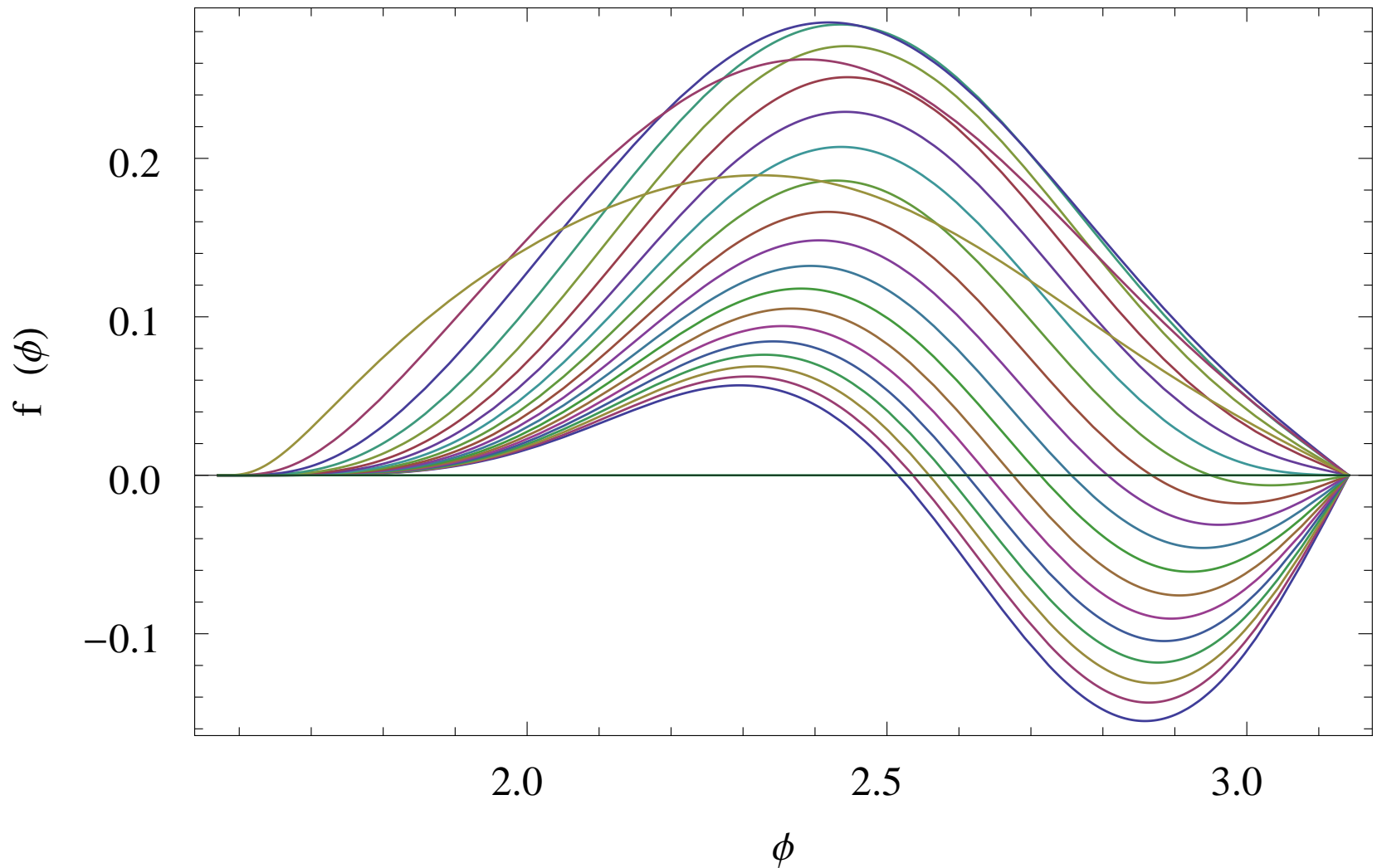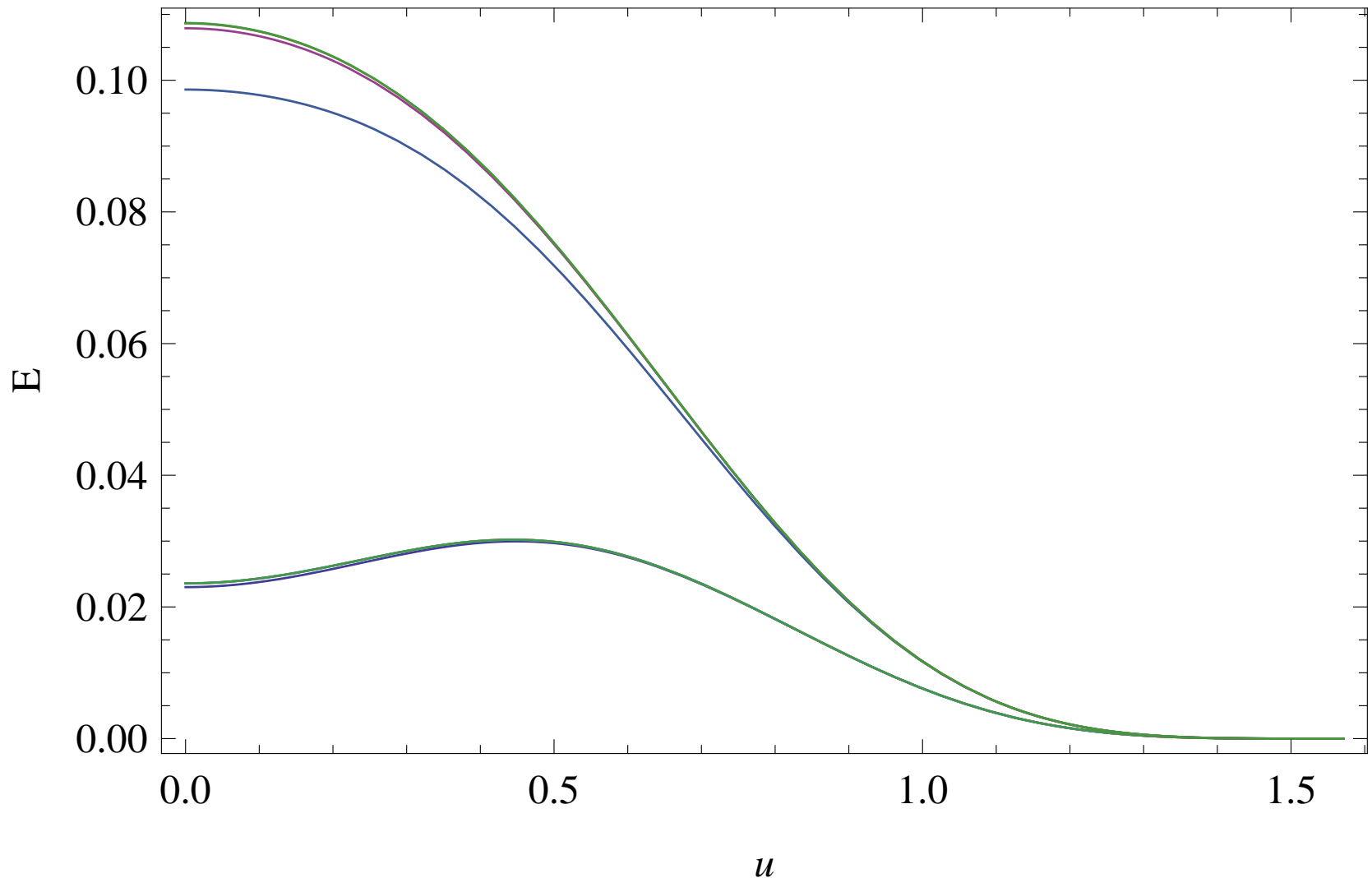
The effect of including the first 2–5 terms.

Plotted is the total CP energy, the upper set being for the distance of closest approach R being 5 times the cylinder radius a, the lower set for the distance of closest approach 10 times the radius. Repulsion is clearly observed when $R/a = 10$, but not for $R/a = 5$. For a conducting sphere, since at large distances it looks like a polarizable atom (with both electric and magnetic polarizabilities), no repulsion on a completely anisotropic atom occurs.

VIII. Conclusions

- Casimir self-energies often exhibit repulsion, but general systematics are not yet worked out.

- Repulsion occurs between electric and magnetic conductors, or materials or metamaterials that mimic this behavior over a wide frequency range.

- Intervening intermediate "density" materials can mimic repulsion.

- But true repulsion can be exhibited in CP situations with suitable anisotropies.

- Stay tuned!