The zeta-regularized spectral determinant and vacuum energy of graph Schrödinger operators

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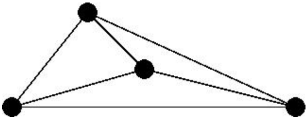
Outline

1. Graph model
2. Dirichlet wire
3. General graph

Quantum graph model



V vertices
B bonds

Metric graph: bond b corresponds to interval $[0, L_b]$.
Eigenproblem on $[0, L_b]$,

$$\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x_b} + A_b\right)^2 \psi_b(x_b) + V_b(x_b)\psi_b(x_b) = E\psi_b(x_b) \ . \tag{1}$$

Matching conditions at vertices so Schrödinger op. self-adjoint on graph. Hilbert space $\mathcal{H} := \bigoplus_{b=1}^{B} L^2([0, L_b])$.

Graph spectrum: $0 < E_0 \leqslant E_1 \leqslant \ldots$

Spectral determinant

$$S(\gamma) = \prod_{j=0}^{\infty} (\gamma + E_j)$$

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(99) Pascaud & Montambaux.

(00) Akkermans, Comtet, Desbois, Montambaux & Texier.

(06) Friedlander: Dirichlet to Neumann map, δ-coupling conditions.

(10) Texier: δ-coupling conditions with potential, conjecture for general matching conditions [3].

(11) JH & Kirsten: Laplacian with general vertex matching conditions [2].

Vacuum energy

$$E_c = \frac{1}{2} \sum_{j=0}^{\infty}{}' \sqrt{E_j}$$

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Models of vacuum energy:

(01) Leboeuf, Monastra, & Bohigas: Riemannium.

(07) Fulling, Kaplan & Wilson: star graphs, repulsive Casimir effect for $B > 3$.

(09) Berkolaiko, JH & Wilson: general graphs, effect of chaotic classical dynamics.

Spectral zeta function

$$\zeta(s,\gamma) = \sum_{j=1}^{\infty} (\gamma + E_j)^{-s} \qquad \mathrm{Re}\, s > 1$$

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Spectral properties

$$S(\gamma) := \exp\Big(-\zeta'(0,\gamma)\Big) = \prod_{j=0}^{\infty} (\gamma + E_j)$$

$$E_c := \frac{1}{2}\zeta(-1/2,0) = \frac{1}{2}\sum_{j=1}^{\infty} \sqrt{E_j}$$

Dirichlet wire

Single interval $[0, L]$ with $\psi(0) = \psi(L) = 0$.

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)\right)\psi(x) = k^2\psi(x) \tag{2}$$

$f_k(x)$ soln. with $f_k(0) = 1$, $f_k(L) = 0$. $\bar{f}_k(\bar{x})$ linearly independent soln. where $\bar{x} = L - x$ & $\bar{f}_k(0) = 1$, $\bar{f}_k(L) = 0$.

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Alternatively $u_k(x)$, $\bar{u}_k(\bar{x})$ solns. with $u_k(0) = 0$ and $u_k'(0) = 1$, $\bar{u}_k(x) = f_k(L - x)/f_k'(L)$.

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Secular eqn.

k^2 is eigenval. of Schrödinger op. iff

$$\bar{u}_k(L) = \frac{-1}{f_k'(L)} = 0\,.$$

General method

Secular eqn. $F(z) = 0$ related to $\zeta(s,\gamma)$ via argument principle,

$$\zeta(s,\gamma) = \int_{\mathcal{C}} \frac{\mathrm{d}z}{2\mathrm{i}\pi} \left(z^2+\gamma\right)^{-s} \frac{\mathrm{d}}{\mathrm{d}z} \log\left(F(z)\right) \ . \qquad (3)$$

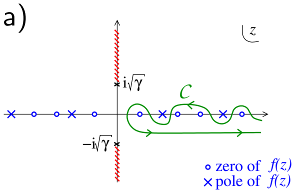

Figure: *Contours (a) before, & (b) after, contour transformation.*

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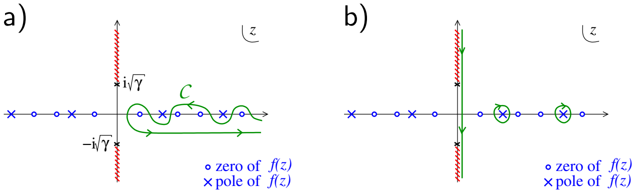

Figure: *Contours (a) before, & (b) after, contour transformation.*

Dirichlet wire: $F(z)=u_z(L)$, on imaginary axis $z=\mathrm{i}t$,

$$\zeta_{\mathrm{Dir}}(s,\gamma)=\frac{\sin\pi s}{\pi}\int_{\sqrt{\gamma}}^{\infty}\mathrm{d}t\left(t^2-\gamma\right)^{-s}\frac{\mathrm{d}}{\mathrm{d}t}\log\left(u_{\mathrm{i}t}(L)\right)\ .$$

Representation of $\zeta_{\mathrm{Dir}}(s,\gamma)$ valid $1/2 < \mathrm{Re} s < 1$.

$$\log u_{\mathrm{i}t}(L) \underset{t\to\infty}{\sim} tL - \log 2t + O(t^0) \tag{4}$$

Subtracting and adding two terms in asymptotic expansion,

$$\zeta_{\mathrm{Dir}}(s,\gamma) = \frac{\sin \pi s}{\pi} \int_{\sqrt{\gamma}}^{\infty} \mathrm{d}t \, (t^2-\gamma)^{-s} \frac{\mathrm{d}}{\mathrm{d}t} \left[\log\left(u_{\mathrm{i}t}(L)\right) - tL + \log(2t)\right]$$
$$+ L \frac{\Gamma(s-1/2)}{2\sqrt{\pi}\Gamma(s)} \gamma^{\frac{1}{2}-s} - \frac{1}{2}\gamma^{-s} \qquad -1/2 < \mathrm{Re}\, s < 1.$$

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Spectral det of Dirichlet wire

$$S_{\mathrm{Dir}}(\gamma) = \exp -\zeta'_{\mathrm{Dir}}(0,\gamma) = 2u_{\mathrm{i}\sqrt{\gamma}}(L) = \frac{-2}{f'_{\mathrm{i}\sqrt{\gamma}}(L)}$$

General graph

- $\boldsymbol{\psi}$, $\boldsymbol{\psi}'$ vectors of values of ψ & its covariant derivative at the $2B$ ends of the B bonds.
- Matching conditions specified by matrices $\mathbb{A}$ & $\mathbb{B}$,

$$\mathbb{A}\boldsymbol{\psi} + \mathbb{B}\boldsymbol{\psi}' = \mathbf{0} \ . \tag{5}$$

- $\mathbb{A}, \mathbb{B}$ define a self-adjoint Schrödinger op. iff $\mathrm{rank}(\mathbb{A}, \mathbb{B}) = 2B$ and $\mathbb{A}\mathbb{B}^\dagger = \mathbb{B}\mathbb{A}^\dagger$ (Kostrykin and Schrader).
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e.g. Neumann like matching at a vertex v: ψ continusous and $\sum_{b\sim v} \psi_b'(0) = 0$.

$$\mathbb{A} = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & -1 \\ 0 & \dots & 0 & 0 & 0 \end{pmatrix} , \qquad \mathbb{B} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ 1 & 1 & \dots & 1 \end{pmatrix} .$$

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Secular equation

$f_{b,k}(x_b), f_{\bar{b},k}(x_{\bar{b}})$ pair of lin. indep. solns of Schrödinger eqn on $[0, L_b]$ s.t. $f_{b,k}(0) = 1$ and $f_{b,k}(L_b) = 0$, $x_{\bar{b}} = L_b - x_b$.
Wavefunction on bond b,

$$\psi_{b,k}(x_b) = c_b \, f_{b,k}(x_b) \, \mathrm{e}^{\mathrm{i}A_b x_b} + c_{\bar{b}} \, f_{\bar{b},k}(x_{\bar{b}}) \, \mathrm{e}^{\mathrm{i}A_{\bar{b}} x_{\bar{b}}} \ . \qquad (6)$$

$\boldsymbol{\psi} = (c_1, \dots, c_B, c_{\bar{1}}, \dots, c_{\overline{B}})^T$ and $\hat{\boldsymbol{\psi}} = M(k)\boldsymbol{\psi}$ where,

$$[M(k)]_{ab} = \delta_{a,b} \, f'_{b,k}(0) - \delta_{a,\bar{b}} \, f'_{\bar{b},k}(L_b) \mathrm{e}^{\mathrm{i}A_b L_b} \ . \qquad (7)$$

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Substitute in matching conditions $\mathbb{A}\boldsymbol{\psi} + \mathbb{B}\boldsymbol{\psi}' = \mathbf{0}$.

Secular equation

$$\det\left(\mathbb{A} + \mathbb{B}M(k)\right) = 0$$

$F(z) := \det(\mathbb{A} + \mathbb{B}M(z))$, poles at eigenvalues of graph with Dirichlet bc's, as $f'_{b,k}(L_b) = -1/u_{b,k}(L)$,

$$\zeta(s,\gamma) = \zeta_{\mathrm{Dir}}(s,\gamma) + \zeta_{\mathrm{Im}}(s,\gamma) \ .$$

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Asymptotics:

$$F(\mathrm{i}t) \underset{t\to\infty}{\sim} \det\Big(\mathbb{A} + \mathbb{B}(-t\mathrm{I} + D(t))\Big) = \sum_{j=0}^{\infty} c_j\, t^{2B-j} \qquad (8)$$

$D(t) = \sum_{j=1}^{\infty} t^{-j} \mathrm{diag}\left\{ s_{1,j}(0), \dots, s_{B,j}(0), s_{\bar{1},j}(0), \dots, s_{\bar{B},j}(0) \right\}$, $s_{b,1}(x_b) = -V_b(x_b)/2$, subsequent fns. determined by recursion relation.

Let c_N be 1^{st} non-zero coeff, $N = 0$ and $c_0 = \det \mathbb{B}$ if $\det \mathbb{B} \neq 0$.

$\zeta_{\mathrm{Im}}(s,\gamma)$ obtained using previous contour integration method.

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Zeta function

$$\zeta(s,\gamma) = \frac{\sin \pi s}{\pi} \int_{\sqrt{\gamma}}^{\infty} (t^2 - \gamma)^{-s} \frac{\mathrm{d}}{\mathrm{d}t} \log \left(F(\mathrm{i}t) t^{N-2B} / c_N \right) \mathrm{d}t$$

$$+ \frac{(2B - N) \sin \pi s}{2\pi s} \gamma^{-s} + \zeta_{\mathrm{Dir}}(s,\gamma) \qquad -1/2 < \operatorname{Re} s < 1$$

$$\zeta'_{\mathrm{Im}}(0,\gamma) = \log c_N - \log \left(F(\mathrm{i}\sqrt{\gamma}) \gamma^{\frac{N-2B}{2}} \right) - \frac{2B-N}{2} \log \gamma$$

$$= \log c_N - \log \det \left(\mathbb{A} + \mathbb{B} M(\mathrm{i}\sqrt{\gamma}) \right)$$

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Spectral determinant

$$\begin{aligned} S(\gamma) &= S_{\mathrm{Dir}}(\gamma) \exp(\zeta'_{\mathrm{Im}}(0,\gamma)) \\ &= \left(\prod_{b=1}^{B} \frac{-2}{f'_{b,\mathrm{i}\sqrt{\gamma}}(L_b)} \right) \frac{\det\left(\mathbb{A} + \mathbb{B}M(\mathrm{i}\sqrt{\gamma})\right)}{c_N} \end{aligned}$$

Vacuum energy

$$\log F(\mathrm{i}t) \underset{t\to\infty}{\sim} \log(c_N t^{2B-N}) + \frac{c_{N+J}}{c_N t^J} + O(t^{-(J+1)}) \qquad (9)$$

c_N and c_{N+J} first 2 non-vanishing coeffs. in expansion of $\det\left(\mathbb{A} + \mathbb{B}(-t\mathrm{I} + D(t))\right)$.

$$\zeta_{\mathrm{Im}}(s,\gamma) = \frac{(2B-N)\sin \pi s}{2\pi s}\gamma^{-s} - \frac{J c_{N+J}\,\Gamma(s+J/2)}{2c_N\,\Gamma(s)\Gamma(J/2+1)}\gamma^{-(2s+J)}$$
$$+ \frac{\sin \pi s}{\pi}\int_{\sqrt{\gamma}}^{\infty}(t^2-\gamma)^{-s}\frac{\mathrm{d}}{\mathrm{d}t}\left(\log\left(\frac{F(\mathrm{i}t)t^{N-2B}}{c_N}\right) - \frac{c_{N+J}}{c_N\, t^J}\right)\mathrm{d}t$$

for $-(J+1)/2 < \operatorname{Re} s < 1$

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for $-(J+1)/2 < \mathrm{Re}\, s < 1$

Generically $J=1$ and $E_c = \zeta(-1/2,0)/2$ divergent.

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Casimir force on bond β

$$
\begin{aligned}
\mathcal{F}_c^\beta &= \frac{\partial}{\partial L_\beta} E_c \\
&= \mathcal{F}_{c,\mathrm{Dir}}^\beta + \int_0^\infty \frac{\partial}{\partial L_\beta} \log F(\mathrm{i}t)\mathrm{d}t \\
F(\mathrm{i}t) &= \det\left(\mathbb{A} + \mathbb{B} M(\mathrm{i}t)\right)
\end{aligned}
$$

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Summary

- Obtained repn of zeta fns of general graph Schrödinger ops.
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Outlook

- Bootstrap results to thin networks.
- Implications for vac energy of chaotic systems.
- Extend to Dirac op.

References

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